Matrix transforms into the set of α-absolutely convergent sequences with speed and the regularity of matrices on the sub-spaces of c

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Abstract: - Let $\alpha > 1$. The α-absolute convergence with speed, where the speed is defined by a monotonically increasing positive sequence μ, has been studied in the present paper. Let l_α^μ be the set of all α-absolutely μ-convergent sequences and X a sequence space defined by another speed λ. Necessary and sufficient conditions for a matrix A (with real or complex entries) to map X into l_α^μ have been presented. It is proved as an example that the Zweier matrix $Z_{1/2}$ satisfies these necessary and sufficient conditions for certain speeds λ and μ. The notion of regularity on the subspace X of the set c of converging sequences is defined, and also, necessary and sufficient conditions for a matrix A to be regular on certain $X \subset c$ are presented. It has also been shown that there exists an irregular matrix, which is regular on the subspace X of c.

Key-Words: Matrix transforms, boundedness with speed, convergence with speed, α-absolute convergence with speed, Zweier matrix, regularity of matrices, regularity of a matrix on the subset of c.

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1 Introduction

Let X, Y be two sequence spaces and $A = (a_{nk})$ be an arbitrary matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. If for each $x = (x_k) \in X$ the series

$$A_n x := \sum_k a_{nk} x_k$$

converge and the sequence $Ax = (A_n x)$ belongs to Y, we say that A transforms X into Y. By (X, Y) we denote the set of all matrices, which transform X into Y. Let ω be the set of all real or complex valued sequences. Further we need the following well-known sub-spaces of ω: c - the space of all convergent sequences, c_0 - the space of all sequences converging to zero, l_∞ - the space of all bounded sequences, and

$$l_\alpha := \{x = (x_n) \,:\, \sum_n |x_n|^\alpha < \infty\},\ \alpha > 0.$$

For estimation and comparison of speeds of convergence of sequences are used different methods, see, for example, [1], [2], [3], [4], [5], [6], [7], [8]. We use the method, introduced in [6] and [7] (see also [1]). Let $\lambda := (\lambda_k)$ be a positive (i.e.; $\lambda_k > 0$ for every k) monotonically increasing sequence. Following [6] and [7] (see also [1]), a convergent sequence $x = (x_k)$ with

$$\lim_k x_k := s \ \text{ and } \ v_k = \lambda_k (x_k - s) \qquad (1)$$

is called bounded with the speed λ (shortly, λ-bounded) if $v_k = O(1)$ (or $(v_k) \in l_\infty$), and convergent with the speed λ (shortly, λ-convergent) if the finite limit

$$\lim_k v_k := b$$

exists (or $(v_k) \in c$). In the following we define the notion of α-absolute convergence with speed.

Definition 1. *We say that a convergent sequence $x = (x_k)$ with the finite* limit *s is called α-absolutely convergent with speed λ (or shortly, α-absolutely λ-convergent), if $(v_k) \in l_\alpha$. For $\alpha = 1$ a sequence x is said to be absolutely convergent with the speed λ (shortly, absolutely λ-convergent).*

We denote the set of all λ-bounded sequences by l_∞^λ, the set of all λ-convergent sequences by c^λ, and the set of all α-absolutely λ-convergent sequences by l_α^λ. Moreover, let

$$c_0^\lambda := \{x = (x_k) \,:\, x \in c^\lambda \text{ and } \lim_k \lambda_k (x_k - s) = 0\}$$

and

$$l_{\infty,0}^\lambda = \{x = (x_k) \,:\, x \in l_\infty^\lambda \cap c_0\}.$$

It is not difficult to see that

$$l_\alpha^\lambda \subset c_0^\lambda \subset c^\lambda \subset l_\infty^\lambda \subset c,\ l_{\infty,0}^\lambda \subset l_\infty^\lambda \subset c.$$

In addition to it, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O(1)$, we get $c^\lambda = l_\infty^\lambda = c$.

Let $e = (1, 1, ...)$, $e^k = (0, ..., 0, 1, 0, ...)$, where 1 is in the k-th position, and $\lambda^{-1} = (1/\lambda_k)$. We note that

$$e, e^k, \lambda^{-1} \in c^\lambda; \quad e, e^k \in l_\alpha^\lambda.$$

A matrix A is said to be regular if $A \in (c, c)$ and $\lim_n A_n x = \lim_n x_n$ for every sequence $x = (x_n) \in c$.

Definition 2. *Let X be a subspace of c*; i.e., *$X \subseteq c$. We say that a matrix A is regular on X, if $\lim_n A_n x = \lim_n x_n$ for every sequence $x = (x_n) \in X$.*

Let $\mu := (\mu_n)$ be another speed of convergence; i.e., a monotonically increasing positive sequence. Matrix transforms between the subsets of c defined by the speeds λ and μ have been studied by the authors of the present work in several papers. For example, in [9] the sets $\left(l_\infty^\lambda, c^\mu\right)$, $\left(l_\infty^\lambda, l_{\infty,0}^\mu\right)$, $\left(l_\infty^\lambda, c_0^\mu\right)$, $\left(c^\lambda, l_{\infty,0}^\mu\right)$, $\left(c^\lambda, c_0^\mu\right)$, $\left(l_{\infty,0}^\lambda, l_\infty^\mu\right)$, $\left(l_{\infty,0}^\lambda, l_{\infty,0}^\mu\right)$, $\left(l_{\infty,0}^\lambda, c^\mu\right)$, $\left(l_{\infty,0}^\lambda, c_0^\mu\right)$, $\left(c_0^\lambda, l_\infty^\mu\right)$, $\left(c_0^\lambda, l_{\infty,0}^\mu\right)$, $\left(c_0^\lambda, c^\mu\right)$ and $\left(c_0^\lambda, c_0^\mu\right)$ have been characterized. A short overview on the convergence with speed has been presented in [1].

The boundedness and convergence with speed are tightly connected with the problems of convergence acceleration and improvement by matrices. These problems have been studied by one author of the present paper (see, for example, [1]), and by several other authors; for example, [10], [11], [12], [13], [14], [15], [16] and [17]. Moreover, in [16] and [17], the λ-convergence and the λ-boundedness in abstract spaces, considering instead of a matrix with real or complex entries a matrix, whose elements are bounded linear operators from a Banach space X into a Banach space Y, have been studied.

We note that the results connected with convergence, absolute convergence, α-absolute λ-convergence and boundedness with speed can be used in several applications. For example, in the theoretical physics such results can be used for accelerating the slowly convergent processes, a good overview of such investigations can be found, for example, from the sources [18] and [19]. These results also have several applications in the approximation theory. Besides, in [1] such results are used for the estimation of the order of approximation of Fourier expansions in Banach spaces.

In the present paper we describe the matrix transforms related to the α-absolute λ-convergence for the case $\alpha > 1$, giving the characterization for the sets $(l_\infty^\lambda, l_\alpha^\mu)$, $(c^\lambda, l_\alpha^\mu)$, $(c_0^\lambda, l_\alpha^\mu)$, $(l_1^\lambda, l_\alpha^\mu)$, and necessary and sufficient conditions for the regularity of a matrix A on l_∞^λ, c^λ and c_0^λ. Also we will present an example of irregular matrix, which is regular on c_0^λ and on c^λ, but not on l_∞^λ for some λ. Moreover, we will prove that this irregular matrix is regular on c_0^λ, on c^λ and on l_∞^λ for another speed λ.

2 Auxiliary results

For the proof of main results we need some auxiliary results.

Lemma 1 ([20], p. 44, see also [21], Proposition 12). *A matrix $A = (a_{nk}) \in (c_0, c)$ if and only if*

$$there\ exists\ limits\ \lim_n a_{nk} := a_k, \tag{2}$$

$$\sum_k |a_{nk}| = O(1). \tag{3}$$

Moreover,

$$\lim_n A_n x = \sum_k a_k x_k \tag{4}$$

for every $x = (x_k) \in c_0$.

Lemma 2 ([20], p. 46-47, see also [21], Proposition 11 or [22], p. 17-19). *A matrix $A = (a_{nk}) \in (c, c)$ if and only if conditions* (2), (3) *are satisfied and*

$$there\ exists\ \tau\ with\ \lim_n \sum_k a_{nk} := \tau. \tag{5}$$

Moreover, if $\lim_k x_k = s$ for $x = (x_k) \in c$, then

$$\lim_n A_n x = s\tau + \sum_k (x_k - s) a_k.$$

A matrix A is regular if and only if conditions (2), (3) *and* (5) *are satisfied with $a_k = 0$ and $\tau = 1$.*

Lemma 3 ([20], p. 51, see also [21], Proposition 10). The following statements are equivalent:

(a) $A = (a_{nk}) \in (l_\infty, c)$.

(b) *The conditions* (2), (3) *are satisfied and*

$$\lim_n \sum_k |a_{nk} - a_k| = 0. \tag{6}$$

(c) *The condition* (2) *holds and*

$$\sum_k |a_{nk}|\ converges\ uniformly\ in\ n. \tag{7}$$

Moreover, if one of statements (a)-(c) *is satisfied, then the equation* (4) *holds for every $x = (x_k) \in l_\infty$.*

Lemma 4 ([21], Proposition 17 or [22], pp. 25-26). *A matrix $A = (a_{nk}) \in (l_1, c)$ if and only if condition* (2) *is satisfied and*

$$a_{nk} = O(1). \tag{8}$$

Moreover, the equation (4) *holds for every $x = (x_k) \in l_1$.*

Lemma 5 ([21], Proposition 21). *A matrix $A = (a_{nk}) \in (l_\infty, c_0)$ if and only if*

$$\lim_n \sum_k |a_{nk}| = 0.$$

Lemma 6 ([21], Proposition 22). *A matrix $A = (a_{nk}) \in (c, c_0)$ if and only if conditions* (2) *and* (5) *with $a_k = 0$, $\tau = 0$, and condition* (3) *are satisfied.*

Lemma 7 ([21], Proposition 23). *A matrix $A = (a_{nk}) \in (c_0, c_0)$ if and only if condition* (2) *with $a_k = 0$, and condition* (3) *are satisfied.*

Lemma 8 ([21], Proposition 68). *A matrix* $A = (a_{nk}) \in (l_1, l_\alpha)$ *for* $\alpha > 1$ *if and only if*

$$\sum_n |a_{nk}|^\alpha = O(1).$$

Lemma 9 ([21], Proposition 63). *A matrix* $A = (a_{nk}) \in (l_\infty, l_\alpha) = (c, l_\alpha) = (c_0, l_\alpha)$ *for* $\alpha > 1$ *if and only if*

$$\sum_n \left| \sum_{k \in K} a_{nk} \right|^\alpha = O(1)$$

for every finite subset K *of* $\mathbf{N} := \{0, 1, 2, ...\}$, *or the series*

$$\sum_n \left| \sum_{k \in K} a_{nk} \right|^\alpha$$

is convergent for arbitrary subset K^* *of* $\mathbf{N}$.

3 Main results

3.1 Matrix transforms into the set l_α^μ

First we prove

Theorem 1. *Let* $\lambda_n \neq O(1)$. *A matrix* $A = (a_{nk}) \in \left(l_\infty^\lambda, l_\alpha^\mu\right)$ *for* $\alpha > 1$ *if and only if condition* (2) *is satisfied, and*

$$Ae = (\tau_n) \in l_\alpha^\mu, \ \tau_n := A_n e = \sum_k a_{nk}, \tag{9}$$

$$\sum_k \frac{|a_{nk}|}{\lambda_k} = O(1), \tag{10}$$

$$\lim_n \sum_k \frac{|a_{nk} - a_k|}{\lambda_k} = 0, \tag{11}$$

$$\sum_n \mu_n^\alpha \left[\sum_{k \in K} \frac{a_{nk} - a_k}{\lambda_k} \right]^\alpha = O(1), \tag{12}$$

where K *is an arbitrary finite subset of* $\mathbf{N}$.

Proof. Necessity. Assume that $A \in \left(l_\infty^\lambda, l_\alpha^\mu\right)$. As $e \in l_\infty^\lambda$ and $e^k \in l_\infty^\lambda$, then conditions (2) and (9) hold. Since, from (1) we have

$$x_k = \frac{v_k}{\lambda_k} + s; \ s := \lim_k x_k, \ (v_k) \in l_\alpha$$

for every $x := (x_k) \in l_\infty^\lambda$, it follows that

$$A_n x = \sum_k \frac{a_{nk}}{\lambda_k} v_k + s\tau_n. \tag{13}$$

As $(\tau_n) \in l_\alpha^\mu$ by (9), then, from (13) we obtain that the matrix

$$A_\lambda := \left(\frac{a_{nk}}{\lambda_k} \right)$$

transforms this sequence $(v_k) \in l_\infty$ into c. In addition, for every sequence $(v_k) \in l_\infty$, the sequence $(v_k/\lambda_k) \in c_0$. But, for (v_k/λ_k), there exists a convergent sequence $x := (x_k)$ with $s := \lim_k x_k$, such that $v_k/\lambda_k = x_k - s$. So we have proved that, for every sequence $(v_k) \in l_\infty$ there exists a sequence $(x_k) \in l_\infty^\lambda$ such that $v_k = \lambda_k (x_k - s)$. Hence $A_\lambda \in (l_\infty, c)$. This implies, by Lemma 3 ((a) and (b)), that conditions (10) and (11) are satisfied, since for A_λ conditions (3) and (6) take correspondingly the forms (10) and (11), and the finite limit

$$\phi := \lim_n A_n x = \sum_k \frac{a_k}{\lambda_k} v_k + s \lim_n \tau_n$$

exists for every $x \in l_\infty^\lambda$. Writing

$$\mu_n(A_n x - \phi) = \mu_n \sum_k \frac{a_{nk} - a_k}{\lambda_k} v_k$$
$$+ s\mu_n(\tau_n - \lim_n \tau_n), \tag{14}$$

we obtain, by (9), that the matrix $A_{\lambda,\mu} \in (l_\infty, l_\alpha)$, where

$$A_{\lambda,\mu} := \left(\mu_n \frac{a_{nk} - a_k}{\lambda_k} \right).$$

Hence condition (12) is satisfied by Lemma 9, since for $A_{\lambda,\mu} \in (l_\infty, l_\alpha)$ the first condition of Lemma 9 takes the form (12).

Sufficiency. Let conditions (2) and (9) - (12) be fulfilled. Then relation (13) also holds for every $x \in l_\infty^\lambda$ and $(\tau_n) \in l_\alpha^\mu$ by (9). Hence, $A_\lambda \in (l_\infty, c)$, and the finite limit ϕ exists for every $x \in l_\infty^\lambda$ by Lemma 3 ((a) and (b)). Hence relation (14) holds for every $x \in l_\infty^\lambda$. As (12) holds, then $A_{\lambda,\mu} \in (l_\infty, l_\alpha)$ by Lemma 9. Therefore, due to (9), $A \in \left(l_\infty^\lambda, l_\alpha^\mu\right)$.

Remark 1. Conditions (10) and (11) can be replaced by the condition

$$\text{the series } \sum_k \frac{|a_{nk}|}{\lambda_k} \text{ converges uniformly in } n \tag{15}$$

in Theorem 1 by Lemma 3 ((a) and (c)).

Remark 2. If $\lambda_n = O(1)$, then a matrix $A = (a_{nk}) \in \left(l_\infty^\lambda, l_\alpha^\mu\right)$ for $\alpha > 1$ if and only if conditions (2), (9), (10) and (12) are satisfied. Indeed, in this case $(v_k) \in c_0$ for every $x := (x_k) \in l_\infty^\lambda$. Hence instead of $A_\lambda \in (l_\infty, c)$ we get $A_\lambda \in (c_0, c)$. Therefore instead of Lemma 3 ((a) and (b)) we use now Lemma

1. Moreover, instead of $A_{\lambda,\mu} \in (l_\infty, c)$ in the present case $A_{\lambda,\mu} \in (c_0, l_\alpha)$. As $(c_0, l_\alpha) = (l_\infty, c)$, then for $A_{\lambda,\mu}$ we can use again Lemma 9 as we did in the proof of Theorem 1.

Corollary 1. *Condition* (10) *can be replaced by condition*

$$\sum_k \frac{|a_k|}{\lambda_k} < \infty \tag{16}$$

in Theorem 1.

Proof. It is easy to see that condition (16) follows from (2) and (10). In the same way, conditions (2), (11) and (16) imply the validity of (10). Indeed, first from condition (11) we obtain that

$$\sum_k \frac{|a_{nk} - a_k|}{\lambda_k} = O(1). \tag{17}$$

Since

$$\frac{a_{nk}}{\lambda_k} = \frac{a_{nk} - a_k}{\lambda_k} + \frac{a_k}{\lambda_k},$$

then

$$\sum_k \frac{|a_{nk}|}{\lambda_k} \leq \sum_k \frac{|a_{nk} - a_k|}{\lambda_k} + \sum_k \frac{|a_k|}{\lambda_k}.$$

Moreover, the finite limits a_k exist by (2). Hence the condition (10) is satisfied by (16) and (17).

Theorem 2. *A matrix* $A = (a_{nk}) \in \left(c_0^\lambda, l_\alpha^\mu\right)$ *for* $\alpha >$ 1 *if and only if conditions* (2), (9), (10) *and* (12) *are satisfied.*

Proof is similar to the proof of Theorem 1. The only difference is that now $A_\lambda \in (c_0, c)$ and $A_{\lambda,\mu} \in (c_0, l_\alpha)$. Therefore instead of Lemma 3 ((a) and (b)) we use Lemma 1 (for $A_{\lambda,\mu} \in (c_0, l_\alpha)$ we use again Lemma 9 as in the proof of Theorem 1).

Theorem 3. *A matrix* $A = (a_{nk}) \in \left(l_1^\lambda, l_\alpha^\mu\right)$ *for* $\alpha >$ 1 *if and only if conditions* (2), (9) *are satisfied and*

$$\frac{a_{nk}}{\lambda_k} = O(1), \tag{18}$$

$$\frac{1}{\lambda_k^\alpha} \sum_n \left[\mu_n |a_{nk} - a_k|\right]^\alpha = O(1). \tag{19}$$

Proof is similar to the proof of Theorem 1. The only difference is that now $A_\lambda \in (l_1, c)$ and $A_{\lambda,\mu} \in (l_1, l_\alpha)$. Therefore instead of Lemma 3 ((a) and (b)) we use Lemma 4, and instead of Lemma 9 we use Lemma 8, considering that for A_λ condition (8) takes the form (18), and for $A_{\lambda,\mu} \in (l_\infty, l_\alpha)$ the condition of Lemma 8 takes the form (19).

Corollary 2. *Condition* (18) *can be replaced by condition*

$$\frac{a_k}{\lambda_k} = O(1) \tag{20}$$

in Theorem 3.

Proof is similar to the proof of Corollary 1, if to consider that condition (20) follows from (2) and (18), and condition (19) implies

$$\frac{a_{nk} - a_k}{\lambda_k} = O(1).$$

Theorem 4. *A matrix* $A = (a_{nk}) \in \left(c^\lambda, l_\alpha^\mu\right)$ *for* $\alpha >$ 1 *if and only if conditions* (10), (12) *are satisfied and*

$$Ae \in l_\alpha^\mu, Ae^k \in l_\alpha^\mu, A\lambda^{-1} \in l_\alpha^\mu. \tag{21}$$

Proof. Necessity. Suppose that $A = (a_{nk}) \in \left(c^\lambda, l_\alpha^\mu\right)$. As $e^k \in c^\lambda$, $e \in c^\lambda$ and $\lambda^{-1} \in c^\lambda$, then condition (21) holds. As equality (13) holds for every $x := (x_k) \in c^\lambda$, and the finite limit

$$\tau := \lim_n \tau_n$$

exists due to $Ae \in l_\alpha^\mu$, then the matrix A_λ transforms this convergent sequence (v_k) into c. Similar to the proof of the necessity of Theorem 1, it is possible to show that, for every sequence $(v_k) \in c$, there exists a sequence $(x_k) \in c^\lambda$ such that $v_k = \lambda_k (x_k - s)$. Hence $A_\lambda \in (c, c)$. This implies by Lemma 2 that the finite limits a_k and

$$a^\lambda := \lim_n \sum_k \frac{a_{nk}}{\lambda_k}$$

exist, and that condition (10) is satisfied. With the help of (13), for every $x \in c^\lambda$, we can write by Lemma 2 that

$$\phi := \lim_n A_n x = a^\lambda b + \sum_k \frac{a_k}{\lambda_k} (v_k - b) + \tau s, \tag{22}$$

where $s := \lim_k x_k$ and $b := \lim_k v_k$. Now, using (13) and (22), we obtain

$$\mu_n(A_n x - \phi) = \mu_n \sum_k \frac{a_{nk} - a_k}{\lambda_k} (v_k - b)$$

$$+\mu_n (\tau_n - \tau) s + \mu_n \left(\sum_k \frac{a_{nk}}{\lambda_k} - a^\lambda\right) b. \tag{23}$$

As $Ae \in l_\alpha^\mu$ and $A\lambda^{-1} \in l_\alpha^\mu$ by (10), then $A_{\lambda,\mu} \in (c_0, l_\alpha)$. Therefore we can conclude by Lemma 9 that condition (12) holds.

Sufficiency. Assume that conditions (10), (12) and (21) are satisfied. First we notice that relation (13) holds for every $x \in c^{\lambda}$ and the finite limits a_k, τ and a^{λ} exist by (21). As (10) also holds, then $A_{\lambda} \in (c, c)$ by Lemma 2, and therefore relations (22) and (23) hold for every $x \in c^{\lambda}$. As condition (12) holds, then $A_{\lambda,\mu} \in (c_0, l_{\alpha})$ by Lemma 9. Moreover, $Ae \in l_{\alpha}^{\mu}$ and $A\lambda^{-1} \in l_{\alpha}^{\mu}$ by (21). Thus, $A \in \left(c^{\lambda}, l_{\alpha}^{\mu}\right)$.

Remark 3. Condition (12) can be replaced by the condition

$$\sum_n \mu_n^{\alpha} \left[\sum_{k\in K^*} \frac{a_{nk} - a_k}{\lambda_k}\right]^{\alpha} < \infty \qquad (24)$$

for arbitrary subset K^* of **N** in Theorems 1, 2 and 4 by Lemma 9.

Example 1. Let us consider the Zweier matrix $Z_{1/2}$, defined by (a_{nk}), where (see [20], p. 14, or [1], p. 3) $a_{00} = 1/2$, $a_{nk} = 1/2$ if $k = n-1$ or $k = n$ for $n \geq 1$, and $a_{nk} = 0$ otherwise. The method $A = Z_{1/2}$ is regular (see [1], p. 3). Let λ be defined by

$$\lambda_n := (n+1)^r,\ r > 0, \qquad (25)$$

and μ by

$$\mu_n := (n+1)^t,\ t > 0. \qquad (26)$$

Case 1: $Z_{1/2} \in \left(l_{\infty}^{\lambda}, l_{\alpha}^{\mu}\right) \cap \left(c_0^{\lambda}, l_{\alpha}^{\mu}\right) \cap \left(c^{\lambda}, l_{\alpha}^{\mu}\right)$ for $\alpha > 1$, if $r < t - 1/\alpha$. For proving it, we show that all conditions of Theorems 1,2 and 4 hold. It is easy to see that in this case $a_k = 0$, $\tau = 1$, and

$$T_n := \sum_k \frac{|a_{nk}|}{\lambda_k} = \sum_k \frac{a_{nk}}{\lambda_k}.$$

As

$$T_0 = \frac{1}{2\lambda_0},\ T_n = \frac{1}{2}\left(\frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_n}\right),\ n \geq 1, \qquad (27)$$

then $\lim_n T_n = a^{\lambda} = 0$, since $\lambda_n \neq O(1)$. Therefore conditions (2), (9) - (11) and (21) hold. Also condition (12) is satisfied. Indeed,

$$S := \sum_n \mu_n^{\alpha} \left[\sum_{k\in K} \frac{a_{nk} - a_k}{\lambda_k}\right]^{\alpha}$$

$$\leq \frac{\mu_0}{2\lambda_0} + \frac{1}{2^{\alpha}} \sum_{n=1}^{\infty} \mu_n^{\alpha} \left(\frac{1}{\lambda_{n-1}} + \frac{1}{\lambda_n}\right)^{\alpha}$$

for every possible K from **N** by (27). Hence, using (25) and (26), we obtain

$$S = O(1) \sum_{n=1}^{\infty} (n+1)^{r\alpha} \frac{1}{(n+1)^{t\alpha}}$$

$$= O(1) \sum_{n=1}^{\infty} \frac{1}{(n+1)^{(t-r)\alpha}} = O(1),$$

if $(t-r)\alpha > 1$ or $r < t - 1/\alpha$. Thus, condition (12) holds.

Case 2: $Z_{1/2} \in \left(l_1^{\lambda}, l_{\alpha}^{\mu}\right)$ for $\alpha > 1$, if $r \leq t$. For proving it, we show that all conditions of Theorem 3 hold. The validity of (2) and (9) are proved in Case 1 of the present example; also it is easy to see that condition (18) is satisfied. Let

$$V := \frac{1}{\lambda_k^{\alpha}} \sum_n \left[\mu_n \left|a_{nk} - a_k\right|\right]^{\alpha}$$

$$= \frac{1}{2^{\alpha}} \frac{1}{\lambda_k^{\alpha}} \left(\mu_k^{\alpha} + \mu_{k+1}^{\alpha}\right).$$

Hence, using (25) and (26), we obtain

$$V = O(1) \frac{1}{(k+1)^{t\alpha}} \left((k+1)^{r\alpha} + (k+2)^{r\alpha}\right)$$

$$= O(1) \frac{1}{(k+1)^{(t-r)\alpha}} = O(1),$$

if $r \leq t$. Thus, condition (21) also holds.

In Example 3.1 r can't be greater than t; i.e., $\mu_n/\lambda_n = O(1)$. In the following example we consider the case, where $\mu_n/\lambda_n \neq O(1)$ also is possible for some collection of parameters.

Example 2. Let $A = (a_{nk})$ be a lower triangular matrix defined by $a_{nk} = 1/(n+1)^c$, $c > 1$, and λ, μ respectively by (25) and (26).

Case 1: $A \in \left(l_{\infty}^{\lambda}, l_{\alpha}^{\mu}\right) \cap \left(c_0^{\lambda}, l_{\alpha}^{\mu}\right) \cap \left(c^{\lambda}, l_{\alpha}^{\mu}\right)$ for $\alpha > 1$, if $t > 1$ and $r < c - 1/\alpha$. For proving it, we show that all conditions of Theorems 1,2 and 4 hold. It is easy to see that in this case $a_k = 0$,

$$\tau_n = \sum_{k=0}^{n} a_{nk} = \frac{1}{(n+1)^{c-1}};\ \tau = 0,$$

since $c > 1$. Thus conditions (2) and (9) hold. As

$$T_n = \sum_k \frac{|a_{nk}|}{\lambda_k} = \sum_k \frac{a_{nk}}{\lambda_k}$$

$$= \frac{1}{(n+1)^c} \sum_{k=0}^{n} \frac{1}{(k+1)^t},$$

then $\lim_n T_n = a^{\lambda} = 0$ (since $t > 1$). Hence conditions (10), (11) and (21) are satisfied. As

$$S = \sum_n \mu_n^{\alpha} \left[\sum_{k\in K} \frac{a_{nk} - a_k}{\lambda_k}\right]^{\alpha}$$

$$\leq \sum_{n=1}^{\infty}(n+1)^{r\alpha}\left[\frac{1}{(n+1)^c}\sum_{k=0}^{n}\frac{1}{(k+1)^t}\right]^{\alpha}$$

for every possible K from **N**, then

$$S = O(1)\sum_{n=1}^{\infty}\frac{1}{(n+1)^{(c-r)\alpha}}\left[\sum_{k=0}^{n}\frac{1}{(k+1)^t}\right]^{\alpha}$$

$$= O(1),$$

if $(c-r)\alpha > 1$ or $r < c - 1/\alpha$. Therefore condition (12) also holds. It is possible to find a collection $\{\alpha, c, r, t\}$ with $r > t$ satisfying conditions $t > 1$ and $r < c - 1/\alpha$. For example, if $\alpha = 2$, $c = 4$ and $t = 2$, then these conditions hold for r, satisfying the relation $2 < r < 3,5$.

Case 2: $A \in \left(l_1^{\lambda}, l_{\alpha}^{\mu}\right)$ for $\alpha > 1$, if $r < c - 1/\alpha$. For proving it, we show that all conditions of Theorem 3 hold. The validity of (2) and (9) are proved in Case 1 of the present example; also it is easy to see that condition (18) is satisfied. As

$$V = \frac{1}{\lambda_k^{\alpha}}\sum_n \left[\mu_n \left|a_{nk} - a_k\right|\right]^{\alpha}$$

$$= \frac{1}{(k+1)^{t\alpha}}\sum_{n=k}^{\infty}(n+1)^{r\alpha}\frac{1}{(n+1)^{c\alpha}}$$

$$= \frac{1}{(k+1)^{t\alpha}}\sum_{n=k}^{\infty}\frac{1}{(n+1)^{(c-r)\alpha}},$$

then $V = O(1)$, if $(c-r)\alpha > 1$ or $r < c - 1/\alpha$. Therefore condition (21) also holds. There exists a collection $\{\alpha, c, r, t\}$ with $r > t$ satisfying the condition $r < c - 1/\alpha$. For example, if $\alpha = 2$ and $c = 4$, then for r, t, satisfying the relation $0 < t < r < 3,5$, this condition holds.

3.2 The regularity of matrices on the sets l_{∞}^{λ}, c_0^{λ} and c^{λ}

We present necessary and sufficient conditions for the regularity of a matrix A on l_{∞}^{λ}, c_0^{λ}, and c^{λ} as the corollaries correspondingly from Theorems 1, 2 and 4.

Corollary 3. *A matrix $A = (a_{nk})$ is regular on l_{∞}^{λ} if and only if condition* (5) *with $\tau = 1$ is satisfied and*

$$\lim_n \sum_k \frac{|a_{nk}|}{\lambda_k} = 0. \tag{28}$$

Proof. Necessity. Assume that A is regular on l_{∞}^{λ}; i.e., $\lim_n A_n x = s$ for every sequence $x \in l_{\infty}^{\lambda}$. Then condition (5) with $\tau = 1$ holds, since $e \in l_{\infty}^{\lambda}$, and relation (13) holds for every $x := (x_k) \in l_{\infty}^{\lambda}$. This implies that A_{λ} transforms every sequence $(v_k) \in l_{\infty}$ into c_0. Hence condition (28) is satisfied by Lemma 5.

Sufficiency. Let all the conditions of the present corollary are satisfied. Then relation (13) also holds for every $x \in l_{\infty}^{\lambda}$. As condition (28) holds, then $A_{\lambda} \in (l_{\infty}, c_0)$ by Lemma 5. Therefore $\lim_n A_n x = s$ for every $x \in l_{\alpha}^{\lambda}$, since $\tau = 1$. Thus A is regular on l_{∞}^{λ}.

Corollary 4. *A matrix $A = (a_{nk})$ is regular on c_0^{λ} if and only if condition* (2) *with $a_k = 0$, condition* (5) *with $\tau = 1$, and condition* (10) *are satisfied.*

Proof is similar to the proof of Corollary 3, if to consider that $\tau = 1$ due to $e \in c_0^{\lambda}$, and instead of Lemma 5 it is necessary to use Lemma 7, since in this case $A_{\lambda} \in (c_0, c_0)$.

Corollary 5. *A matrix $A = (a_{nk})$ is regular on c^{λ} if and only if condition* (2) *with $a_k = 0$, condition* (5) *with $\tau = 1$ and condition* (10) *are satisfied*, and $a^{\lambda} = 0$.

Proof is similar to the proof of Corollary 3, if to consider that $\tau = 1$ due to $e \in c^{\lambda}$, and instead of Lemma 5 it is necessary to use Lemma 6, since in this case $A_{\lambda} \in (c, c_0)$.

Now we prove that there exists an irregular matrix, which is regular on l_{∞}^{λ}, c_0^{λ} and c^{λ}.

Example 3. Let $A = (a_{nk})$ be a matrix, where $a_{nk} = n+1$ if $k = n$, $a_{nk} = -n$ if $k = n+1$, and $a_{nk} = 0$ otherwise. Then obviously $a_k = 0$ and $\tau = 1$; i.e., conditions (2) and (5) hold. But condition (3) does not hold, since

$$\sum_k |a_{nk}| = 2n + 1 \neq O(1).$$

Thus the matrix A is not regular by Lemma 2.

Case 1: Let λ be defined by (25) with $r = 1$. Then

$$T_n = \sum_k \frac{|a_{nk}|}{\lambda_k} = 1 + \frac{n}{n+2} = O(1);$$

i.e., conditions (10) is satisfied. Hence A is regular on c_0^{λ} by Corollary 4. Moreover,

$$\sum_k \frac{a_{nk}}{\lambda_k} = 1 - \frac{n}{n+2},$$

$$a^{\lambda} = \lim_n \left(1 - \frac{n}{n+2}\right) = 0.$$

Therefore A is regular on c^{λ} by Corollary 5. We note that A is not regular on l_{∞}^{λ} by Corollary 3, since condition (11) does not hold.

Case 2: Let λ be defined by (25) with $r = 2$. Then also condition (10) holds, $a^{\lambda} = 0$, and in addition to it, $\lim_n T_n = 0$. Hence condition (11) is also satisfied and in addition to regularity on c_0^λ and on c^λ, A is also regular on l_∞^λ by Corollary 3.

4 Conclusion

In this paper we consider the α-absolute convergence with speed, where the speed is defined by a monotonically increasing positive sequence μ and $\alpha > 1$. The notions of ordinary convergence and boundedness with speed are known earlier. Let λ be another speed of convergence, and l_∞^λ, c_0^λ, c^λ and l_α^μ be respectively the sets of all λ-bounded, all λ-convergent to zero, all λ-convergent and all α-absolutely μ-convergent sequences.

Let A be a matrix with real or complex entries. We found necessary and sufficient conditions for the transforms $A : l_\infty^\lambda \to l_\alpha^\mu$, $A : c_0^\lambda \to l_\alpha^\mu$, $A : c^\lambda \to l_\alpha^\mu$ and $A : l_1^\lambda \to l_\alpha^\mu$ for the case, when $\alpha > 1$. As an example we show that the Zweier matrix $Z_{1/2}$ satisfies these necessary and sufficient conditions for certain speeds λ and μ.

Also we define the notion of regularity on the subspace X of the set of convergent sequences c, and present necessary and sufficient conditions for a matrix A to be regular on l_∞^λ, c_0^λ and c^λ. We presented an example of irregular matrix, which is regular on c_0^λ and on c^λ, but not on l_∞^λ for some λ. Also we proved that this irregular matrix is regular on c_0^λ, c^λ and l_∞^λ for another λ.

Further we intend to define the notion of α-absolute summability with speed by a matrix (with real or complex entries), and to study the matrix transforms between the sets of sequences, α-absolutely summable with speeds by matrices.

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