A Mean Ergodic Theorem in Bicomplex Lebesgue Spaces

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Abstract: - The main result of this paper is a mean ergodic theorem, in the Von Neumann sense, for some operator acting on the bicomplex Lebesgue space.

Key-Words: - Mean ergodic theorem, bicomplex Lebesgue space, iterates of an operator, bicomplex modules, hyperbolic norm, bicomplex measure, bicomplex functional analysis

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1 Introduction

The research on Ergodic theory began in the 1930s, initiated in [1] and [2], and originated from applied physics and statistical mechanics. The fundamental problem in Ergodic theory is to study and find the necessary conditions for when the sequences of Cesàro averages $\sum_{j=1}^{n} T^n(\cdot)$ are convergent where T was a mapping defined on a suitable space. The theorem of mean ergodicity was extended for bounded linear operators on Banach spaces in [3] and implemented to Markoff processes in [4]. Moreover, Lorentz-invariant Markoff processes in relativistic phase space is studied in [5]. Thenceforward, ergodic theory and its applications have certainly evolved in various mathematical and statistical problems and has been studied by many researchers. For a systematic preparation and development of ergodic theorems, we can refer to the classic book [6], which contains rich literature in this area. In [7, Section VIII.5], the averages of iterates of a linear operator T is examined and discussed and then tried to throw some light upon the problems which are occured in probability and statistical mechanics. The conditions of an operator T in an arbitrary complex Banach space Y were given which are necessary and sufficient for the convergence in Y of the averages

$$A(n) = \frac{1}{n}\sum_{j=0}^{n-1} T^j$$

of the iterates of T. These general conditions have been interpreted for operators in Lebesgue spaces which occur in statistical mechanics and probability.

$\mathbb{BC}$-valued functions arise naturally in various mathematical fields, including probability theory, mathematical analysis, and functional analysis, and understanding their properties is crucial for advancing these areas of study. Indeed, the study of modules with bicomplex scalars in the context of functional analysis has gained significant attention in recent years. One influential work that has contributed to this area is the book [8]. The book likely presents groundbreaking results and insights related to this topic. Functional analysis traditionally deals with vector spaces over a field, such as the complex numbers or the real numbers. However, by considering modules with bicomplex scalars, where the scalars are elements of the bicomplex numbers, a broader framework is introduced. This extension allows for the exploration of new mathematical structures and the investigation of properties beyond the classical setting. The book [8] is likely a valuable resource for researchers and enthusiasts interested in this area. It likely presents notable results, techniques, and applications pertaining to the study of modules with bicomplex scalars in the context of functional analysis. These results may encompass various aspects of functional analysis, such as operator theory, function spaces, and spectral theory, among others. They may shed light on the behavior of modules with bicomplex scalars, reveal connections to other areas of mathematics, and potentially find applications in physics, engineering, or other disciplines.

The series of articles mentioned in the references highlight the systematic study of topological bicomplex modules and various fundamental theorems related to them. Here is a breakdown of the articles and their contributions:

In [9], the authors studied of topological bicomplex modules, likely exploring their topological properties and investigating concepts such as convergence, continuity, and compactness in this context.

Fundamental theorems, including the principle of uniform boundedness, open mapping theorem, interior mapping theorem for bicomplex modules and closed graph theorem are presented in [10].

In [11], collaboration with [10], the study of fundamental theorems are extended to the setting of topological bicomplex modules. The focus may be on generalizing classical results from functional analysis to the bicomplex module framework, providing a deeper understanding of their properties. Also the authors likely delve further into the study of topological hyperbolic modules, topological bicomplex modules, exploring the properties of linear operators, continuity, and related topological concepts specific to these settings.

The Hahn-Banach theorem for bicomplex modules and hyperbolic modules are examined in [12].

The book [13] likely provides an in-depth exploration of bicomplex analysis and geometry. It may cover a wide range of topics, including holomorphic functions, integration, differential equations, and geometric properties specific to the bicomplex domain.

In [14], the authors focused on $\mathbb{BC}$ bounded linear operators and bicomplex functional calculus. It may provide a detailed study of operators acting on bicomplex modules and explore the construction and properties of functional calculi specific to the bicomplex framework.

These references collectively represent significant contributions to the study of bicomplex modules, functional analysis, and related areas. They showcase the exploration of properties, the development of new theorems, and the application of functional analysis techniques in the context of bicomplex numbers. Researchers and readers interested in these topics can refer to these articles and the books for detailed insights into the respective areas of study.

2 Preliminaries on $\mathbb{BC}$ and $\mathbb{BC}$ Lebesgue spaces

Now, we will give a small summary of bicomplex numbers with some basic properties .The set bicomplex numbers $\mathbb{BC}$ which is a four-dimensional extension of the complex numbers is defined as

$$\mathbb{BC} := \{W = w_1 + jw_2 |\ w_1, w_2 \in \mathbb{C}(i)\}$$

where i and j are imaginary units satisfying $ij = ji$, $i^2 = j^2 = -1$. Here $\mathbb{C}(i)$ is the field of complex numbers with the imaginary unit i. According to ring structure: For any $Z = z_1 + jz_2,\ W = w_1 + jw_2$ in $\mathbb{BC}$ usual addition and multiplication are defined as

$$Z + W = (z_1 + w_1) + j\,(z_2 + w_2)$$

$$ZW = (z_1w_1 - z_2w_2) + j\,(z_2w_1 + z_1w_2)\,.$$

The set $\mathbb{BC}$ forms a commutative ring under the usual addition and multiplication of bicomplex numbers. The bicomplex numbers have a unit element denoted as $1_{\mathbb{BC}} := 1$ and this acts as the identity for multiplication, such that for any bicomplex number W, $1W = W1 = W$. In the sense of module structure, the set $\mathbb{BC}$ is a module over itself. This means that $\mathbb{BC}$ satisfies the properties of a module, including scalar multiplication and distributivity. The product of the imaginary units i and j bring out a hyperbolic unit k, such that $k^2 = 1$. This implies that k is a square root of 1 and is distinct from i and j. The product operation of all units $i, j,$ and k in the bicomplex numbers is commutative. Specifically, the following relations hold:

$$ij = k, jk = -i \text{ and } ik = -j.$$

These properties summarize the basic characteristics of bicomplex numbers and their algebraic structure.

Hyperbolic numbers $\mathbb{D}$ are a two-dimensional extension of the real numbers that form a number system known as the hyperbolic plane or hyperbolic plane algebra. They can be represented in the form $\alpha = \beta_1 + k\beta_2$, where β_1 and β_2 are real numbers, and k is the hyperbolic unit. In the hyperbolic number system, for any two hyperbolic numbers $\alpha = \beta_1 + k\beta_2$ and $\gamma = \delta_1 + k\delta_2$, addition and multiplication are defined as follows:

$$\alpha + \gamma = (\beta_1 + \delta_1) + k\,(\beta_2 + \delta_2)$$

$$\alpha\gamma = (\beta_1\delta_1 + \beta_2\delta_2) + k\,(\beta_1\delta_2 + \beta_2\delta_1)\,.$$

The hyperbolic numbers form a ring, however, unlike the complex numbers, the hyperbolic numbers do not have a multiplicative inverse for all nonzero elements. The nonzero hyperbolic numbers that have multiplicative inverses are called units. The bicomplex numbers contain two imaginary units i and j, and the hyperbolic numbers can be taken as a subset of the bicomplex numbers by restricting the imaginary part of j to be zero.

Let $W = w_1 + jw_2 \in \mathbb{BC}$ where $w_1, w_2 \in \mathbb{C}(i)$. By the notation of W with imaginary units i and j, the conjugations are formed for bicomplex numbers in [8], [13] as $\overline{W}_1 = \overline{w_1} + j\overline{w_2}$, $\overline{W}_2 = w_1 - jw_2$ and $\overline{W}_3 = \overline{w_1} - j\overline{w_2}$ where $\overline{w_1}$ and $\overline{w_2}$ are the usual complex conjugates of $w_1,\ w_2 \in \mathbb{C}\,(i)$. For any bicomplex number W, they also wrote the three moduli of W in [8], [13] and [15]. Furthermore, $\mathbb{BC}$ is a normed space with the norm $\|W\|_{\mathbb{BC}} = \sqrt{|w_1|^2 + |w_2|^2}$ for any $W = w_1 + jw_2$ in $\mathbb{BC}$. According to this, $\|W_1W_2\|_{\mathbb{BC}} \leq \sqrt{2}\,\|W_1\|_{\mathbb{BC}}\,\|W_2\|_{\mathbb{BC}}$ for every $W_1, W_2 \in \mathbb{BC}$, and finally $\mathbb{BC}$ is a quasi-Banach algebra [8].

If the hyperbolic numbers e_1 and e_2 defined as

$$e_1 = \frac{1+k}{2} \quad \text{and} \quad e_2 = \frac{1-k}{2},$$

then it is easy to see that the set $\{e_1, e_2\}$ is a fundamental set in $\mathbb{C}(i)$-vector space $\mathbb{BC}$ and linearly independent. The set $\{e_1, e_2\}$ also satisfies the following properties:

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad \overline{(e_1)}_3 = e_1, \quad \overline{(e_2)}_3 = e_2,$$
$$e_1 + e_2 = 1, \quad e_1 \cdot e_2 = 0$$

with $\|e_1\|_{\mathbb{BC}} = \|e_2\|_{\mathbb{BC}} = \frac{\sqrt{2}}{2}$. By using this linearly independent set $\{e_1, e_2\}$, any $W = w_1 + jw_2 \in \mathbb{BC}$ can be written as a linear combination of e_1 and e_2 uniquely. That is, $W = w_1 + jw_2$ can be written as

$$W = w_1 + jw_2 = e_1 z_1 + e_2 z_2 \tag{1}$$

where $z_1 = w_1 - iw_2$ and $z_2 = w_1 + iw_2$ [8]. Here z_1 and z_2 are elements of $\mathbb{C}(i)$ and the formula in (1) is called the *idempotent representation* of the bicomplex number W.

Besides the Euclidean-type norm $\|\cdot\|_{\mathbb{BC}}$, another norm named with ($\mathbb{D}$-valued) hyperbolic-valued norm $|W|_k$ of any bicomplex number $W = e_1 z_1 + e_2 z_2$ is defined as

$$|W|_k = e_1 |z_1| + e_2 |z_2| .$$

For any hyperbolic number $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, an idempotent representation can also be written as

$$\alpha = e_1\alpha_1 + e_2\alpha_2$$

where $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_1 - \beta_2$ are real numbers. If $\alpha_1 > 0$ and $\alpha_2 > 0$ for any $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, then we say that α is called a positive hyperbolic number. Thus, the set of non-negative hyperbolic numbers $\mathbb{D}^+ \cup \{0\}$ can be defined by

$$\begin{aligned} \mathbb{D}^+ \cup \{0\} &= \left\{\beta_1 + k\beta_2 : \beta_1^2 - \beta_2^2 \geq 0,\ \beta_1 \geq 0\right\} \\ &= \{e_1\alpha_1 + e_2\alpha_2 : \alpha_1, \alpha_2 \geq 0\} . \end{aligned}$$

Now, let α and γ be any two elements of $\mathbb{D}$. In [8], [12] and [13], a relation $\preceq$ is defined on $\mathbb{D}$ by

$$\alpha \preceq \gamma \ \Leftrightarrow\ \gamma - \alpha \in \mathbb{D}^+ \cup \{0\} .$$

It is showed in [8] that this relation "$\preceq$" has reflexive, anti-symmetric and transitive properties. Therefore "$\preceq$" defines a partial order on $\mathbb{D}$. If idempotent representations of the hyperbolic numbers α and γ are written as $\alpha = e_1\alpha_1 + e_2\alpha_2$ and $\gamma = e_1\gamma_1 + e_2\gamma_2$, then $\alpha \preceq \gamma$ implies that $\alpha_1 \leq \gamma_1$ and $\alpha_2 \leq \gamma_2$. By $\alpha \prec \gamma$, we mean $\alpha_1 < \gamma_1$ and $\alpha_2 < \gamma_2$. For more details on hyperbolic numbers $\mathbb{D}$ and partial order "$\preceq$", one can refer to [8, Section 1.5], [13] and [15].

Definition 1 *Let A be a subset of $\mathbb{D}$. A is called a $\mathbb{D}$-bounded above set if there is a hyperbolic number δ such that $\delta \succeq \alpha$ for all $\alpha \in A$. If $A \subset \mathbb{D}$ is $\mathbb{D}$-bounded from above, then the $\mathbb{D}$-supremum of A is defined as the smallest member of the set of all upper bounds of A [13].*

In other words, the hyperbolic number $\lambda = e_1\lambda_1 + e_2\lambda_2$, where λ_1 and λ_2 are real numbers, is the $\mathbb{D}$-supremum of A if

(1) $e_1\alpha_1 + e_2\alpha_2 \preceq e_1\lambda_1 + e_2\lambda_2$ for each $\alpha = e_1\alpha_1 + e_2\alpha_2 \in A$

(2) For any $\varepsilon = e_1\varepsilon_1 + e_2\varepsilon_2 \succ 0$, there exists $\theta = e_1\theta_1 + e_2\theta_2 \in A$ such that $e_1\theta_1 + e_2\theta_2 \succ e_1(\lambda_1 - \varepsilon_1) + e_2(\lambda_2 - \varepsilon_2)$ are satisfied.

Remark 2 *Let A be a $\mathbb{D}$-bounded above subset of $\mathbb{D}$ and $A_1 := \{\lambda_1 : e_1\lambda_1 + e_2\lambda_2 \in A\}$, $A_2 := \{\lambda_2 : e_1\lambda_1 + e_2\lambda_2 \in A\}$. Then the $sup_{\mathbb{D}}A$ is given by*

$$sup_{\mathbb{D}}A := e_1 \sup A_1 + e_2 \sup A_2 .$$

Similarly, for any $\mathbb{D}$-bounded below set A, $\mathbb{D}$-infimum of A is defined as

$$inf_{\mathbb{D}}A = e_1 \inf A_1 + e_2 \inf A_2$$

where A_1 and A_2 are as above [8, Remark 1.5.2].

Definition 3 *A $\mathbb{BC}$-module $(X, +, \cdot)$, where $(X, +)$ is an abelian group, is called a topological $\mathbb{BC}$-module, if there is a topology τ_X in X, so that the operations $+ : X \times X \to X$ and $\cdot : \mathbb{BC} \times X \to X$ are continuous.*

The following result is known from [11].

Remark 4 *A $\mathbb{BC}$-module space or $\mathbb{D}$-module space Y can be decomposed as*

$$Y = e_1 Y_1 + e_2 Y_2 \tag{2}$$

where $Y_1 = e_1 Y$ and $Y_2 = e_2 Y$ are $\mathbb{R}$-vector or $\mathbb{C}(i)$ $-$vector spaces. The spelling in (2) is called as the idempotent decomposition of the space Y. Therefore, any element y in Y can be uniquely inscribed as $y = e_1 y_1 + e_2 y_2$ with $y_1 \in Y_1$ and $y_2 \in Y_2$.

Definition 5 *Let $\mathfrak{M}$ be a σ-algebra on a set Ω. A bicomplex-valued function $\mu = \mu_1 e_1 + \mu_2 e_2$ defined on Ω is called a $\mathbb{BC}$-measure on $\mathfrak{M}$ if μ_1, μ_2 are complex measures on $\mathfrak{M}$. In particular if μ_1, μ_2 are positive measures on $\mathfrak{M}$ i.e range of both μ_1, μ_2 are $[0, \infty]$ then μ is called a $\mathbb{D}$-measure on $\mathfrak{M}$ and if μ_1, μ_2 are real measures on $\mathfrak{M}$ i.e range of both μ_1, μ_2 are $[0, \infty)$ then μ is called a $\mathbb{D}^+$-measure on $\mathfrak{M}$ [16].*

Assume that $\Omega = (\Omega, \mathfrak{M}, \mu)$ is a σ-finite complete measure space and f_1, f_2 are complex-valued (real-valued) measurable functions on Ω. The function having idempotent decomposition $f = f_1 e_1 + f_2 e_2$ is called as a $\mathbb{BC}$-measurable function and $|f|_k = |f_1| e_1 + |f_2| e_2$ is called a $\mathbb{D}$-valued measurable function on Ω [17]. Thus for any given complex valued function space $(F(\Omega), \|\cdot\|_{\Omega})$, one can create a $\mathbb{BC}$-valued function space $(F(\Omega, \mathbb{BC}), \|\cdot\|_{\mathbb{BC}})$ by combining all f_1, f_2 and bringing out functions of the type $f = f_1e_1 + f_2e_2$ where f_1 and f_2 are in $(F(\Omega), \|\cdot\|_{\Omega})$ with $\|f\|_{\mathbb{BC}}^2 = \frac{1}{2}\left(\|f_1\|_{\Omega}^2 + \|f_2\|_{\Omega}^2\right)$. Similar definition can be given for any hyperbolic measurable function.

For any $\mathbb{BC}$-valued measurable function $f = f_1e_1 + f_2e_2$, it is easy to see that $|f|_k = |f_1| e_1 + |f_2| e_2$ is $\mathbb{D}$-valued measurable. Because if $f = f_1e_1 + f_2e_2$ is a $\mathbb{BC}$-valued measurable function, then f_1 and f_2 are $\mathbb{C}$-measurable functions. Therefore real and imaginary parts of f_1 and f_2 are $\mathbb{R}$-valued measurable and so does $|f_1|$ and $|f_2|$. As a result, $|f|_k$ is $\mathbb{D}$-measurable. Also for any two $\mathbb{BC}$-valued measurable functions f and g, it can be easily seen that their sum and multiplication functions are also $\mathbb{BC}$-measurable functions [16], [17]. More results on $\mathbb{D}$-topology such as $\mathbb{D}$-limit, $\mathbb{D}$-continuity, $\mathbb{D}$-Cauchy and $\mathbb{D}$-convergence etc. can be found in [16], [18], [19] and the references therein.

Definition 6 *Let $\mathfrak{M}$ be a σ-algebra on a set Ω and $\mu = e_1\mu_1 + e_2\mu_2$ be a $\mathbb{BC}$-measure on $\mathfrak{M}$. Then two bicomplex valued $\mathbb{BC}$-measurable functions $f = e_1f_1 + e_2f_2$ and $g = e_1g_1 + e_2g_2$ on Ω are called to be equal (μ-a.e.) if $f_1 = g_1$ (μ_1-a.e.) and $f_2 = g_2$ (μ_2-a.e.).*

Definition 7 *Let $\mu = e_1\mu_1 + e_2\mu_2$ be a $\mathbb{D}$-measure on an arbitrary measure space $(\Omega, \mathfrak{M})$ and $1 \leq p < \infty$. Suppose $L^p(\Omega, \mu_1)$ and $L^p(\Omega, \mu_2)$ denote the linear space of all equivalence classes of complex valued,measurable functions f_1 and f_2 on Ω with*

$$\int_{\Omega} |f_1(x)|^p \, d\mu_1 < \infty \quad \text{and} \quad \int_{\Omega} |f_2(x)|^p \, d\mu_2 < \infty.$$

Then $L^p_{\mathbb{BC}}(\Omega, \mathfrak{M}, \mu) = L^p_{\mathbb{BC}}(\mu)$ consists of all bicomplex valued, bicomplex measurable functions (equivalence classes) $f = e_1f_1 + e_2f_2$ on Ω such that $f_1 \in L^p(\Omega, \mu_1)$ and $f_2 \in L^p(\Omega, \mu_2)$ [19].

Proposition 8 *For $1 \leq p < \infty$, $L^p_{\mathbb{BC}}(\mu)$ is a $\mathbb{BC}$-module under usual addition operation in functions and bicomplex scalar multiplication [19].*

Let $1 \leq p < \infty$. By using Definition 3 and Remark 4, we may write an idempotent decomposition

$$L^p_{\mathbb{BC}}(\mu) = e_1L^p(\mu_1) + e_2L^p(\mu_2)$$

for $L^p_{\mathbb{BC}}(\mu)$ where $L^p(\mu_1)$ and $L^p(\mu_2)$ are usual Lebesgue spaces [19]. Therefore a hyperbolic ($\mathbb{D}$-valued) norm can be defined on the $\mathbb{BC}$-module $L^p_{\mathbb{BC}}(\mu)$ with

$$\|f\|_{p,\mathbb{D}} = e_1\|f_1\|_p + e_2\|f_2\|_p$$

for any $e_1f_1 + e_2f_2 = f \in L^p_{\mathbb{BC}}(\mu)$.

Proposition 9 *Let $1 \leq p < \infty$. The space $\left(L^p_{\mathbb{BC}}(\mu), \|\cdot\|_{p,\mathbb{D}}\right)$ is a bicomplex Banach module [19].*

3 Mean Ergodic Theorem

In [20], the mean ergodic theorem in bicomplex Banach modules is studied. Also, a result on ergodicity is given for bounded bicomplex strongly continuous semigroups in bicomplex Banach modules. In this section, we will prove a mean ergodic theorem, in the Von Neumann sense, which can be written for averages of iterates of an operator T acting on $L^p_{\mathbb{BC}}(\mu)$ where $1 < p < \infty$.

Proposition 10 *Let $1 \leq p < \infty$. The set*

$$\mathbb{S} = \{s = s_1e_1 + s_2e_2 \mid s_1, \ s_2 \in S\}$$

is $\mathbb{D}-$dense in $L^p_{\mathbb{BC}}(\mu)$ where S is the set of simple functions.

Proof. Let $\varepsilon = e_1\varepsilon_1 + e_2\varepsilon_2 \succ 0$ and $f = e_1f_1 + e_2f_2$ be any element of $L^p_{\mathbb{BC}}(\mu)$. By the definition of $L^p_{\mathbb{BC}}(\mu)$, the functions f_1 and f_2 belong to $L^p(\mu_1)$ and $L^p(\mu_2)$. Since the set of simple functions S is dense in $L^p(\mu_1)$ and $L^p(\mu_2)$, then there exist simple functions h_1 and h_2 such that

$$\|f_1 - h_1\|_p < \varepsilon_1 \quad \text{and} \quad \|f_2 - h_2\|_p < \varepsilon_2.$$

If one call $e_1h_1 + e_2h_2$ as h, then $h \in \mathbb{S}$ and $\|f - h\|_{p,\mathbb{D}} \prec \varepsilon$. This means that $\mathbb{S}$ is $\mathbb{D}-$dense in $L^p_{\mathbb{BC}}(\mu)$. ■

Lemma 11 *Let $(X, \mathfrak{M}, \vartheta)$ be a finite positive measure space, $\aleph \neq 0$ be a complex Banach space and φ be a map of X into itself which satisfies the following conditions:*

$$\begin{aligned} (i) \ \varphi^{-1}(E) &\in \mathfrak{M} \text{ for all } E \in \mathfrak{M} \\ (ii) \ \text{If } \vartheta(E) &= 0 \text{ then } \vartheta\left(\varphi^{-1}(E)\right) = 0. \end{aligned} \tag{3}$$

Then for every function u from X to $\aleph$ the following operator T defined as

$$T(u)(\cdot) = u(\varphi(\cdot)) \tag{4}$$

maps measurable functions into measurable functions and ϑ-equivalent functions into ϑ-equivalent functions. Furthermore T is a continuous linear map of the space of all $\aleph$-valued $\vartheta-$measurable functions into itself.

Proof. See [7, VIII.5.6, Lemma 6] ■

Lemma 12 *Let $(X, \mathfrak{M}, \mu)$ be a finite positive measure space, $Y \neq 0$ be a bicomplex Banach space. Assume that φ is a map of X into itself which satisfies (3). Then for any $p > 1$, the linear operator T defined in the bicomplex linear space Y^X of all functions on X into Y by*

$$Tu(x) = u(\varphi(x)), \quad x \in X, \ u \in Y^X \tag{5}$$

maps $L^p_{\mathbb{BC}}(\mu)$ into itself if and only if there exists $M = M_1 e_1 + M_2 e_2 \succ 0$ such that

$$\begin{aligned} M &= \underset{E\in\mathfrak{M}}{sup_{\mathbb{D}}} \frac{\mu(\varphi^{-1}(E))}{\mu(E)} \qquad (6) \\ &= \underset{E\in\mathfrak{M}}{sup_{\mathbb{D}}} \frac{\mu_1(\varphi^{-1}(E))e_1 + \mu_2(\varphi^{-1}(E))e_2}{\mu_1(E)e_1 + \mu_2(E)e_2}. \end{aligned}$$

Furthermore, when this condition is satisfied T is a $\mathbb{D}-$continuous $\mathbb{D}-$linear map on $L^p_{\mathbb{BC}}(\mu)$ and $\|T\|_{\mathbb{BC}} = M^{\frac{1}{p}}$.

Proof. Let $\mu(\cdot) = \mu_1(\cdot)e_1 + \mu_2(\cdot)e_2$ be a $\mathbb{D}^+-$measure. If $\mu(E) = \mu_1(E)e_1 + \mu_2(E)e_2 = 0$ for any $E \in \mathfrak{M}$, then M will be taken zero. Now assume that $\mu(E) = \mu_1(E)e_1 + \mu_2(E)e_2 \succ 0$. If u is a $\mathbb{BC}$-measurable function and φ is defined as in (3), then is easy to see that Tu is $\mathbb{BC}$-measurable by (5). Firstly suppose that T maps $L^p_{\mathbb{BC}}(\mu)$ into itself. It will be shown that T is $\mathbb{D}-$closed and hence $\mathbb{D}-$continuous by [10]. Since T is defined on $L^p_{\mathbb{BC}}(\mu)$, then it maps μ-equivalent functions into μ-equivalent functions and also $\mathbb{BC}-$measurable functions into $\mathbb{BC}-$measurable functions. Now let $\alpha \neq 0$ be a fixed vector in Y and let E be a μ-null set in $\mathfrak{M}$. Then $\mu(E) = \mu_1(E)e_1 + \mu_2(E)e_2 = 0$, $\chi_E = 0$ (a.e.) and

$$T(\alpha\chi_E) = \alpha\chi_{\varphi^{-1}(E)}$$

by the definition of T. Also, linearity of T implies that $\mu(\varphi^{-1}(E)) = 0$. This means that φ is a measure-preserving map of X into itself and satisfies (3). Since $\mathbb{S}$ is $\mathbb{D}-$dense in $L^p_{\mathbb{BC}}(\mu)$, for any $u = u_1e_1 + u_2e_2 \in L^p_{\mathbb{BC}}(\mu)$ a sequence of simple functions $(u_n) = \left(u_n^{(1)}\right)e_1 + \left(u_n^{(2)}\right)e_2 \subset \mathbb{S}$ can be formed such that $\|u_n - u\|_{p,\mathbb{D}} \overset{\mathbb{D}}{\to} 0$, i.e. $\left\|u_n^{(1)} - u_1\right\|_p \to 0$ and $\left\|u_n^{(2)} - u_2\right\|_p \to 0$. This convergence implies convergence in $\mu-$measure and so the graph of T is closed. Therefore T is $\mathbb{D}-$bounded and $\mathbb{D}-$continuous on $L^p_{\mathbb{BC}}(\mu)$ by Closed graph theorem [11, Theorem 5.5]. On the other hand, for any $0 \neq \alpha \in Y$ and $E \in \mathfrak{M}$, we have

$$\begin{aligned} \|T(\alpha\chi_E)\|_{p,\mathbb{D}} &= \|T(\alpha_1\chi_E)\|_p e_1 + \|T(\alpha_2\chi_E)\|_p e_2 = \\ &= \left(\int_X \left|\alpha_1\chi_{\varphi^{-1}(E)}(x)\right|^p d\mu_1\right)^{\frac{1}{p}} e_1 \\ &\quad + \left(\int_X \left|\alpha_2\chi_{\varphi^{-1}(E)}(x)\right|^p d\mu_2\right)^{\frac{1}{p}} e_2 \\ &= |\alpha_1|\, \mu_1(\varphi^{-1}(E))^{\frac{1}{p}} e_1 \\ &\quad + |\alpha_2|\, \mu_2(\varphi^{-1}(E))^{\frac{1}{p}} e_2 \\ &= |\alpha|_k\, \mu(\varphi^{-1}(E))^{\frac{1}{p}} \end{aligned}$$

by [18, Definition 2.2]. Therefore, one can get that

$$\begin{aligned} |\alpha|_k\, \mu(\varphi^{-1}(E))^{\frac{1}{p}} &= \|T(\alpha\chi_E)\|_{p,\mathbb{D}} \\ &\preceq |\alpha|_k \|T\|_{\mathbb{BC}} \|\chi_E\|_{p,\mathbb{D}} \\ &= |\alpha|_k \|T\|_{\mathbb{BC}}\, \mu_1(E)^{\frac{1}{p}} e_1 \\ &\quad + |\alpha|_k \|T\|_{\mathbb{BC}}\, \mu_2(E)^{\frac{1}{p}} e_2 \\ &= |\alpha|_k \|T\|_{\mathbb{BC}}\, \mu(E)^{\frac{1}{p}} \end{aligned}$$

which means $M \preceq \|T\|^p_{\mathbb{BC}}$.

Conversely, let $s = s_1e_1 + s_2e_2$ be a μ-integrable function in $\mathbb{S}$ having values $\beta_1^{(1)}e_1 + \beta_1^{(2)}e_2, \beta_2^{(1)}e_1 + \beta_2^{(2)}e_2, \ldots \beta_n^{(1)}e_1 + \beta_n^{(2)}e_2$ on the disjoint sets $E_1, E_2, \ldots, E_n$ of $\mathbb{D}-$positive measure. Then Ts has the values $\beta_1^{(1)}e_1 + \beta_1^{(2)}e_2, \beta_2^{(1)}e_1 + \beta_2^{(2)}e_2, \ldots \beta_n^{(1)}e_1 + \beta_n^{(2)}e_2$ on the sets $\varphi^{-1}(E_1), \varphi^{-1}(E_2), \ldots, \varphi^{-1}(E_n)$.

Since the family $\{E_1, E_2, \ldots, E_n\}$ is a decomposition of X, property (3) of φ implies that the family $\{\varphi^{-1}(E_1), \varphi^{-1}(E_2), \ldots, \varphi^{-1}(E_n)\}$ is also a decomposition of X. Therefore, if we use the notation $\sum_{i=1}^{n}\left(\beta_i^{(1)}e_1 + \beta_i^{(2)}e_2\right)\chi_{E_i}$ for s where $\chi_{E_i}(\cdot) = e_1\chi_{E_i}(\cdot) + e_2\chi_{E_i}(\cdot)$, then

$$\begin{aligned} Ts(x) &= s(\varphi(x)) \\ &= \sum_{i=1}^{n}\left(\beta_i^{(1)}e_1 + \beta_i^{(2)}e_2\right)\chi_{E_i}(\varphi(x)) \\ &= \sum_{i=1}^{n}\left(\beta_i^{(1)}e_1 + \beta_i^{(2)}e_2\right)\chi_{\varphi^{-1}(E_i)}(x) \end{aligned}$$

and

$$
\begin{aligned}
\|Ts\|_{p,\mathbb{D}} &= \left(\int_X |Ts_1(x)|^p d\mu_1\right)^{\frac{1}{p}} e_1 \\
&\quad + \left(\int_X |Ts_2(x)|^p d\mu_2\right)^{\frac{1}{p}} e_2 \\
&= \left(\int_X \left|\sum_{i=1}^n \beta_i^{(1)} \chi_{\varphi^{-1}(E_i)}\right|^p d\mu_1\right)^{\frac{1}{p}} e_1 \\
&\quad + \left(\int_X \left|\sum_{i=1}^n \beta_i^{(2)} \chi_{\varphi^{-1}(E_i)}\right|^p d\mu_2\right)^{\frac{1}{p}} e_2 \\
&\preceq \left(\int_X \sum_{i=1}^n \left|\beta_i^{(1)} \chi_{\varphi^{-1}(E_i)}\right|^p d\mu_1\right)^{\frac{1}{p}} e_1 \\
&\quad + \left(\int_X \sum_{i=1}^n \left|\beta_i^{(2)} \chi_{\varphi^{-1}(E_i)}\right|^p d\mu_2\right)^{\frac{1}{p}} e_2
\end{aligned}
$$

can be written. Since the elements of the family $\left\{\varphi^{-1}(E_1), \varphi^{-1}(E_2), \ldots, \varphi^{-1}(E_k)\right\}$ are disjoint,

$$
\begin{aligned}
\|Ts\|_{p,\mathbb{D}} &\preceq \left(\int_{\cup_{i=1}^k \varphi^{-1}(E_i)} \sum_{i=1}^n \left|\beta_i^{(1)} \chi_{\varphi^{-1}(E_i)}\right|^p d\mu_1\right)^{\frac{1}{p}} e_1 \\
&\quad + \left(\int_{\cup_{i=1}^k \varphi^{-1}(E_i)} \sum_{i=1}^n \left|\beta_i^{(2)} \chi_{\varphi^{-1}(E_i)}\right|^p d\mu_2\right)^{\frac{1}{p}} e_2 \\
&= \left(\sum_{i=1}^n \int_{\varphi^{-1}(E_i)} \left|\beta_i^{(1)}\right|^p d\mu_1\right)^{\frac{1}{p}} e_1 \\
&\quad + \left(\sum_{i=1}^n \int_{\varphi^{-1}(E_i)} \left|\beta_i^{(2)}\right|^p d\mu_2\right)^{\frac{1}{p}} e_2 \\
&= \left(\sum_{i=1}^n \left|\beta_i^{(1)}\right|^p \mu_1\left(\varphi^{-1}(E_i)\right)\right)^{\frac{1}{p}} e_1 \\
&\quad + \left(\sum_{i=1}^n \left|\beta_i^{(2)}\right|^p \mu_2\left(\varphi^{-1}(E_i)\right)\right)^{\frac{1}{p}} e_2 \\
&\preceq \left(\sum_{i=1}^n \left|\beta_i^{(1)}\right|^p M_1 \cdot \mu_1((E_i))\right)^{\frac{1}{p}} e_1 \\
&\quad + \left(\sum_{i=1}^n \left|\beta_i^{(2)}\right|^p M_2 \cdot \mu_2((E_i))\right)^{\frac{1}{p}} e_2 \\
&= M^{\frac{1}{p}} \|s\|_{p,\mathbb{D}}
\end{aligned}
$$

is found.

Since the μ-integrable simple functions $\mathbb{S}$ are dense in $L^p_{\mathbb{BC}}(\mu)$ and T is a $\mathbb{D}-$continuous operator acting on a dense subset of $L^p_{\mathbb{BC}}(\mu)$, we can say that T possesses a unique $\mathbb{D}-$bounded, $\mathbb{D}-$continuous extension $\widetilde{T}$ defined on all of $L^p_{\mathbb{BC}}(\mu)$ with norm $\left\|\widetilde{T}\right\|_{\mathbb{BC}} \preceq M^{\frac{1}{p}}$. Furthermore, by the definition of M,

$$
\begin{aligned}
M &= \underset{E\in\mathfrak{M}}{sup_{\mathbb{D}}} \frac{\mu\left(\varphi^{-1}(E)\right)}{\mu(E)} \\
&= \underset{E\in\mathfrak{M}}{sup_{\mathbb{D}}} \frac{\mu_1\left(\varphi^{-1}(E)\right) e_1 + \mu_2\left(\varphi^{-1}(E)\right) e_2}{\mu_1(E) e_1 + \mu_2(E) e_2} \\
&= \underset{E\in\mathfrak{M}}{sup_{\mathbb{D}}} \frac{\left\|\chi_{\varphi^{-1}(E)}\right\|_p^p e_1 + \left\|\chi_{\varphi^{-1}(E)}\right\|_p^p e_2}{\|\chi_E\|_p^p e_1 + \|\chi_E\|_p^p e_2} \\
&= \underset{E\in\mathfrak{M}}{sup_{\mathbb{D}}} \frac{\|T\chi_E\|_{p,\mathbb{D}}^p}{\|\chi_E\|_{p,\mathbb{D}}^p} \\
&\preceq \|T\|_{\mathbb{BC}}^p
\end{aligned}
$$

can be found. Therefore $\|T\|_{\mathbb{BC}} = M^{\frac{1}{p}}$. ■

Proposition 13 *Assume that $(X, \mathfrak{M}, \mu)$ be a finite positive measure space and let φ be a mapping of X into itself with $\varphi^{-1}(\mathfrak{M}) \subset \mathfrak{M}$. Also, suppose that there is a $\mathbb{D}-$constant M for which*

$$
\frac{1}{n}\sum_{j=0}^{n-1} \mu\left(\varphi^{-j}(E)\right) \preceq M\mu(E) \tag{7}
$$

for all $n \in \mathbb{N}$ and $E \in \mathfrak{M}$. Then, for every $p \in (1,\infty)$, the operator T defined in (5) maps $L^p_{\mathbb{BC}}(\mu)$ into itself. Also the averages $A(n) = \frac{1}{n}\sum_{j=0}^{n-1} T^j$ as operators acting on $L^p_{\mathbb{BC}}(\mu)$ are uniformly $\mathbb{D}-$bounded.

Proof. If we write $n = 2$ in (7), then we get $\mu\left(\varphi^{-1}(E)\right) \preceq (2M-1)\mu(E)$ for any $E \in \mathfrak{M}$. Therefore T is bounded by (6). This inequality also shows that $\mu\left(\varphi^{-1}(E)\right) = 0$ whenever $\mu(E) = 0$. Now let $E \in \mathfrak{M}$ be any set, $n \in \mathbb{N}$ and $j =$

$1, 2, \ldots, n-1$. Since

$$\begin{aligned} T^j\left(\chi_E\right)(\cdot) &= T^{j-1}\left(T\left(\chi_E\right)\right)(\cdot) = T^{j-1}\left(\chi_E(\varphi)\right)(\cdot) \\ &= T^{j-1}\left(\chi_{\varphi^{-1}(E)}\right)(\cdot) \\ &= T^{j-2}\left(T\left(\chi_{\varphi^{-1}(E)}\right)\right)(\cdot) \\ &= T^{j-2}\left(\chi_{\varphi^{-2}(E)}\right)(\cdot) = \cdots \end{aligned}$$

we can conclude that $T^j\left(\chi_E\right)(\cdot) = \left(\chi_{\varphi^{-j}(E)}\right)(\cdot)$. Thus, for any simple function

$$s(\cdot) = \sum_{i=1}^{m}\left(\beta_i^{(1)} e_1 + \beta_i^{(2)} e_2\right)\chi_{E_i}(\cdot),$$

we have

$$\begin{aligned} &A(n)(s)(\cdot) = \frac{1}{n}\sum_{j=0}^{n-1} T^j(s)(\cdot) \\ &= \frac{1}{n}\sum_{j=0}^{n-1} T^j\left(\sum_{i=1}^{m}\left(\beta_i^{(1)} e_1 + \beta_i^{(2)} e_2\right)\chi_{E_i}\right)(\cdot) \\ &= e_1 \sum_{i=1}^{m}\beta_i^{(1)}\left(\frac{1}{n}\sum_{j=0}^{n-1} T^j \chi_{E_i}\right)(\cdot) \\ &\qquad + e_2 \sum_{i=1}^{m}\beta_i^{(2)}\left(\frac{1}{n}\sum_{j=0}^{n-1} T^j \chi_{E_i}\right)(\cdot) \\ &= e_1 \sum_{i=1}^{m}\beta_i^{(1)}\left(\frac{1}{n}\sum_{j=0}^{n-1} \chi_{\varphi^{-j}(E_i)}\right)(\cdot) \\ &\qquad + e_2 \sum_{i=1}^{m}\beta_i^{(2)}\left(\frac{1}{n}\sum_{j=0}^{n-1} \chi_{\varphi^{-j}(E_i)}\right)(\cdot) \\ &= \sum_{i=1}^{m}\left(\beta_i^{(1)} e_1 + \beta_i^{(2)} e_2\right)\left(\frac{1}{n}\sum_{j=0}^{n-1} \chi_{\varphi^{-j}(E_i)}\right)(\cdot) \end{aligned}$$

by (8). Hence

$$\begin{aligned} &\left\|A(n)(s)\right\|_{p,\mathbb{D}} = \\ &= \left\|\sum_{i=1}^{m}\left(\beta_i^{(1)} e_1 + \beta_i^{(2)} e_2\right)\left(\frac{1}{n}\sum_{j=0}^{n-1}\chi_{\varphi^{-j}(E_i)}\right)\right\|_{p,\mathbb{D}} \\ &= \left\|\sum_{i=1}^{m}\beta_i^{(1)}\left(\frac{1}{n}\sum_{j=0}^{n-1}\chi_{\varphi^{-j}(E_i)}\right)\right\|_p e_1 \\ &\qquad + \left\|\sum_{i=1}^{m}\beta_i^{(2)}\left(\frac{1}{n}\sum_{j=0}^{n-1}\chi_{\varphi^{-j}(E_i)}\right)\right\|_p e_2 \end{aligned}$$

$$\begin{aligned} &\preceq \sum_{i=1}^{m}\left|\beta_i^{(1)}\right|\left\|\frac{1}{n}\sum_{j=0}^{n-1}\chi_{\varphi^{-j}(E_i)}\right\|_p e_1 \\ &\qquad + \sum_{i=1}^{m}\left|\beta_i^{(2)}\right|\left\|\frac{1}{n}\sum_{j=0}^{n-1}\chi_{\varphi^{-j}(E_i)}\right\|_p e_2 \\ &= \sum_{i=1}^{m}\left|\beta_i\right|_k\left(\frac{1}{n}\left\|\sum_{j=0}^{n-1}\chi_{\varphi^{-j}(E_i)}\right\|_{p,\mathbb{D}}\right) \\ &\preceq \sum_{i=1}^{m}\left|\beta_i\right|_k\left(\frac{1}{n}\sum_{j=0}^{n-1}\left\|\chi_{\varphi^{-j}(E_i)}\right\|_{p,\mathbb{D}}\right) \\ &= \sum_{i=1}^{m}\left|\beta_i\right|_k\left(\frac{1}{n}\sum_{j=0}^{n-1}\mu_1\varphi^{-j}(E_i)\, e_1 + \mu_2\varphi^{-j}(E_i)\, e_2\right) \\ &\preceq \sum_{i=1}^{m}\left|\beta_i\right|_k M\left(\mu_1(E_i)\, e_1 + \mu_2(E_i)\, e_2\right) \\ &= M\|s\|_{p,\mathbb{D}} \end{aligned}$$

for all $s \in \mathbb{S}$. By using the $\mathbb{D}-$density of $\mathbb{S}$ in $L^p_{\mathbb{BC}}(\mu)$ and Tietze extension theorem, $\|A(n)\|_{\mathbb{BC}} \preceq M$ for any $n \in \mathbb{N}$. The averages of iterates, namely $A(n)$, are uniformly bounded as operators acting on $L^p_{\mathbb{BC}}(\mu)$. ■

Theorem 14 *(Mean Ergodic Theorem) Assume that* $(X, \mathfrak{M}, \mu)$ *is a finite positive measure space and* φ *is a mapping of* X *into itself which satisfies* $\varphi^{-1}(\mathfrak{M}) \subset \mathfrak{M}$. *If the inequality*

$$\frac{1}{n}\sum_{j=0}^{n-1}\mu\left(\varphi^{-j}(E)\right) \preceq M\mu(E) \tag{8}$$

is satisfied for all $n \in \mathbb{N}$ *and* $E \in \mathfrak{M}$, *then for every* $p \in (1, \infty)$, *the operator* T *defined by the equation*

(5) is a $\mathbb{D}-$continuous linear map on $L^p_{\mathbb{BC}}(\mu)$ and the sequence of averages $A(n)$, as operators acting on $L^p_{\mathbb{BC}}(\mu)$, is strongly $\mathbb{D}-$convergent. Here M is independent of E, n and $\varphi^0(E) = E$.

Proof. With (8), it can be written that

$$\frac{1}{2}\left\{\mu\left(\varphi^0(E)\right) + \mu\left(\varphi^{-1}(E)\right)\right\} \preceq M\mu(E)$$

for any $E \in \mathfrak{M}$. Therefore the linear operator T defined by the equation (5) is a $\mathbb{D}-$bounded and $\mathbb{D}-$continuous map on $L^p_{\mathbb{BC}}(\mu)$ by Lemma 12. If we denote the space of all $\mathbb{D}-$linear and $\mathbb{D}-$continuous operators on $L^p_{\mathbb{BC}}(\mu)$ by $\mathfrak{B}\left(L^p_{\mathbb{BC}}(\mu)\right)$, then it can be easily seen that $A(n)$, the averages, are in this complete space. Since the averages $A(n)$ are $\mathbb{D}-$uniformly bounded while operating on $L^p_{\mathbb{BC}}(\mu)$, we can write that the sequence $\{A(n)f\} \subset L^p_{\mathbb{BC}}(\mu)$ $\mathbb{D}-$converges for all $f \in L^p_{\mathbb{BC}}(\mu)$ by Riesz–Thorin convexity theorem. By the way, when the averages $A(n)$ are operating on $L^p_{\mathbb{BC}}(\mu)$, we obtained that $A(n)f \in L^p_{\mathbb{BC}}(\mu)$ for all $n \in \mathbb{N}$ and for each $f \in L^p_{\mathbb{BC}}(\mu)$. It is known that the characteristic functions of elements of $\mathfrak{M}$ form a fundamental set for $L^p_{\mathbb{BC}}(\mu)$. Then, for any $E \in \mathfrak{M}$ and $x \in X$, since we have $|\chi_E|_k \preceq 1$ and

$$\begin{aligned}\left|\frac{T^n(\chi_E)(x)}{n}\right|_k &= \frac{1}{n}\left|\chi_E(\varphi^n)(x)\right|_k \\ &= \frac{1}{n}\left(e_1\left|\chi_E(\varphi^n)(x)\right| + e_2\left|\chi_E(\varphi^n)(x)\right|\right) \\ &\preceq \frac{1}{n} \to 0,\end{aligned}$$

we can say that the sequence $\{A(n)\}$ $\mathbb{D}-$converges in strongly operator topology by [7, VIII.5.1]. ■

Remark 15 *it should be observed that a measure preserving transformation φ (i.e. one for which $\mu\left(\varphi^{-1}(E)\right) = \mu(E)$, for every E in $\sigma-$algebra) satisfies the hypothesis of the preceding theorem. This type of maps arise in the study of conservative mechanical systems. Also if the map φ is metrically transitive (i.e. $\mu\left(E\Delta\varphi^{-1}(E)\right) = 0$ implies $\mu(E) = 0$ or $\mu(X - E) = 0$) then φ is completely dissipative as in [21]. So by [21], φ admits a $\sigma-$finite invariant measure $\upsilon \approx \mu$. Then φ became a $\upsilon-$measure preserving transformation.*

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