The Reflexive Edge Strength of Cycles Plus One Edge

[1]UTHOOMPORN MATO, [2*]TANAWAT WICHIANPAISARN

[1]Department of Mathematics, Faculty of Science,
Srinakharinwirot University, Bangkok 10110,
THAILAND

[2]Department of Mathematics, Faculty of Applied Science,
King Mongkut's University of Technology North Bangkok,
Bangkok 10800,
THAILAND

*Corresponding Author

Abstract: Let G be a simple graph with a vertex set $V(G)$ and edge set $E(G)$. Given a vertex labeling $f_V : V(G) \to \{0, 2, 4, \ldots, 2k_v\}$ and an edge labelings $f_E : E(G) \to \{1, 2, 3, \ldots, 2k_e\}$. Define a function f by $f(x) = f_V(x)$ if $x \in V(G)$ and $f(x) = f_E(x)$ if $x \in E(G)$. We call f be the total k-labeling where $k = \max\{k_e, k_v\}$. A total k-labeling f is called an edge irregular reflexive k-labeling of G if every two distinct edge xy and $x'y'$, we have $wt_f(xy) \neq wt_f(x'y')$ where $wt_f(uv) = f(u) + f(uv) + f(v)$ if uv is an edge of G. The reflexive edge strength of G, denoted by $res(G)$ is the minimum k for G which has an edge irregular reflexive k-labeling. In this paper, we give the exact value of $res(C_n + e)$ where $C_n + e$ is a cycle of order n plus one edge which contains a triangle.

Key-Words: Reflexive edge strength, edge irregular reflexive, total labeling, graph, label, cycle

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1 Introduction

Throughout this paper we consider finite, simple and undirected graphs. Notations and terminologies not defined here are followed from [1].

A *labeling* is a one-to-one mapping that carries a set of graph elements into a set of non negative integers, called *labels*. If a domain is the vertex (or edge) set, the labeling is called a *vertex (or edge) labeling*, respectively. If the domain is both vertex and edge sets, then the labeling is call a *total labeling*.

Graph labelings were introduced for the first time by [2] in 1963. The most complete recent survey of graph labeling can be found in the Survey of Graph Labeling written by [3]. The topics of graph labelings have been studied by a lot of mathematicians in area of graph theory. The applications of graph labelings are the benefits indicate in some branch of science, for examples, coding theory, X-ray, circuit design and communication network design etc, see [4].

The notation of the irregularity strength of graphs was introduced in 1988. Then the idea of irregularity strength was extended to irregular total k-labeling by [5], [6]. Some previous results of vertex or edge irregular total k-labeling can be found in [7], [8], [9] and [10]. After that they extend the concept into an edge irregular reflexive k-labeling, the complete definition is defined as the following.

Let $G = (V, E)$ be a simple graph. Given a vertex labeling $f_V : V(G) \to \{0, 2, 4, \ldots, 2k_v\}$ and an edge labeling $f_E : E(G) \to \{1, 2, 3, \ldots, 2k_e\}$.

Define a function f by $f(x) = f_V(x)$ if $x \in V(G)$ and $f(x) = f_E(x)$ if $x \in E(G)$. We call f be the *total k-labeling* where $k = \max\{k_e, k_v\}$. Let $wt_f(uv)$ be the weight of the edge uv where $wt_f(uv) = f(u) + f(uv)+f(v)$. A total k-labeling f is called an *edge irregular reflexive k-labeling* of G if every two distinct edge xy and $x'y'$, we have $wt_f(xy) \neq wt_f(x'y')$. If G has an edge irregular reflexive k-labeling then the minimum number of k is called *reflexive edge strength* of G, denoted by $res(G)$.

In 2017, [11] determined the exact value of $res(G)$ where G is a wheel, prism, basket, and fan graph. Then [12] determined the exact value of $res(G)$ where G is a cycle, cartesian product of two cycles, and join graph of the path and cycle with $2K_2$ in 2019. Some more relevant results are found in [13] and [14].

In [12], the value of reflexive edge strength of a cycle C_n was found. We would like to know that if we add one edge to a cycle, then the value of its reflexive edge strength is whether same or not. We use the notation $C_n + e$ for a graph by adding one edge to a cycle C_n. In this paper, we will determine the exact value of $res(C_n+e)$ where C_n+e contains a triangle.

2 Main Result

The following lemma, proved in [15], shows the lower bound of the reflexive edge strength for any graphs.

Lemma 2.1. *For every graph G,*

$$res(G) \geq \begin{cases} \lceil \frac{|E(G)|}{3} \rceil & \text{if } |E(G)| \equiv 0,1,4,5 \pmod 6, \\ \lceil \frac{|E(G)|}{3} \rceil + 1 & \text{if } |E(G)| \equiv 2,3 \pmod 6. \end{cases}$$

Lemma 2.1 gives us the lower bound for $C_n + e$ that will be used in the proof of Theorem 2.3. To investigate the value of reflexive edge strength of the grpah $C_n + e$ for $n \geq 4$, we first consider the graphs $C_4 + e$ and $C_5 + e$ are as follows.

Proposition 2.2. $res(C_4 + e) = res(C_5 + e) = 3$.

Proof. The graphs $C_4 + e$ and $C_5 + e$ have edge irregular reflexive 3-labelings as follow (Figure 1).

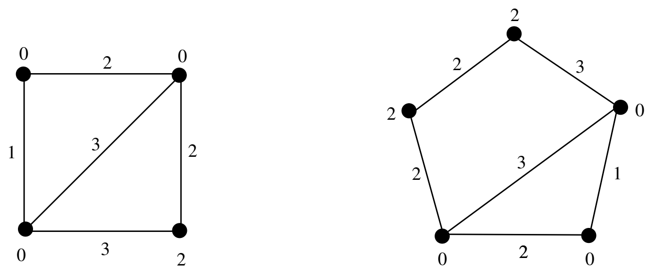

Figure 1: Edge irregular reflexive 3-labelings of $C_4 + e$ and $C_5 + e$

Thus $res(C_4 + e) \leq 3$ and $res(C_5 + e) \leq 3$.

Suppose that there is an edge irregular reflexive 2-labeling f of $C_4 + e$. If there is an induced subgraph P_3 of $C_4 + e$ such that all its vertices are labeled by the same number (see Figure 2), then we have three remaining edges of $C_4 + e$ incident to v where $v \in V(C_4 + e) \setminus V(P_3)$. Since edges of $C_4 + e$ must be labeled by 1 or 2, by pigeonhole principle, there are two edges which their weight are not different. This contradicts the assumption.

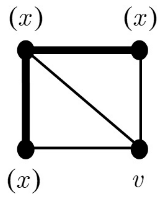

Figure 2: An induced subgraph P_3 of $C_4 + e$

Assume that there is no an induced subgraph P_3 of $C_4 + e$ such that all its vertices are labeled by the same number. Then there are three edges of C_4+e that their two endpoints are labeled in the same way. Since edges of $C_4 + e$ must be labeled by 1 or 2, there are two edges which their weight are not different. This contradicts the assumption. Hence $res(C_4 + e) \geq 3$.

For the graph $C_5 + e$, we can consider in the same way of C_4+e, and then we have $res(C_5+e) \geq 3$. □

We give the exact value of reflexive edge strength of the graph $C_n + e$ which contains a triangle in the next theorem.

Theorem 2.3. *For positive integer n, $n \geq 4$. Let G be a graph $C_n + e$ which contains a triangle. Then*

$$res(G) = \begin{cases} 3 & \text{if } n = 4, 5, \\ \lceil \frac{n+1}{3} \rceil & \text{if } n \equiv 0,3,4,5 \pmod 6 \\ & \quad \textit{and } n \geq 6, \\ \lceil \frac{n+1}{3} \rceil + 1 & \text{if } n \equiv 1,2 \pmod 6. \end{cases}$$

Proof. By Proposition 2.2, we have $res(C_4 + e) = res(C_5 + e) = 3$.

Assume $n \geq 6$. By Lemma 2.1 and $|E(G)| = n + 1$, we have

$$res(G) \geq \begin{cases} \lceil \frac{n+1}{3} \rceil & \text{if } n \equiv 0,3,4,5 \pmod 6 \\ & \quad \textit{and } n \geq 6, \\ \lceil \frac{n+1}{3} \rceil + 1 & \text{if } n \equiv 1,2 \pmod 6. \end{cases}$$

Case 1 : $n \equiv 0,1,2,3 \pmod 6$.

Let G be a graph $C_n = (x_1, x_2, ..., x_n, x_1)$ add the edge $x_{\lceil \frac{n}{2} \rceil} x_{\lceil \frac{n}{2} \rceil + 2}$. Define the total labeling f of G by

$$f(x_i) = 2\left(\left\lceil \frac{i+1}{3} \right\rceil - 1\right), \quad i = 1, 2, ..., \left\lceil \frac{n}{2} \right\rceil,$$

$$f(x_{n-i+1}) = 2\left\lceil \frac{i-1}{3} \right\rceil, \qquad i = 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

$$f(x_{\lceil \frac{n}{2} \rceil + 1}) = \begin{cases} 2\lceil \frac{n}{6} \rceil & n \equiv 0,2,3 \pmod 6, \\ 2\lfloor \frac{n}{6} \rfloor & n \equiv 1 \pmod 6, \end{cases}$$

$$f(x_i x_{i+1}) = 2\left\lceil \frac{i}{3} \right\rceil - 1, \quad i = 1, 2, ..., \left\lceil \frac{n}{2} \right\rceil - 1,$$

$$f(x_{n-i} x_{n-i+1}) = 2\left\lceil \frac{i+1}{3} \right\rceil, \quad i = 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor - 2,$$

$$f(x_n x_1) = 2,$$

$$f(x_{\lceil \frac{n}{2} \rceil} x_{\lceil \frac{n}{2} \rceil + 1}) = \begin{cases} 2\lceil \frac{n}{6} \rceil + 1 & n \equiv 0 \pmod 6, \\ 2\lceil \frac{n}{6} \rceil - 1 & n \equiv 1,2,3 \pmod 6, \end{cases}$$

$$f(x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 2}) = \begin{cases} 2\lceil \frac{n}{6} \rceil & n \equiv 0,1,3 \pmod 6, \\ 2\lfloor \frac{n}{6} \rfloor & n \equiv 2 \pmod 6, \end{cases}$$

$$f(x_{\lceil \frac{n}{2} \rceil} x_{\lceil \frac{n}{2} \rceil + 2}) = \begin{cases} 2\lfloor \frac{n+3}{6} \rfloor & n \equiv 1,3 \pmod 6, \\ 2\lceil \frac{n}{6} \rceil - 1 & n \equiv 0,2 \pmod 6. \end{cases}$$

Note that $\max(f[V(G) \cup E(G)])$

$$= \begin{cases} \lceil \frac{n+1}{3} \rceil & \text{if}\ ,\ n \equiv 0,3 \pmod 6, \\ \lceil \frac{n+1}{3} \rceil + 1 & \text{if}\ ,\ n \equiv 1,2 \pmod 6. \end{cases}$$

For case $n \equiv 0 \pmod 6$, the weights of the edges in $C_n + e$ under the labeling f are the following.

For $i = 1, 2, ..., \lceil \frac{n}{2} \rceil - 1$,
$wt_f(x_i x_{i+1}) = f(x_i) + f(x_i x_{i+1}) + f(x_{i+1})$

$$= 2\left(\left\lceil \frac{i+1}{3} \right\rceil - 1\right) + 2\left(\left\lceil \frac{i}{3} \right\rceil - 1\right) + 2\left(\left\lceil \frac{i+2}{3} \right\rceil - 1\right)$$
$$= 2\left(\left\lceil \frac{i+1}{3} \right\rceil + \left\lceil \frac{i}{3} \right\rceil + \left\lceil \frac{i+2}{3} \right\rceil\right) - 5$$
$$= 2(i+2) - 5$$
$$= 2i - 1.$$

For $i = 1, 2, ..., \lfloor \frac{n}{2} \rfloor - 2$,
$wt_f(x_{n-i} x_{n-i+1})$

$$= f(x_{n-i}) + f(x_{n-1} x_{n-i+1}) + f(x_{n-i+1})$$
$$= 2\left\lceil \frac{i}{3} \right\rceil + 2\left\lceil \frac{i+1}{3} \right\rceil + 2\left\lceil \frac{i-1}{3} \right\rceil$$
$$= 2\left(\left\lceil \frac{i-1}{3} \right\rceil + \left\lceil \frac{i}{3} \right\rceil + \left\lceil \frac{i+1}{3} \right\rceil\right)$$
$$= 2i + 2.$$

$$wt_f(x_n x_1) = f(x_n) + f(x_n x_1) + f(x_1) = 2.$$

$wt_f(x_{\lceil \frac{n}{2} \rceil} x_{\lceil \frac{n}{2} \rceil + 1})$

$$= f(x_{\frac{n}{2}}) + f(x_{\frac{n}{2}} x_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1})$$
$$= 2\left(\left\lceil \frac{\frac{n}{2}+1}{3} \right\rceil - 1\right) + \left(2\left\lceil \frac{n}{6} \right\rceil + 1\right) + 2\left\lceil \frac{n}{6} \right\rceil$$
$$= 2\left(\frac{n}{6}\right) + 4\left(\frac{n}{6}\right) + 1$$
$$= n + 1.$$

$wt_f(x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 2})$

$$= f(x_{\frac{n}{2}+1}) + f(x_{\frac{n}{2}+1} x_{\frac{n}{2}+2}) + f(x_{\frac{n}{2}+2})$$
$$= 2\left\lceil \frac{n}{6} \right\rceil + 2\left\lceil \frac{n}{6} \right\rceil + f(x_{n-(\frac{n}{2}-1)+1})$$
$$= 4\left\lceil \frac{n}{6} \right\rceil + 2\left\lceil \frac{(\frac{n}{2}-1)-1}{3} \right\rceil$$
$$= \frac{4n}{6} + \frac{2n}{6} = n.$$

$wt_f(x_{\lceil \frac{n}{2} \rceil} x_{\lceil \frac{n}{2} \rceil + 2})$

$$= f(x_{\frac{n}{2}}) + f(x_{\frac{n}{2}} x_{\frac{n}{2}+2}) + f(x_{\frac{n}{2}+2})$$
$$= 2\left(\left\lceil \frac{\frac{n}{2}+1}{3} \right\rceil - 1\right) + \left(2\left\lceil \frac{n}{6} \right\rceil - 1\right) + f(x_{n-(\frac{n}{2}-1)+1})$$
$$= \frac{2n}{6} + \left(\frac{2n}{6} - 1\right) + \frac{2n}{6}$$
$$= n - 1.$$

An example of the total labeling f of $C_{12} + e$ is shown in Figure 3.

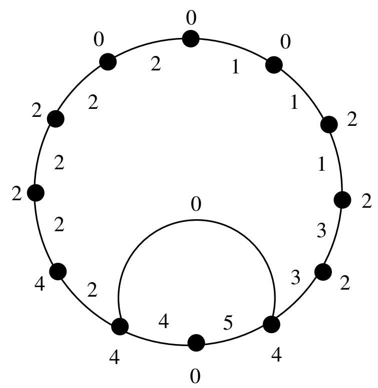

Figure 3: An edge irregular reflexive 5-labeling of $C_{12} + e$

The weights of the edges in case $n \equiv 1, 2, 3 \pmod 6$ can be considered similarly.

For case $n \equiv 1, 3 \pmod 6$, we have that
$wt_f(x_i x_{i+1}) = 2i - 1$ for $i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$,
$wt_f(x_{n-i} x_{n-i+1}) = 2i+2$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 2$,
$wt_f(x_n x_1) = 2$,
$wt_f(x_{\lceil \frac{n}{2} \rceil} x_{\lceil \frac{n}{2} \rceil + 1}) = n$,
$wt_f(x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 2}) = n + 1$,
$wt_f(x_{\lceil \frac{n}{2} \rceil} x_{\lceil \frac{n}{2} \rceil + 2}) = n - 1$.

For case $n \equiv 2 \pmod 6$, we have that
$wt_f(x_i x_{i+1}) = 2i - 1$ for $i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$,
$wt_f(x_{n-i} x_{n-i+1}) = 2i+2$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 2$,
$wt_f(x_n x_1) = 2$,
$wt_f(x_{\lceil \frac{n}{2} \rceil} x_{\lceil \frac{n}{2} \rceil + 1}) = n + 1$,
$wt_f(x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 2}) = n$,
$wt_f(x_{\lceil \frac{n}{2} \rceil} x_{\lceil \frac{n}{2} \rceil + 2}) = n - 1$.

We have that the weights of edges in G are distinct numbers from the set $\{1, 2, ..., n + 1\}$.

Hence f is an edge irregular reflexive $\lceil \frac{n+1}{3} \rceil$-labeling of G for case $n \equiv 0, 3 \pmod 6$ and an edge irregular reflexive $\left(\lceil \frac{n+1}{3} \rceil + 1\right)$-labeling of G for case $n \equiv 1, 2 \pmod 6$.

Case 2 : $n \equiv 4, 5 \pmod 6$.

Let G be a graph $C_n = (x_1, x_2, ..., x_n, x_1)$ with adding the edge $x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 3}$.

Define the total labeling f of G by

$$
\begin{aligned}
f(x_i) &= 2\left(\left\lceil \frac{i+1}{3} \right\rceil - 1\right),\ i = 1, 2, ..., \left\lceil \frac{n}{2} \right\rceil, \\
f(x_{n-i+1}) &= 2\left\lceil \frac{i-1}{3} \right\rceil,\ i = 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor - 2, \\
f(x_{\lceil \frac{n}{2} \rceil + 1}) = f(x_{\lceil \frac{n}{2} \rceil + 2}) &= \left\lceil \frac{n+1}{3} \right\rceil, \\
f(x_i x_{i+1}) &= 2\left\lceil \frac{i}{3} \right\rceil - 1, \\
f(x_{n-i} x_{n-i+1}) &= 2\left\lceil \frac{i+1}{3} \right\rceil, \\
f(x_n x_1) &= 2, \\
f(x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 3}) &= \left\lceil \frac{n+1}{3} \right\rceil - 2.
\end{aligned}
$$

Note that $\max(f[V(G) \cup E(G)]) = \lceil \frac{n+1}{3} \rceil$.

For case $n \equiv 4 \pmod 6$, the weights of the edges in $C_n + e$ under the labeling f are the following.

For $i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil$,

$$
\begin{aligned}
& wt_f(x_i x_{i+1}) \\
&= f(x_i) + f(x_i x_{i+1}) + f(x_{i+1}) \\
&= \left(2\left\lceil \frac{i+1}{3} \right\rceil - 1\right) + \left(2\left\lceil \frac{i}{3} \right\rceil + 1\right) + \left(2\left\lceil \frac{i+2}{3} \right\rceil - 1\right) \\
&= 2\left(\left\lceil \frac{i}{3} \right\rceil + \left\lceil \frac{i+1}{3} \right\rceil + \left\lceil \frac{i+2}{3} \right\rceil\right) - 5 \\
&= 2i - 1.
\end{aligned}
$$

For $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 3$,

$$
\begin{aligned}
& wt_f(x_{n-i} x_{n-i+1}) \\
&= f(x_{n-i}) + f(x_{n-i} x_{n-i+1}) + f(x_{n-i+1}) \\
&= 2\left\lceil \frac{i}{3} \right\rceil + 2\left\lceil \frac{i+1}{3} \right\rceil + 2\left\lceil \frac{i-1}{3} \right\rceil \\
&= 2\left(\left\lceil \frac{i-1}{3} \right\rceil + \left\lceil \frac{i}{3} \right\rceil + \left\lceil \frac{i+1}{3} \right\rceil\right) \\
&= 2i + 2.
\end{aligned}
$$

$$wt_f(x_n x_1) = f(x_n) + f(x_n x_1) + f(x_1) = 2.$$

$$
\begin{aligned}
& wt_f(x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 2}) \\
&= f(x_{\lceil \frac{n}{2} \rceil + 1}) + f(x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 2}) + f(x_{\lceil \frac{n}{2} \rceil + 2}) \\
&= 2\left(\left\lceil \frac{\lceil \frac{n}{2} \rceil + 2}{3} \right\rceil - 1\right) + \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lceil \frac{n+1}{3} \right\rceil \\
&= 2\left(\left\lceil \frac{\frac{n}{2} + 2}{3} \right\rceil - 1\right) + \frac{n-1}{3} + \frac{n+2}{3} \\
&= 2\left(\left\lceil \frac{n+4}{6} \right\rceil - 1\right) + \frac{2n+1}{3} \\
&= 2\left(\frac{n+2}{6}\right) + \frac{2n+1}{3} \\
&= n + 1.
\end{aligned}
$$

$$
\begin{aligned}
& wt_f(x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 3}) \\
&= f(x_{\lceil \frac{n}{2} \rceil + 1}) + f(x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 3}) + f(x_{\lceil \frac{n}{2} \rceil + 3}) \\
&= 2\left(\left\lceil \frac{n+4}{6} \right\rceil - 1\right) + \frac{n-4}{3} + 2\left\lceil \frac{n-6}{6} \right\rceil \\
&= \frac{n+8}{3} - 2 + \frac{n-4}{3} + \frac{n-4}{3} \\
&= n - 2.
\end{aligned}
$$

$$
\begin{aligned}
& wt_f(x_{\lceil \frac{n}{2} \rceil + 2} x_{\lceil \frac{n}{2} \rceil + 3}) \\
&= wt_f(x_{\lceil \frac{n}{2} \rceil + 2} x_{n - (\lfloor \frac{n}{2} \rfloor - 2) + 1}) \\
&= f(x_{\lceil \frac{n}{2} \rceil + 2}) + f(x_{\lceil \frac{n}{2} \rceil + 2} x_{\lceil \frac{n}{2} \rceil + 3}) + f(x_{n - (\lfloor \frac{n}{2} \rfloor - 2) + 1}) \\
&= \left\lceil \frac{n+1}{3} \right\rceil - 1 + \left\lceil \frac{n+1}{3} \right\rceil + 2\left\lceil \frac{\lfloor \frac{n}{2} \rfloor - 2 - 1}{3} \right\rceil \\
&= \frac{2n+4}{3} + 2\left\lceil \frac{n-6}{6} \right\rceil \\
&= \frac{2n+4}{3} + \frac{n+2}{3} - 2 \\
&= n.
\end{aligned}
$$

For case $n \equiv 5 \pmod 6$, we can compute in the same way of case $n \equiv 4 \pmod 6$. Then
$wt_f(x_i x_{i+1}) = 2i - 1$ for $i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil$,
$wt_f(x_{n-i} x_{n-i+1}) = 2i + 2$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 3$,
$wt_f(x_n x_1) = 2$,
$wt_f(x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 2}) = n + 1$,
$wt_f(x_{\lceil \frac{n}{2} \rceil + 1} x_{\lceil \frac{n}{2} \rceil + 3}) = n - 3$,
$wt_f(x_{\lceil \frac{n}{2} \rceil + 2} x_{\lceil \frac{n}{2} \rceil + 3}) = n - 1$.

We have that the weights of edges in G are distinct numbers from the set $\{1, 2, ..., n + 1\}$. Hence f is an edge irregular reflexive $\lceil \frac{n+1}{3} \rceil$-labeling of G. □

3 Conclusion

In this paper, we obtained the exact values of the reflexive edge strength of $C_n + e$ containing a triangle for all $n \geq 4$. In general when we added an edge to a cycle, the graph might not be contained a triangle, this issue is still an open problem.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Uthoomporn conceived of the presented idea. Uthoomporn and Tanawat applied the theory to the research. Tanawat verified the analytical methods. All authors discussed the results and contributed to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript.

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Conflicts of Interest

The authors have no conflicts of interest.