Evaluating complex inverse formulas for q-Sumudu transforms

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Abstract: In this paper, q-analogues of the Sumudu transform, along with an inversion formula and some explicit computations, are presented. This work essentially focuses on q-analogues of the inverse Sumudu transform and the construction method of the inversion formula via a path integral along a Bromwich contour. It is also shown how the complex inversion formulas considered in this paper admit q-expansions that yield various inverse q-Sumudu transforms for q-series.

Key-Words: Integral transforms, Sumudu transform, q-Sumudu transforms, Complex inversion formula,

Received: May 12, 2022. Revised: August 13, 2023. Accepted: September 15, 2023. Available online: November 16, 2023.

1 Introduction

Integral transforms have gained importance and are ubiquitous, mainly because of their tremendous ability to be used in various fields of applied sciences and engineering. The best known and mostly used integral transforms are Laplace, Fourier, Mellin and Hankel. In 1993, Watugala, [1], added another dimension to this research by proposing a new integral transform, which is called the Sumudu transform, and he used it in control engineering problems to obtain the solutions of certain ordinary differential equations. In this way, Weerakoon, [2], provide the Sumudu transform of partial derivatives as well as the complex inversion formula for this transform, and put it to use in the solution of partial differential equations. Despite the fact that the Sumudu transform is the hypothetical dual of the Laplace transform, it has a wide range of applications in science and engineering due to its special core properties. Its main advantage is that it can be used to solve problems without having to use a new frequency domain, since it has scale and unit conservation properties [3]. Readers are recommended to refer to [4], [5], [6] for more information on this matter.

The origin of q-calculus dates back to the late 18th century. In some works, q-calculus is also referred to as limitless calculus. The letter q comes from quantum. In recent years, q-calculus has found its applications in many fields, especially quantum mechanics. In q-calculus the concepts of q-series, q-derivatives and q-integrals have as much importance as series, derivatives and integrals in classical calculus [7]. The q-series has been applied in numerous areas of mathematics and physics, such as optimal control problems, [8], arbitrary order (fractional) computation, [9], q-transform analysis, [10], geometric function theory, [11], and the discovery of solutions to the q-difference equations [12], [13]. q-differential (or q-difference) equations arise as a result of mathematical modeling in the solution of many problems in mathematics and physics, just like the classical ones. One of the most comprehensive techniques for solving q-differential (or q-difference) equations is the q-integral transform method. In this method, the most commonly used q-integral transforms to solved the q-differential (or q-difference) equations are q-Laplace, q-Fourier, q-Mellin transforms. (see, [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24])

Through this method, a q-differential (or q-difference) equation is reduced to an algebraic problem that is easier to solve with the help of the transformed function, rather than the q-derivative. The inverse q-integral transform is then used to obtain the solution of the original problem. Motivated by the applications of classical Sumudu transform, Albayrak et al. considered and studied basic (or q-) analogues of Sumudu transform. They also considered basic properties and gave q-Sumudu transforms of some q-functions and their special cases (see [25, 26]). Certain inversion and representation theorems and their applications to q-Sumudu transforms are also discussed in [27]. More recently, Purohit and Uçar, [28], using the q-Sumudu transforms gave an alternative solution to the q-kinetic equation with fractional q-integral operators of the Riemann-Liouville category.

Though it is acceptable to rely on lists and tables of function transforms while looking for solutions to differential equations, having access to inverse transform formulas is always a significantly better tool for engineers and mathematicians. Using the Cauchy hypothesis together with the residue theorem, Belgacem and Karaballi [4] proved a theorem (using Bromwich contour integral) with respect to the complex inverse Sumudu transform. In this article, we aim to present certain complex inversion formulas for the q-Sumudu transforms. Our main result is a q-extension of the consequence of Belgacem and Karaballi [4] and can be used to derive certain intriguing consequences and exceptional cases.

2 Preliminaries

To deal with this work effectively, we use some basics of quantum theory. For more subtleties of the q-calculus, one can refer to [29]. Throughout this paper, we will consider q as a particular quantity satisfying the agreement $0 < |q| < 1$.

The q-derivative of a given function φ is defined by

$$(D_q\varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x}. \tag{1}$$

This definition holds when x is not equal to zero. If the function φ is differentiable, it is clear that as the parameter q approaches 1 from the left, the q-derivative converges to the classical derivative.

The q-shifted factorial is defined by

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ \prod_{k=0}^{n} \left(1 - aq^k\right), & n = 1, 2, ... \end{cases} \tag{2}$$

and has properties as follows:

$$(a;q)_\infty = \prod_{n=0}^{\infty} \left(1 - aq^n\right), \tag{3}$$

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{4}$$

$$\left(\frac{1}{q};\frac{1}{q}\right)_n = (-1)^n q^{-\binom{n+1}{2}} (q;q)_n . \tag{5}$$

where $a \neq 0, |q| < 1, \alpha \in \mathbb{R}$. We recall that the basic analogues of the classical exponential function are defined by

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty}, \tag{6}$$

where $|t| < 1$ and

$$E_q(t) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(-t)^n}{(q;q)_n} = (t;q)_\infty, \tag{7}$$

where $t \in \mathbb{C}$ and the relationship between $e_q(t)$ and $E_q(t)$ is as follows:

$$E_q(qt) = e_{1/q}(t) . \tag{8}$$

By virtue of the (6) and (7), q-trigonometric functions can be defined as follows [29]:

$$\begin{aligned} \sin_q t &= \frac{e_q(it) - e_q(-it)}{2i} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(q;q)_{2n+1}}, \end{aligned} \tag{9}$$

$$\begin{aligned} \operatorname{Sin}_q t &= \frac{E_q(-it) - E_q(it)}{2i} \\ &= \sum_{n=0}^{\infty} (-1)^n q^{n(2n+1)} \frac{t^{2n+1}}{(q;q)_{2n+1}}, \end{aligned} \tag{10}$$

$$\begin{aligned} \cos_q t &= \frac{e_q(it) + e_q(-it)}{2} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(q;q)_{2n}}, \end{aligned} \tag{11}$$

$$\begin{aligned} \operatorname{Cos}_q t &= \frac{E_q(it) + E_q(-it)}{2} \\ &= \sum_{n=0}^{\infty} (-1)^n q^{n(2n-1)} \frac{t^{2n}}{(q;q)_{2n}}. \end{aligned} \tag{12}$$

Note that, classic exponential function e^t and trigonometric functions $\sin t$ and $\cos t$ can be obtained as the limit case $(q \to 1^-)$ of $e_q((1-q)t)$, $E_q((q-1)t)$, $\sin_q((1-q)t)$, $\operatorname{Sin}_q((q-1)t)$, $\cos_q((1-q)t)$, $\operatorname{Cos}_q((q-1)t)$.

A basic hypergeometric series is defined by the following summation [29, p.4]:

$$\begin{aligned} &{}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z) \\ &\equiv {}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; q, z \right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots (a_r;q)_n}{(b_1;q)_n (b_1;q)_n \ldots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r+1} \frac{z^n}{(q;q)_n} \end{aligned}$$

where $q \neq 0, r > s+1$. Jackson, [30], introduced the following q-analogues of Bessel function and therefore they are referred as Jackson's q-Bessel functions (or basic Bessel function) in some papers:

$$\begin{aligned} J_\nu^{(1)}(z;q) &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \left(\frac{z}{2}\right)^\nu {}_2\Phi_1 \left[\begin{matrix} 0 & 0 \\ q^{\nu+1} & \end{matrix} ; q, -\frac{z^2}{4} \right], \\ &= \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-z/2)^{2n}}{(q;q)_{\nu+n} (q;q)_n}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} J_\nu^{(2)}(z;q) &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \left(\frac{z}{2}\right)^\nu {}_0\Phi_1 \left[\begin{matrix} - \\ q^{\nu+1} \end{matrix} ; q, -\frac{q^{\nu+1} z^2}{4} \right] \\ &= \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{q^{n(n+\nu)} (-z/2)^{2n}}{(q;q)_{\nu+n} (q;q)_n} \end{aligned} \tag{14}$$

where the q-shifted factorials are defined as in (2). The simple association among q-Bessel functions is as

$$J_\nu^{(2)}(z;q) = \left(-\frac{z^2}{4};q\right)_\infty J_\nu^{(1)}(z;q), \qquad |z| < 2.$$

The q-Bessel functions mentioned above can be considered as q-extensions of the first kind Bessel function. The third kind q-analogue of the Bessel function, some authors refer to this function as Hahn-Exton q-Bessel function, is given by the subsequent formula

$$\begin{aligned} J_\nu^{(3)}(z;q) &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} z^\nu \, {}_1\Phi_1\left[\begin{matrix} 0 \\ q^{\nu+1} \end{matrix} ;q, qz^2\right] \\ &= z^\nu \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (qz^2)^n}{(q;q)_{\nu+n} (q;q)_n}. \quad (15) \end{aligned}$$

Here we note that, in the limiting cases for $q \to 1^-$ we obtain

$$\lim_{q\to1^-} J_\nu^{(k)}((1-q)z;q) = J_\nu(z) \quad (k=1,2)$$

and

$$\lim_{q\to1^-} J_\nu^{(3)}((1-q)z;q) = J_\nu(2z).$$

Jackson, [31], introduced q-definite integrals, namely the Jackson integral is defined by

$$\int_0^x f(t) d_q t = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k), \quad (16)$$

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{k\in\mathbb{Z}} \frac{q^k}{A} f\left(\frac{q^k}{A}\right). \quad (17)$$

Hahn, [18], defined the q-analogues of the well-known classical Laplace transform by means of the following q-integrals

$$\begin{aligned} F_1(s) &= L_q\{f(t);s\} \\ &= \frac{1}{1-q} \int_0^{s^{-1}} E_q(qst) f(t) d_q t, \quad (18) \end{aligned}$$

and

$$\begin{aligned} F_2(s) &= \mathbb{L}_q\{f(t);s\} \\ &= \frac{1}{1-q} \int_0^{\infty} e_q(-st) f(t) d_q t, \quad (19) \end{aligned}$$

where $\mathrm{Re}(s) > 0$ and q-analogues of the classical exponential functions are defined by (6) and (8). Albayrak et al. [25] characterized q-analogues of the Sumudu transform via the succeeding q-integrals

$$\begin{aligned} G_1(s) &= S_q\{f(t);s\} \\ &= \frac{1}{(1-q)s} \int_0^s E_q\left(\frac{q}{s}t\right) f(t) d_q t, \quad (20) \end{aligned}$$

where $s \in (-\tau_1, \tau_2)$ for the set of functions

$$\begin{aligned} A = \{f(t) \,| \exists M, \tau_1, \tau_2 > 0, |f(t)| < M E_q(|t|/\tau_j), \\ t \in (-1)^j \times [0,\infty)\}, \end{aligned}$$

and

$$\begin{aligned} G_2(s) &= \mathbb{S}_q\{f(t);s\} \\ &= \frac{1}{(1-q)s} \int_0^{\infty} e_q\left(-\frac{1}{s}t\right) f(t) d_q t, \quad (21) \end{aligned}$$

where $s \in (-\tau_1, \tau_2)$ considering the set of functions

$$\begin{aligned} B = \{f(t) \,| \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e_q(|t|/\tau_j), \\ t \in (-1)^j \times [0,\infty)\}. \end{aligned}$$

As a result of (16) and (17), q-Laplace and q-Sumudu transforms can be asserted as

$$L_q\{f(t);s\} = \frac{(q;q)_\infty}{s} \sum_{k=0}^{\infty} \frac{q^k f(s^{-1}q^k)}{(q;q)_k}, \quad (22)$$

$$S_q\{f(t);s\} = (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k f(sq^k)}{(q;q)_k}, \quad (23)$$

and

$$\begin{aligned} &\mathbb{L}_q\{f(t);s\} \\ &= \frac{1}{(-s;q)_\infty} \sum_{k=0}^{\infty} q^k f(q^k) (-s;q)_k, \quad (24) \\ &\mathbb{S}_q\{f(t);s\} \\ &= \frac{s^{-1}}{(-\frac{1}{s};q)_\infty} \sum_{k\in\mathbb{Z}} q^k f(q^k) \left(-\frac{1}{s};q\right)_k. \quad (25) \end{aligned}$$

In [18], [25] the following limit cases were proved:

$$\begin{aligned} \lim_{q\to1^-} L_q\{f(t);(1-q)s\} &= \lim_{q\to1^-} \mathbb{L}_q\{f(t);(q-1)s\} \\ &= L\{f(t);s\} \end{aligned}$$

and

$$\begin{aligned} \lim_{q\to1^-} S_q\{f(t);(1-q)s\} &= \lim_{q\to1^-} \mathbb{S}_q\{f(t);(q-1)s\} \\ &= S\{f(t);s\} \end{aligned}$$

where

$$F_0(s) = L\{f(t);s\} = \int_0^\infty e^{-st} f(t)\, dt, \quad \mathrm{Re}\, s > 0,$$

and

$$G_0(s) = S\{f(t);s\} = \frac{1}{s} \int_0^\infty e^{-t/s} f(t)\, dt, \quad s \in (-\tau_1, \tau_2),$$

over the set of functions

$$A = \left\{ f(t) \,\middle|\, \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j} \right., \quad \left. t \in (-1)^j \times [0, \infty) \right\}.$$

The relationship between classical Laplace and Sumudu transform and the relationship between their q-versions for $i = 0, 1, 2$ can be written as follows:

$$G_i(s) = \frac{1}{s} F_i \left(\frac{1}{s} \right) \text{ or } F_i(s) = \frac{1}{s} G_i \left(\frac{1}{s} \right).$$

In [4] the authors proved a theorem for defining the inverse Sumudu transform using the Cauchy theorem, the residue theorem, and the Bromwich contour as shown in Figure 1. However, they used the relationship between classical Laplace and Sumudu transforms, as well as the definition of the classical inverse Laplace transform, to establish inverse Sumudu transform. In our work, we did not need to q-versions of this relationship or the definition of the q-inverse Laplace transform to find q-variants of inverse Sumudu transform. By leveraging the following theorems, the Bromwich contour illustrated in Figure 1, and the definitions of the q-Sumudu transforms, we have successfully defined the q-inverse transforms.

3 Complex Inversion Formula

Though it is acceptable to rely on lists and tables of function transforms while looking for solutions to differential equations, having access to inverse transform formulas is always a significantly better tool for engineers and mathematicians. In this section, we will give complex inversion formulas for q-Sumudu transforms, starting with q-version of the results given in [4]. The following theorem gives us the complex inverse q-Sumudu transforms, derived from the Cauchy and residue theorems, using a Bromwich contour.

Theorem 1. *Let* $S_q \{f(t); s\} = G_1(s)$ *and* $\lim_{s\to\infty} G_1(s) = 0$. *Assume that* $G_1(s)/s$ *is a meromorphic function with singularities in the region where* $\operatorname{Re}(s) < c$. *If a circular domain* γ *exists with radius* R *and positive constants,* M *and* k, *with*

$$\left| \frac{G_1(s)}{s} \right| < M R^{-k},$$

then $f(t)$ *is given by*

$$S_q^{-1} \{G_1(s); t\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e_q\left(\frac{t}{s}\right) \frac{G_1(s)}{s} ds$$
$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q,q)_n} G_1(tq^n). \quad (26)$$

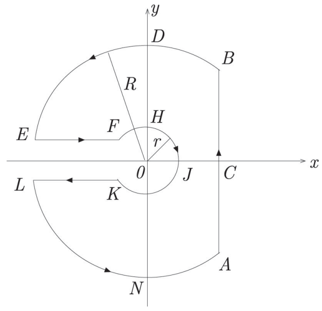

Figure 1: Bromwich contour

Proof of Theorem 1. Let $s \in \mathbb{C}$ and integration takes place in the complex plane along the line $s = c$. To show this and from the conditions of the theorem, c, which is the abscissa of point C, is chosen to replace the real part of all singularities of $\frac{G_1\left(\frac{1}{s}\right)}{s}$, including branch points, principal singularities and poles. For $\frac{G_1\left(\frac{1}{s}\right)}{s}$, one can compute the integral over the Bromwich contour as shown in Figure 1 to accommodate all cases, including infinitely many singularities. Therefore, without loss of generality, assuming that the singularities $\frac{G_1\left(\frac{1}{s}\right)}{s}$ are poles, that they all lie to the left of the real line $s = c$.

Let us start by evaluating the following integral along a path described in Figure 1:

$$S_q^{-1} \{G_1(s); t\} = \frac{1}{2\pi i} \int_{AB+\gamma} e_q\left(\frac{t}{s}\right) \frac{G_1(s)}{s} ds$$
$$= \frac{1}{2\pi i} \left(\int_{AB} + \int_{BDE} + \int_{EF} + \int_{FJK} + \int_{KL} + \int_{LNA} \right) e_q\left(\frac{t}{s}\right) \frac{G_1(s)}{s} ds.$$

Along AB: $T = \sqrt{R^2 - c^2}$ and $G_1(s)/s$ is analytic on AB and for sufficently large values of R, the following is imply

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{AB} e_q\left(\frac{t}{s}\right) \frac{G_1(s)}{s} ds$$
$$= \lim_{R\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} e_q\left(\frac{t}{s}\right) \frac{G_1(s)}{s} ds$$
$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e_q\left(\frac{t}{s}\right) \frac{G_1(s)}{s} ds$$
$$= \sum \operatorname{Res}\left[e_q\left(\frac{t}{s}\right) \frac{G_1(s)}{s} \right].$$

From the definition of (6), we have

$$\sum \operatorname{Res}\left[e_q\left(\frac{t}{s}\right) \frac{G_1(s)}{s} \right] = \sum \operatorname{Res}\left[\frac{1}{\left(\frac{t}{s}; q\right)_\infty} \frac{G_1(s)}{s} \right].$$

Since $\frac{1}{\left(\frac{t}{s};q\right)_\infty}\frac{G_1(s)}{s}$ has poles at $s_n = tq^n$ $(n = 0, 1, 2, ...)$, we get

$$\begin{aligned}
&\sum \operatorname{Res}\left[\frac{1}{\left(\frac{t}{s};q\right)_\infty}\frac{G_1(s)}{s}\right] \\
&= \sum_{n=0}^{\infty}\lim_{s\to s_n}\left((s-tq^n)\prod_{k=0}^{\infty}\frac{1}{\left(1-\frac{t}{s}q^k\right)}\frac{G_1(s)}{s}\right) \\
&= \sum_{n=0}^{\infty}\lim_{s\to s_n}\left(\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{1}{\left(1-\frac{t}{s}q^k\right)}G_1(s)\right) \\
&= \sum_{n=0}^{\infty}\prod_{\substack{k=0\\k\neq n}}^{\infty}\left(\frac{1}{(1-q^{k-n})}G_1(tq^n)\right) \\
&= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{(q,q)_n}G_1(tq^n).
\end{aligned}$$

Along BDE and LNA: We can find some parameters $M > 0,\ k > 0$ such that on BDE and LNA

$$\left|\frac{G_1(s)}{s}\right| < MR^{-k},$$

then the integral along BDE and LNA of $e_q\left(\frac{t}{s}\right)G_1(s)/s$ approaches zero as $R\to\infty$

$$\begin{aligned}
\lim_{R\to\infty}\frac{1}{2\pi i}\int_{BDE} e_q\left(\frac{t}{s}\right)\frac{G_1(s)}{s}ds = 0,\\
\lim_{R\to\infty}\frac{1}{2\pi i}\int_{LNA} e_q\left(\frac{t}{s}\right)\frac{G_1(s)}{s}ds = 0.
\end{aligned}$$

Along EF: $s = xe^{\pi i},\ ds = e^{\pi i}dx$ and in the process of s moves from $-R$ to $-r$, x goes from R to r :

$$\begin{aligned}
&\lim_{\substack{R\to\infty\\r\to 0}}\frac{1}{2\pi i}\int_{EF} e_q\left(\frac{t}{s}\right)\frac{G_1(s)}{s}ds \\
&= \lim_{\substack{R\to\infty\\r\to 0}}\frac{1}{2\pi i}\int_{R}^{r} e_q\left(-\frac{t}{x}\right)\frac{G_1(-x)}{x}dx.
\end{aligned}$$

Along FJK: $s = re^{i\theta},\ ds = ire^{i\theta}d\theta$, and since $\lim_{s\to\infty}G_1(s) = 0$, we have for sufficently small r

$$\begin{aligned}
&\lim_{r\to 0}\frac{1}{2\pi i}\int_{FJK} e_q\left(\frac{t}{s}\right)\frac{G_1(s)}{s}ds \\
&= \lim_{r\to 0}\frac{1}{2\pi i}\int_{\pi}^{-\pi} e_q\left(\frac{t}{re^{i\theta}}\right)G_1\left(re^{i\theta}\right)d\theta = 0.
\end{aligned}$$

Along KL: $s = xe^{-\pi i},\ ds = e^{-\pi i}dx$ and as s goes from $-r$ to $-R$, x goes from r to R :

$$\begin{aligned}
&\lim_{\substack{R\to\infty\\r\to 0}}\frac{1}{2\pi i}\int_{KL} e_q\left(\frac{t}{s}\right)\frac{G_1(s)}{s}ds \\
&= \lim_{\substack{R\to\infty\\r\to 0}}\frac{1}{2\pi i}\int_{r}^{R} e_q\left(-\frac{t}{x}\right)\frac{G_1(-x)}{x}dx.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
S_q^{-1}\{G_1(s);t\} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e_q\left(\frac{t}{s}\right)\frac{G_1(s)}{s}ds \\
&= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{(q,q)_n}G_1(tq^n).
\end{aligned}$$

□

Theorem 2. *Let $\mathbb{S}_q\{f(t);s\} = G_2(s)$ and its value zero at ∞. Assume that $G_2(s)/s$ is a meromorphic function with singularities in the region where* Re$(s) < c$. *If a circular domain γ exists with radius R and positive constants, M and k, with*

$$\left|\frac{G_2(s)}{s}\right| < MR^{-k},$$

then $f(t)$ is given by

$$\begin{aligned}
\mathbb{S}_q^{-1}\{G_2(s);t\} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds \\
&= \frac{1}{\left(\frac{1}{q};\frac{1}{q}\right)_\infty}\sum_{n=0}^{\infty}\frac{1}{(q,q)_n}G_2\left(-\frac{t}{q^n}\right). \qquad (27)
\end{aligned}$$

Proof of Theorem 2. The theorem is proved in the same way and under the same assumptions as in the preceding one. We evaluate integrals along the same path described in previous theorem:

$$\begin{aligned}
\mathbb{S}_q^{-1}\{G_2(s);t\} &= \frac{1}{2\pi i}\int_{AB+\gamma} E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds \\
&= \frac{1}{2\pi i}\left(\int_{AB}+\int_{BDE}+\int_{EF}+\int_{FJK}\right. \\
&\left.+\int_{KL}+\int_{LNA}\right)E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds.
\end{aligned}$$

Along AB: $T = \sqrt{R^2-c^2}$ and $G_2(s)/s$ is analytic on AB and for sufficently large values of R, we have

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{AB}E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds$$
$$=\lim_{R\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds$$
$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds$$
$$=\sum\text{Res}\left[E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}\right].$$

Making use of (8) and (6) we have

$$\sum\text{Res}\left[E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}\right]$$
$$=\sum\text{Res}\left[e_{1/q}\left(-\frac{t}{s}\right)\frac{G_2(s)}{s}\right]$$
$$=\sum\text{Res}\left[\frac{1}{\left(-\frac{t}{s};\frac{1}{q}\right)_\infty}\frac{G_2(s)}{s}\right]$$

Since $\frac{1}{\left(-\frac{t}{s};\frac{1}{q}\right)_\infty}\frac{G_2(s)}{s}$ has poles at $s_n=-\frac{t}{q^n}$ $(n=0,1,2,...)$, we get

$$\sum\text{Res}\left[\frac{1}{\left(-\frac{t}{s};\frac{1}{q}\right)_\infty}\frac{G_2(s)}{s}\right]$$
$$=\sum_{n=0}^{\infty}\lim_{s\to s_n}\left(\left(s+\frac{t}{q^n}\right)\prod_{k=0}^{\infty}\frac{1}{\left(1+\frac{t}{sq^k}\right)}\frac{G_2(s)}{s}\right)$$
$$=\sum_{n=0}^{\infty}\lim_{s\to s_n}\left(\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{1}{\left(1+\frac{t}{sq^k}\right)}G_2(s)\right)$$
$$=\sum_{n=0}^{\infty}\prod_{\substack{k=0\\k\neq n}}^{\infty}\left(\frac{1}{(1-q^{n-k})}G_2\left(-\frac{t}{q^n}\right)\right)$$
$$=\frac{1}{\left(\frac{1}{q};\frac{1}{q}\right)_\infty}\sum_{n=0}^{\infty}\frac{1}{(q,q)_n}G_2\left(-\frac{t}{q^n}\right).$$

Along BDE and LNA: We can find constans $M>0, k>0$ such that on BDE and LNA

$$\left|\frac{G_2(s)}{s}\right|<MR^{-k},$$

then the integral along BDE and LNA of $E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}$ approaches zero as $R\to\infty$

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{BDE}E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds=0,$$
$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{LNA}E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds=0.$$

Along EF: $s=xe^{\pi i}, ds=e^{\pi i}dx$ and as s goes from $-R$ to $-r$, x goes from R to r :

$$\lim_{\substack{R\to\infty\\r\to0}}\frac{1}{2\pi i}\int_{EF}E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds$$
$$=\lim_{\substack{R\to\infty\\r\to0}}\frac{1}{2\pi i}\int_{R}^{r}E_q\left(\frac{qt}{x}\right)\frac{G_2(-x)}{x}dx.$$

Along FJK: $s=re^{i\theta}, ds=rie^{i\theta}$, and since $\lim_{s\to\infty}G_2(s)=0$, for sufficently small r, we have

$$\lim_{r\to0}\frac{1}{2\pi i}\int_{FJK}E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds$$
$$=\lim_{r\to0}\frac{1}{2\pi i}\int_{\pi}^{-\pi}E_q\left(-\frac{qt}{re^{i\theta}}\right)G_2\left(re^{i\theta}\right)d\theta=0.$$

Along KL: $s=xe^{-\pi i}, ds=e^{-\pi i}dx$ and as s goes from $-r$ to $-R$, x goes from r to R :

$$\lim_{\substack{R\to\infty\\r\to0}}\frac{1}{2\pi i}\int_{KL}E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds$$
$$=\lim_{\substack{R\to\infty\\r\to0}}\frac{1}{2\pi i}\int_{r}^{R}E_q\left(\frac{qt}{x}\right)\frac{G_2(-x)}{x}dx.$$

Thus, we obtain

$$\mathbb{S}_q^{-1}\{G_2(s);t\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}E_q\left(-\frac{qt}{s}\right)\frac{G_2(s)}{s}ds$$
$$=\frac{1}{\left(\frac{1}{q};\frac{1}{q}\right)_\infty}\sum_{n=0}^{\infty}\frac{1}{(q;q)_n}G_2\left(-\frac{t}{q^n}\right).$$

□

4 Applications

In the present section, we will give some examples and consider inverse q-Sumudu transforms of the q-series by the help of main theorems. Throughout this section, we will assume that $G_1(s)=S_q\{f(t);s\}$ and $G_2(s)=\mathbb{S}_q\{f(t);s\}$.

Example 1. *Let*

$$G_i(s)=s^j\quad(\forall j\in\mathbb{Z},\ i=1,2).$$

Let us consider $S_q^{-1}\{G_1(s);t\}$ first. Using (26) we obtain

$$S_q^{-1}\{G_1(s);t\} = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{(q,q)_n}G_1(tq^n) = \frac{t^j}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q,q)_n}\left(q^{j+1}\right)^n.$$

Hence by (7) and (4), we have

$$S_q^{-1}\{G_1(s);t\} = \frac{t^j\left(q^{j+1};q\right)_\infty}{(q;q)_\infty} = \frac{t^j}{(q;q)_j}.$$

Now, we consider $\mathbb{S}_q^{-1}\{G_2(s);t\}$. Similarly, as in the previous one, using (27) we obtain

$$\mathbb{S}_q^{-1}\{G_2(s);t\} = \frac{1}{\left(\frac{1}{q};\frac{1}{q}\right)_\infty}\sum_{n=0}^{\infty}\frac{1}{(q,q)_n}G_2\left(-\frac{t}{q^n}\right) = \frac{(-1)^j t^j}{\left(\frac{1}{q};\frac{1}{q}\right)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n+1}{2}}}{\left(\frac{1}{q};\frac{1}{q}\right)_n}\left(q^{-j}\right)^n.$$

Making use of (7), (4) and (5), we have

$$\mathbb{S}_q^{-1}\{G_2(s);t\} = \frac{(-1)^j t^j\left(\frac{1}{q^{j+1}};\frac{1}{q}\right)_\infty}{\left(\frac{1}{q};\frac{1}{q}\right)_\infty} = \frac{(-1)^j t^j}{\left(\frac{1}{q};\frac{1}{q}\right)_j} = \frac{q^{\binom{j+1}{2}}t^j}{(q;q)_j}.$$

Consequently, we obtain

$$S_q^{-1}\left\{s^j;t\right\} = \frac{1}{(q;q)_j}t^j, \tag{28}$$

$$\mathbb{S}_q^{-1}\left\{s^j;t\right\} = \frac{q^{\binom{j+1}{2}}}{(q;q)_j}t^j. \tag{29}$$

Now, we consider an interesting application of the main results, which provides the inverse q-Sumudu transforms of q-series functions. Furthermore, we also provide a comprehensive list of inverse q-Sumudu transforms for certain functions available in the literature.

Example 2. *If $G_i(s)$ $(i=1,2)$ has in the form*

$$G_i(s) = \sum_{j=0}^{\infty}a_j s^{aj+b},\ \left(i=1,2,\ \left|\frac{a_{j+1}}{a_j}\right|<1,\ a,b\in\mathbb{Z}\right)$$

then

$$S_q^{-1}\{G_1(s);t\} = \sum_{j=0}^{\infty}a_j\frac{t^{aj+b}}{(q;q)_{aj+b}}, \tag{30}$$

$$\mathbb{S}_q^{-1}\{G_2(s);t\} = \sum_{j=0}^{\infty}a_j\frac{q^{\binom{aj+b+1}{2}}t^{aj+b}}{(q;q)_{aj+b}}. \tag{31}$$

To show (30), we consider $S_q^{-1}\{G_1(s);t\}$. From (26), we have

$$\begin{aligned} S_q^{-1}\{G_1(s);t\} &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{(q,q)_n}G_1(tq^n) \\ &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{(q,q)_n}\sum_{j=0}^{\infty}a_j(tq^n)^{aj+b} \\ &= \sum_{j=0}^{\infty}a_j\frac{t^{aj+b}}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q,q)_n}\left(q^{aj+b+1}\right)^n. \end{aligned}$$

Using (7) and (4), we get

$$S_q^{-1}\{G_1(s);t\} = \sum_{j=0}^{\infty}a_j\frac{t^{aj+b}\left(q^{aj+b+1};q\right)_\infty}{(q;q)_\infty} = \sum_{j=0}^{\infty}a_j\frac{t^{aj+b}}{(q;q)_{aj+b}}.$$

Now, we consider $\mathbb{S}_q^{-1}\{G_2(s);t\}$. Similarly, by (27), we have

$$\begin{aligned} \mathbb{S}_q^{-1}\{G_2(s);t\} &= \frac{1}{\left(\frac{1}{q};\frac{1}{q}\right)_\infty}\sum_{n=0}^{\infty}\frac{1}{(q,q)_n}G_2\left(-\frac{t}{q^n}\right) \\ &= \frac{1}{\left(\frac{1}{q};\frac{1}{q}\right)_\infty}\sum_{n=0}^{\infty}\frac{1}{(q,q)_n}\sum_{j=0}^{\infty}a_j\left(-\frac{t}{q^n}\right)^{aj+b} \\ &= \sum_{j=0}^{\infty}a_j\frac{(-1)^{aj+b}t^{aj+b}}{\left(\frac{1}{q};\frac{1}{q}\right)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{-\binom{n}{2}}}{\left(\frac{1}{q};\frac{1}{q}\right)_n}\left(q^{-aj-b-1}\right)^n. \end{aligned}$$

Making use of (7), (4) and (5), we obtain

$$\begin{aligned} \mathbb{S}_q^{-1}\{G_2(s);t\} &= \sum_{j=0}^{\infty}a_j\frac{(-1)^{aj+b}t^{aj+b}\left(\frac{1}{q^{aj+b+1}};\frac{1}{q}\right)_\infty}{\left(\frac{1}{q};\frac{1}{q}\right)_\infty} \\ &= \sum_{j=0}^{\infty}a_j\frac{(-1)^{aj+b}t^{aj+b}}{\left(\frac{1}{q};\frac{1}{q}\right)_{aj+b}} = \sum_{j=0}^{\infty}a_j\frac{q^{\binom{aj+b+1}{2}}t^{aj+b}}{(q;q)_{aj+b}}. \end{aligned}$$

Example 3. *If we set* $a_j = \frac{(-1)^j a^j}{(q;q)_j}$, $a = 1, b = 0$ *in the previous example, we have*

$$G_i(s) = \sum_{j=0}^{\infty} a_j s^{aj+b} = \sum_{j=0}^{\infty} \frac{(-1)^j a^j}{(q;q)_j} s^j = e_q(-as)$$

for $i = 1, 2$. *Thus, using the results (30) and (31), we get*

$$\begin{aligned} S_q^{-1}\{e_q(-as);t\} &= \sum_{j=0}^{\infty} \frac{(-1)^j a^j}{(q;q)_j} \frac{t^j}{(q;q)_j} \\ &= J_0^{(1)}\left(2\sqrt{at};q\right), \\ \mathbb{S}_q^{-1}\{e_q(-as);t\} &= \sum_{j=0}^{\infty} \frac{(-1)^j a^j}{(q;q)_j} \frac{q^{\binom{j+1}{2}} t^j}{(q;q)_j} \\ &= J_0^{(3)}\left(2\sqrt{at};q\right). \end{aligned}$$

We have summarized the results obtained with some specific choices of a_j in Examples 2 and 3 in Table 1 (Appendix) and Table 2 (Appendix).

5 Concluding Remark

In this paper we focused on evaluating complex inversion formulas for the q-Sumudu transforms. The special cases of (30) and (31) can be found in Table 1 (Appendix) and Table 2 (Appendix). As a final remark it can be said that, the complex inversion formulas, deduced in the previous section for q-Sumudu transforms, are significant and can yield numerous inverse q-Sumudu transforms for variety of q-functions. The results obtained in this paper provide q-extensions of the outcome given in, [4], as mentioned earlier. q-difference equations are q-functional equations which relate q-functions with their q-derivatives. We think that as an application of the our results obtained here, finding the solutions of q-difference equations such as non-homogenous q-difference equations with linear constant coefficients, non-homogenous q-difference equations with linear variable coefficients will be the main subject of our future studies.

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Appendix

Table 1: Special cases of (30)

a_j	a	b	$G_1(s)$	$f(t)=S_q^{-1}\{G_1(s);t\}$
a^j	1	0	$\frac{1}{1-as}$	$e_q(at)$
$(-1)^j a^{2j+1}$	2	1	$\frac{as}{1+a^2s^2}$	$\sin_q(at)$
$(-1)^j a^{2j}$	2	0	$\frac{1}{1+a^2s^2}$	$\cos_q(at)$
$(-1)^j a^j q^{\binom{j}{2}}$	1	0	${}_1\Phi_1(q;0;q;as)$	$E_q(at)$
$(-1)^j q^{j(2j+1)}a^{2j+1}$	2	1	$as{}_1\Phi_1\left(q^4;0;q^4;a^2s^2q^3\right)$	$\mathrm{Sin}_q(at)$
$(-1)^j q^{j(2j-1)}a^{2j}$	2	0	${}_1\Phi_1\left(q^4;0;q^4;a^2s^2q\right)$	$\mathrm{Cos}_q(at)$
$\frac{(-1)^j a^j}{(q;q)_j}$	1	0	$e_q(-as)$	$J_0^{(1)}\left(2\sqrt{at};q\right)$

Table 2: Special cases of (31)

a_j	a	b	$G_2(s)$	$f(t)=\mathbb{S}_q^{-1}\{G_2(s);t\}$
$\left(\frac{-a}{q}\right)^j$	1	0	$\frac{q}{q+as}$	$E_q(at)$
$(-1)^j\frac{a^{2j+1}}{q^{2j+1}}$	2	1	$\frac{qas}{q^2+a^2s^2}$	$\mathrm{Sin}_q(at)$
$(-1)^j\frac{a^{2j}}{q^{2j}}$	2	0	$\frac{q^2}{q^2+a^2s^2}$	$\mathrm{Cos}_q(at)$
$\frac{(-1)^j a^j}{(q;q)_j}$	1	0	$e_q(-as)$	$J_0^{(3)}\left(\sqrt{at};q\right)$

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.