A Generalized Hybrid Method for Handling Fractional Caputo Partial Differential Equations via Homotopy Perturbed Analysis

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Abstract: - This article describes a novel hybrid technique known as the Sawi transform homotopy perturbation method for solving Caputo fractional partial differential equations. Combining the Sawi transform and the homotopy perturbation method, this innovative technique approximates series solutions for fractional partial differential equations. The Sawi transform is a recently developed integral transform that may successfully manage recurrence relations and integro-differential equations. Using a homotopy parameter, the homotopy perturbation method is a potent semi-analytical tool for constructing approximate solutions to nonlinear problems. The suggested method offers various advantages over existing methods, including high precision, rapid convergence, minimal computing expense, and broad applicability. The new method is used to solve the convection–reaction–diffusion problem using fractional Caputo derivatives.

Key-Words: - Sawi transform, Homotopy perturbation method, Fractional partial differential equations, Caputo fractional derivative, Series solution, Nonlinear partial differential equations.

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1 Introduction

Various phenomena in physics, engineering, biology, and other disciplines are typically modeled using fractional partial differential equations (FPDEs), [1], [2], [3]. Due to the nonlocal and singular nature of fractional derivatives, however, accurate or numerical solutions to FPDEs are frequently difficult to discover. The fractional power series method, the fast convolution algorithm, the fractional differential transform method, the finite difference method, and the fixed point and upper and lower solution methods, [4], [5], [6], [7], [8] are examples of analytical and numerical methods that have been developed to solve FPDEs.

In this paper, the Sawi transform homotopy perturbation method (STHPM) is introduced as a novel hybrid strategy for solving FPDEs with Caputo fractional derivatives. The Caputo fractional derivative, which is widely recognized as one of the most significant definitions of fractional derivatives, offers the distinct advantage of maintaining the beginning conditions in the classical sense, as supported by references [9], [10] and [11]. Approximate series solutions for FPDEs are generated using the Sawi transform-homotopy perturbation method (STHPM) combination, [12], [13], [14], [15], [16].

In this study, homotopy perturbation methods are applied to solve fractional Caputo partial differential equations (PDEs) in a novel manner, as noted in references [17] and [18].

According to references [19] and [20], the STHPM produces extremely precise results and saves a significant amount of calculation time when compared to other techniques like the variational iteration approach and the Adomian decomposition method.

Additionally, a variety of complex and non-linear partial differential equations (PDEs) can be solved quickly and effectively with STHPM's flexibility, something that is challenging to achieve with conventional numerical approaches, [21], [22] and [23]. Scholars and professionals alike will find the STHPM to be a beneficial tool as it provides a novel perspective on the analysis and solution of fractional Caputo PDEs. The solution of the convection-reaction-diffusion equation shows the practicality and efficacy of the STHPM. We compare our results with the accuracy, computational expense, and convergence of known numerical methods or solutions, as documented in references [24], [25] and [26]. As shown in references [27], [28] and [29], we also go over potential STHPM extensions and uses to handle other types of FPDEs.

It is crucial to acknowledge that the method has limits, including the need for meticulous selection of homotopy parameters. These constraints will be thoroughly described in the subsequent portions of this study, as referenced by [30], [31], [32], [33], [34].

The Sawi transform homotopy perturbation method is a significant improvement in the field of fractional differential equations, addressing a critical gap in the current literature and providing a more generic, efficient, and accurate method for solving fractional Caputo PDEs, [35].

2 Basic Concepts of Sawi Transform

This section is concerned with the presentation of the Sawi transform. We out line some basic properties regarding the existence conditions, linearity and the inverse of this transform. Moreover, some essential properties and results regarding Sawi transform are discussed. We introduce the Sawi convolution theorem and the derivative properties. For more details about Sawi transform see, [17], [18].

Definition 2.1. If $w(t)$ is a function defined over a positive domain. Then, Sawi transform of $w(t)$, denoted by $\mathbb{S}[w(t)]$, is given by

$$\mathbb{S}[w(t)] = \Psi(v) = \frac{1}{v^2}\int_0^\infty w(t)e^{\frac{-t}{v}}dt, t \geq 0. \quad (1)$$

The inverse Sawi transformation is provided as

$$\mathbb{S}^{-1}[\Psi(v)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{v^2}e^{\frac{1}{v}t}\Psi(v)dv \quad (2)$$
$$= w(t), \qquad t > 0\,.$$

Theorem 2.1. If $w(t)$ is continuous function defined for $t > 0$ and of exponential order ρ. Then $\mathbb{S}[w(t)]$ exists for $v > \rho$ and satisfies

$$|w(t)| \leq Me^{\rho t},$$

where $M > 0$, then Sawi transformation exists for $v > \rho$.

Suppose that $\mathbb{S}[w(t)] = \Psi(v)$ and $\mathbb{S}[h(t)] = H(v)$ and $\imath, \jmath \in \mathbb{R}$, then the following properties hold:

$$\mathbb{S}[\imath\, w(t) + \jmath\, h(t)] = \imath\, \mathbb{S}[w(t)] + \jmath\, \mathbb{S}[h(t)].$$

$$\mathbb{S}^{-1}[\imath\, \Psi(v) + \jmath\, H(v)] = \imath\, \mathbb{S}^{-1}[\Psi(v)] + \jmath\, \mathbb{S}^{-1}[H(v)].$$

$$\mathbb{S}[t^{\jmath}] = v^{\jmath-1}\Gamma(\jmath+1).$$

$$\mathbb{S}[e^{\jmath t}] = \frac{1}{v(1-\jmath v)}.$$

$$\mathbb{S}[\cos(\jmath t)] = \frac{1}{v(1+\jmath^2 v^2)}.$$

$$\mathbb{S}[\sin(\jmath t)] = \frac{\jmath}{1+\jmath^2 v^2}.$$

$$\mathbb{S}[\cosh(\jmath t)] = \frac{1}{v(1-\jmath^2 v^2)}.$$

$$\mathbb{S}[\sinh(\jmath t)] = \frac{\jmath}{1-\jmath^2 v^2}.$$

$$\mathbb{S}\left[\frac{d^{\jmath} w(t)}{dt^{\jmath}}\right] = \frac{\Psi(v)}{v^{\jmath}} - \sum_{i=0}^{\jmath-1} \frac{w^{(i)}(0)}{v^{\jmath-i+1}}.$$

Theorem 2.2. Let $\mathbb{S}[w(t)] = \Psi(v)$. Then,

$$\mathbb{S}[w(t-\jmath)H(t-\jmath)] = e^{-\frac{1}{v}\jmath}\,\Psi(v), \quad (3)$$

where $H(t)$ denotes the unit step function defined by

$$H(t-\jmath) = \begin{cases} 1, & t > j, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.3. (Sawi Convolution Theorem). If $\mathbb{S}[w(t)] = \Psi(v)$ and $[h(t)] = \mathrm{H}(v)$, then

$$\mathbb{S}[(w * h)(t)] = v^2\,\Psi(v)H(v), \quad (4)$$

where

$$(w * h)(t) = \int_0^t w(\tau)h(\,t-\tau)d\tau.$$

3 Fundamental Facts of Fractional Calculus

In this section, some definitions and properties of fractional calculus that will be used in this work are presented.

Definition 3.1. [35], The three - parameters Mittag-Leffler function is defined as:

$$E_{r,j}^{\delta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{(\delta)_n}{\Gamma(rn + j)} \quad , t, r, j, \delta \in \mathbb{C} \quad , \mathrm{Re}(\delta) > 0, Re(r) > 0, Re(j) > 0, \tag{5}$$

where $(\delta)_n$ is the Pochhammer symbol.

Putting $\delta = 1$ in Eq. (5), we have the new function turns into the two - parameters Mittag-Leffler function:

$$E_{r,j}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(rn + j)} \quad , t, r, j \in \mathbb{C}, \quad \mathrm{Re}(r) > 0, Re(j) > 0. \tag{6}$$

Putting $j = 1$ in Eq. (6), we have the new function turns into the classical Mittag-Leffler function:

$$E_r(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(rn + 1)}, t, r \in \mathbb{C}, \mathrm{Re}(r) > 0. \tag{7}$$

we note that $E_{1,1}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t$.

Definition 3.2. [36], Let $w(t)$ be a continuous function, and $n - 1 < r \leq n$. Then the Caputo fractional derivative of the function $w(t)$ with respect to t of the order r is defined as:

$$D_t^r w(t) = \frac{1}{\Gamma(n - r)} \int_a^t (t - \tau)^{n-r-1} w^{(n)}(\tau) d\tau, a \in \mathbb{R}. \tag{8}$$

Theorem 3.1. Let $\Psi(v)$ be the Sawi transform of $w(t)$ .Then the Sawi transform of Caputo fractional derivative of $w(t)$ is expressed as:

$$\mathbb{S}[D_t^r w(t)] = \frac{1}{v^r} \left(\Psi(v) - \sum_{i=0}^{n-1} \frac{w^{(i)}(0)}{v^{1-i}} \right), \quad n - 1 < r \leq n. \tag{9}$$

Proof. By the definition of convolution integral, we have:

$$\int_0^t (t - \tau)^{n-r-1} w^{(n)}(\tau) d\tau = t^{n-r-1} * w^{(n)}(t).$$

Therefore,

$$\begin{aligned}
\mathbb{S}[D_t^r w(t)] &= \frac{1}{v^2} \int_0^{\infty} \left(\frac{1}{\Gamma(n - r)} \int_0^t (t - \tau)^{n-r-1} w^{(n)}(\tau) d\tau \right) e^{\frac{-t}{v}} dt \\
&= \frac{1}{\Gamma(n - r)} \mathbb{S}\left[t^{n-r-1} * w^{(n)}(t)\right] \\
&= \frac{1}{\Gamma(n - r)} \left(v^2 \, \mathbb{S}[t^{n-r-1}] \, \mathbb{S}\left[w^{(n)}(t)\right]\right).
\end{aligned}$$

Using the properties of Sawi transform, we have

$$\begin{aligned}
\mathbb{S}[D_t^r w(t)] &= \frac{v^2}{\Gamma(n - r)} \left(v^{n-r-2} \Gamma(n - r) \left(\frac{\Psi(v)}{v^n} - \sum_{i=0}^{n-1} \frac{w^{(i)}(0)}{v^{n-i+1}} \right) \right) \\
&= \frac{\Psi(v)}{v^r} - \sum_{i=0}^{n-1} \frac{w^{(i)}(0)}{v^{r-i+1}}.
\end{aligned}$$

Thus,

$$\mathbb{S}[D_t^r w(t)] = \frac{1}{v^r} \left(\Psi(v) - \sum_{i=0}^{n-1} \frac{w^{(i)}(0)}{v^{1-i}} \right), \quad n - 1 < r \leq n.$$

Corollary3.1. Let $\Psi(u, v)$ be the Sawi transform of $w(u, t)$ and $0 < r \leq 1$ .Then the Sawi transform of Caputo fractional derivative of $w(u, t)$ is expressed as

$$\mathbb{S}[D_t^r w(u, t)] = \frac{1}{v^r} \left(\Psi(u, v) - \frac{1}{v} w(u, 0) \right). \tag{10}$$

4 Analysis of Sawi Transform Homotopy Perturbation Method

In this part of the paper, we present the fundamental idea of Sawi transform homotopy perturbation method for solving fractional Caputo partial differential equations. In order to show the fundamental plan of the STHPM, we consider the following general partial differential equation

$$D_t^r w(u, t) + L(w(u, t)) + N\big(w(u, t)\big) = R(u, t), \quad (u, t) \in [0,1] \times [0, T], n - 1 < r < n, \text{ and } T > 0, \tag{11}$$

subject to the conditions

$$\frac{\partial^i w(u, 0)}{\partial t^i} = k_i(u) \quad , \quad i = 0,1, \dots, n - 1, \tag{12}$$

where L, N are linear and nonlinear differential operators, D_t^r denotes the Caputo fractional derivative with respect to the variable t, $w(u,t)$ is the unknown function and $R(u,t)$ is a given function.

Applying the Sawi transform for Eq. (11), with respect to t, we obtain

$$W(u,v) = v^r\left(\mathbb{S}\left[R(u,t) - L(w(u,t)) + N\left(w(u,t)\right)\right]\right) + \sum_{i=0}^{n-1} \frac{1}{v^{1-i}}\left(\frac{\partial^i \mathrm{w}(u,0)}{\partial t^i}\right). \quad (13)$$

Thus, the homotopy parameter q is defined as

$$w(u,t) = \sum_{z=0}^{\infty} q^z w_z(u,t). \quad (14)$$

The nonlinear terms in Eq.(11) can be written as

$$N\left(w(u,t)\right) = \sum_{z=0}^{\infty} q^z \mathcal{H}_z. \quad (15)$$

Where $\mathcal{H}_z$ are He's polynomials, which can be calculated by using the following formula

$$\mathcal{H}_z = \frac{1}{z!}\frac{\partial^z}{\partial q^z} N\left(\sum_{i=0}^{\infty} q^i w_i(u,t)\right)\Bigg|_{q=0}, z = 0,1,2,\ldots \quad (16)$$

We carry out the component of the Caputo operator result by substituting Eqs. (14) and (15) into Eq. (13).

$$\sum_{z=0}^{\infty} q^z W_z(u,v) = v^r \mathbb{S}[R(u,t)] + \sum_{i=0}^{n-1} \frac{1}{v^{1-i}}\left(\frac{\partial^i w(u,0)}{\partial t^i}\right) - q v^r\left(\mathbb{S}\left[L\left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right) + \sum_{z=0}^{\infty} q^z \mathcal{H}_z\right]\right). \quad (17)$$

Appling the inverse Sawi transform to Eq. (17), we have:

$$\sum_{z=0}^{\infty} q^z w_z(u,t) = \mathbb{S}^{-1}\left[v^r \mathbb{S}[R(u,t)] + \sum_{i=0}^{n-1} \frac{1}{v^{1-i}}\left(\frac{\partial^i w(u,0)}{\partial t^i}\right)\right] - q\,\mathbb{S}^{-1}\left[v^r \mathbb{S}\left[L\left(\sum_{z=0}^{\infty} q^z w_z(u,t)\right) + \sum_{z=0}^{\infty} q^z \mathcal{H}_z\right]\right]. \quad (18)$$

Thus, Eq. (18) , when solved with respect to q, are defined as

$$\begin{aligned}
q^0 &: w_0(u,t) = \mathbb{S}^{-1}\left[v^r \mathbb{S}[R(u,t)] + \sum_{i=0}^{n-1} \frac{1}{v^{1-i}}\left(\frac{\partial^i w(u,0)}{\partial t^i}\right)\right],\\
q^1 &: w_1(u,t) = -\mathbb{S}^{-1}\left[v^r \mathbb{S}\left[L\left(w_0(u,t)\right) + \mathcal{H}_0\right]\right],\\
q^2 &: w_2(u,t) = -\mathbb{S}^{-1}\left[v^r \mathbb{S}\left[L\left(w_1(u,t)\right) + \mathcal{H}_1\right]\right],\\
&\vdots\\
q^{z+1} &: w_{z+1}(u,t) = -\mathbb{S}^{-1}\left[v^r \mathbb{S}\left[L\left(w_z(u,t)\right) + \mathcal{H}_z\right]\right], \quad z \geq 0,
\end{aligned} \quad (19)$$

when $q \to 1$ is applied, suppose that Eq .(19) is the approximated solution to Eq. (11) , and the solution is

$$w(u,t) = w_0(u,t) + w_1(u,t) + w_2(u,t) + \cdots. \quad (20)$$

5 Applications

In this section of this paper, we present some examples to show the efficiency of the presented method.

Application 5.1. Consider the following convection–reaction–diffusion equation

$$\frac{\partial^{r} w(u,t)}{\partial t^{r}}=\frac{\partial^{2} w(u,t)}{\partial u^{2}}+w(u,t)-\frac{\partial w(u,t)}{\partial u}+w(u,t)\frac{\partial w(u,t)}{\partial u}-w^{2}(u,t), \tag{21}$$

subject to the conditions

$$w(u,0)=e^{u}. \tag{22}$$

Applying Sawi transform homotopy perturbation method for Eq. (21), we obtain:

$$\sum_{z=0}^{\infty} q^{z} W_{z}(u,v)=\frac{1}{v} w(u,0)+q v^{r}\left(\mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^{z} w_{z}(u,t)\right)_{uu}+\sum_{z=0}^{\infty} q^{z} w_{z}(u,t)-\left(\sum_{z=0}^{\infty} q^{z} w_{z}(u,t)\right)_{u}\right]\right)+q v^{r}\left(\mathbb{S}\left[\sum_{z=0}^{\infty} q^{z}\mathcal{H}_{z}(u,t)\right]\right). \tag{23}$$

Taking the inverse Sawi transform to Eq. (23) ,we get

$$\sum_{z=0}^{\infty} q^{z} w_{z}(u,t)=\mathbb{S}^{-1}\left[\frac{1}{v} w(u,0)\right]+q\,\mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^{z} w_{z}(u,t)\right)_{uu}+\sum_{z=0}^{\infty} q^{z} w_{z}(u,t)-\left(\sum_{z=0}^{\infty} q^{z} w_{z}(u,t)\right)_{u}\right]\right]+q\,\mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[\sum_{z=0}^{\infty} q^{z}\mathcal{H}_{z}(u,t)\right]\right] \tag{24}$$

Note that, the first few terms of $\mathcal{H}_z$ in this case is given by:

$$\begin{aligned}
\mathcal{H}_0&=w_0 w_{0_u}-(w_0)^2,\\
\mathcal{H}_1&=w_0 w_{1_u}+w_1 w_{0_u}-2w_0 w_1,\\
\mathcal{H}_2&=w_0 w_{2_u}+w_1 w_{1_u}+w_2 w_{0_u}-2w_0 w_2-(w_2)^2,\\
&\ \vdots
\end{aligned} \tag{25}$$

The function of the Caputo derivative result is achieved by calculating the powers of q

$$q^{0}: w_0(u,t)=\mathbb{S}^{-1}\left[\frac{1}{v} w(u,0)\right]=\mathbb{S}^{-1}\left[\frac{1}{v} e^{u}\right]=e^{u}, \tag{26}$$

$$q^{n+1}: w_{n+1}(u,t)=\mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[w_{n_{uu}}(u,t)+w_n(u,t)-w_{n_u}(u,t)+\mathcal{H}_n\right]\right]. \tag{27}$$

Putting $n=0$ into Eq. (27) , we get

$$\begin{aligned}
q^{1}: w_1(u,t)&=\mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[w_{0_{uu}}(u,t)+w_0(u,t)-w_{0_u}(u,t)+\mathcal{H}_0\right]\right]\\
&=\mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[e^{u}+e^{u}-e^{u}+e^{2u}-e^{2u}\right]\right]=\mathbb{S}^{-1}\left[v^{r-1}e^{u}\right]\\
&=e^{u}\frac{t^{r}}{\Gamma(r+1)}.
\end{aligned}$$

Putting $n=1$ into Eq. (27), we get

$$\begin{aligned}
q^{2}: w_2(u,t)&=\mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[w_{1_{uu}}(u,t)+w_1(u,t)-w_{1_u}(u,t)+\mathcal{H}_1\right]\right]\\
&=\mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[\frac{t^{r}e^{u}}{\Gamma(r+1)}+\frac{t^{r}e^{u}}{\Gamma(r+1)}-\frac{t^{r}e^{u}}{\Gamma(r+1)}+\frac{t^{r}e^{2u}}{\Gamma(r+1)}-\frac{t^{r}e^{2u}}{\Gamma(r+1)}\right]\right]\\
&=\mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[\frac{t^{r}e^{u}}{\Gamma(r+1)}\right]\right]\\
&=\mathbb{S}^{-1}\left[v^{2r-1}e^{u}\right]=\frac{t^{2r}e^{u}}{\Gamma(2r+1)},
\end{aligned}$$

in the same way, we get

$$\begin{gathered}
q^{3}: w_3(u,t)=\frac{t^{3r}e^{u}}{\Gamma(3r+1)},\\
\vdots\\
q^{n}: w_n(u,t)=\frac{t^{nr}e^{u}}{\Gamma(nr+1)}.
\end{gathered}$$

Therefore, the solution of Eq.(21) is given by

$$
\begin{aligned}
w(u,t) &= w_0(u,t) + w_1(u,t) + w_2(u,t) \\
&\quad + \cdots + w_n(u,t) + \cdots \\
&= e^{u} + e^{u}\frac{t^{r}}{\Gamma(r+1)} \\
&\quad + e^{u}\frac{t^{2r}}{\Gamma(2r+1)} \\
&\quad + e^{u}\frac{t^{3r}}{\Gamma(3r+1)} + \cdots \\
&\quad + \frac{t^{nr}e^{u}}{\Gamma(nr+1)} + \cdots \\
&= e^{u}\left(1 + \frac{t^{r}}{\Gamma(r+1)} \right. \\
&\quad + \frac{t^{2r}}{\Gamma(2r+1)} + \frac{t^{3r}}{\Gamma(3r+1)} \\
&\quad \left. + \cdots + \frac{t^{nr}}{\Gamma(nr+1)} + \cdots \right) \\
&= e^{u}\sum_{z=0}^{\infty}\frac{t^{zr}}{\Gamma(zr+1)}.
\end{aligned} \tag{28}
$$

at $r = 1$, then the exact solution is $w(u,t) = e^{u+t}$. Below, we sketch the graph of the exact solution $w(u,t) = e^{u+t}$ in Figure 1, and the approximate solution in Eq. (28) with different values of r, $r = 1, 0.9, 0.8, 0.6$ in Figure 2.

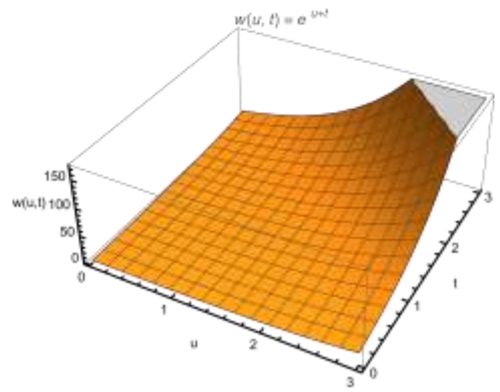
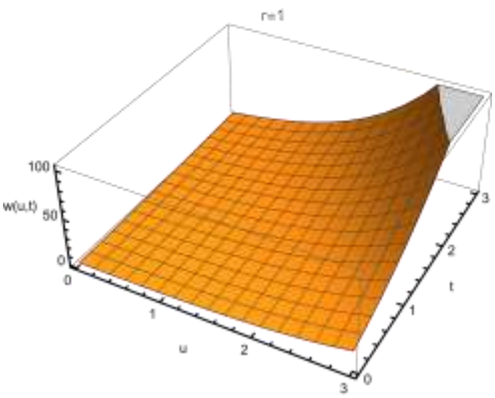


Fig. 1: The exact solution convection–reaction–diffusion equation (21)



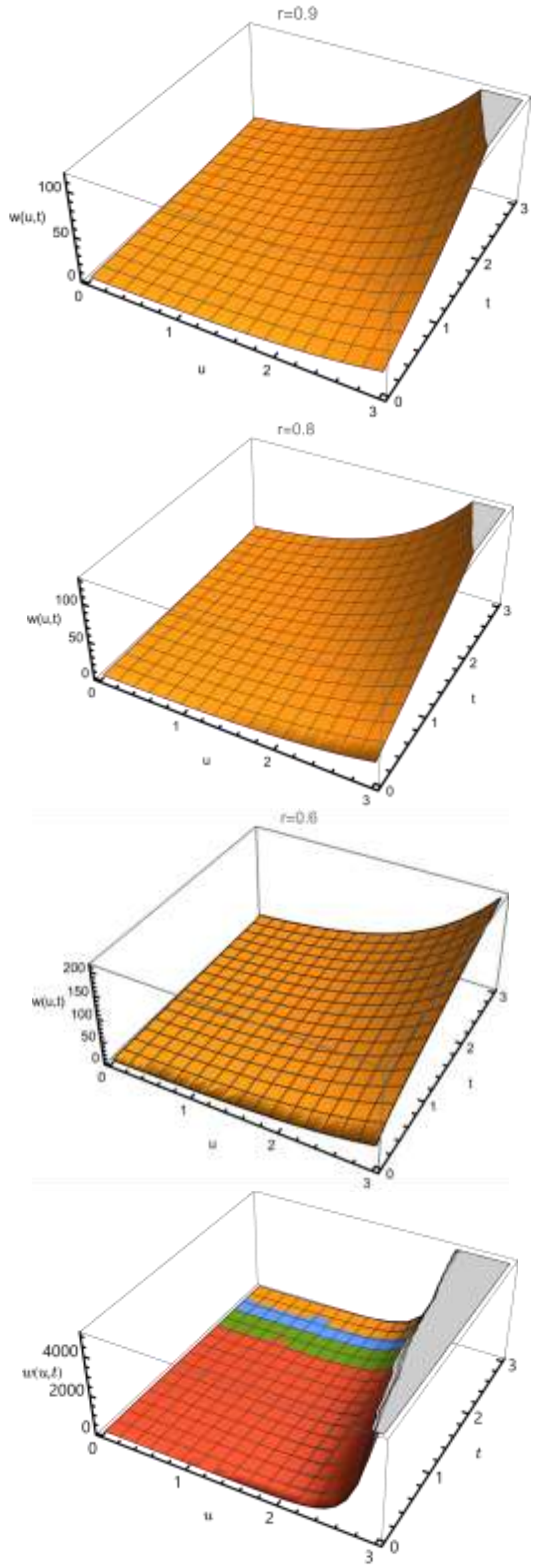


Fig. 2: The approximate solution in Eq. (28) with different values of r

In Figure 3, the 3D plots showing the absolute error between the solution of Eq.(21) and the exact solution e^{u+t} for each specified value of r.

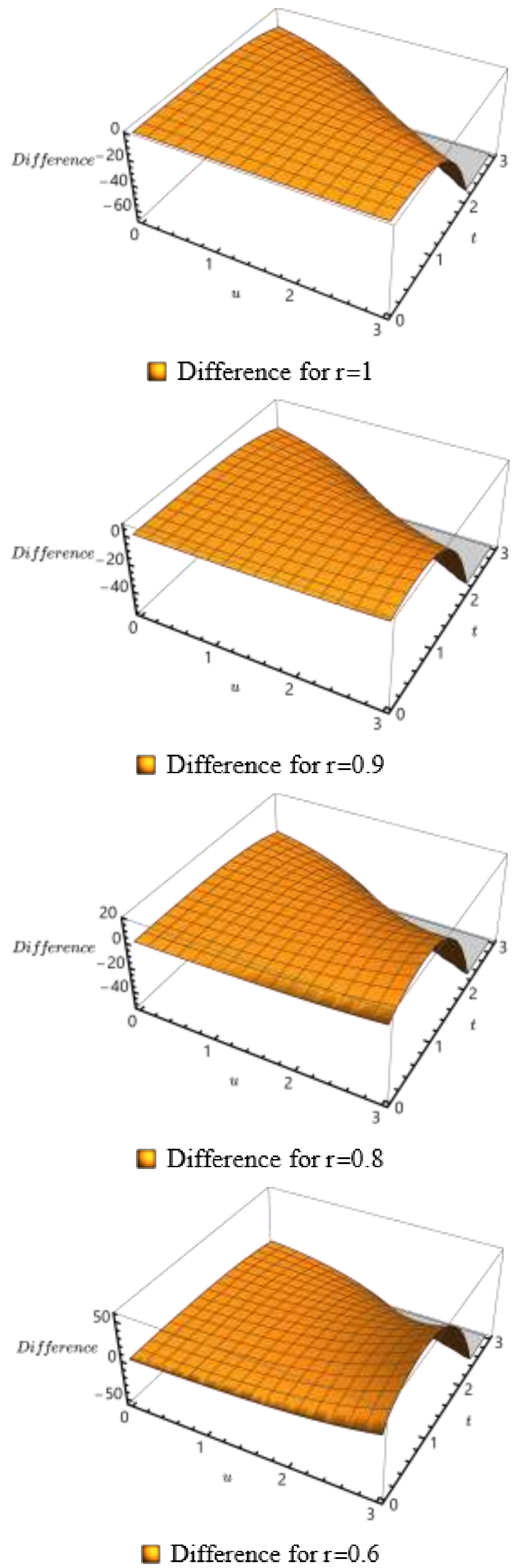


Fig. 3: The absolute error of Application 5.1

The above plots show the difference between the approximate solution and the exact solution e^{u+t} for different values of r. This difference can be interpreted as the deviation of the exact function from a simple exponential function e^{u+t}.

As r decreases, the difference becomes more pronounced, especially for larg values of u and t. This suggests that the approximate solution deviates more from e^{u+t} as r decreases. The complexity of the surface increases as r decreases, indicating that the function becomes more sensitive to changes in u and t.

Application 5.2. Consider the following convection–reaction–diffusion equation:

$$\frac{\partial^r w(u,t)}{\partial t^r} = \frac{\partial^2 w(u,t)}{\partial u^2} - (1+4u^2)w(u,t), \quad (29)$$

subject to the conditions:

$$w(u,0) = e^{u^2}. \quad (30)$$

Applying Sawi transform homotopy perturbation method for Eq. (29), we obtain:

$$\sum_{z=0}^{\infty} q^z W_z(u,v) = \frac{1}{v} w(u,0) + q v^r \left(\mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^z w_z(u,t) \right)_{uu} - (1 + 4u^2) \left(\sum_{z=0}^{\infty} q^z w_z(u,t) \right) \right] \right). \quad (31)$$

Taking inverse Sawi transform to Eq.(31) ,we get:

$$\sum_{z=0}^{\infty} q^z w_z(u,t) = \mathbb{S}^{-1}\left[\frac{1}{v} w(u,0)\right] + q\, \mathbb{S}^{-1}\left[v^r \mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^z w_z(u,t) \right)_{uu} - (1 + 4u^2) \left(\sum_{z=0}^{\infty} q^z w_z(u,t) \right) \right] \right].$$

Thus, the function of the Caputo derivative result is achieved by calculating the powers of:

$$q^0: w_0(u,t) = \mathbb{S}^{-1}\left[\frac{1}{v}w(u,0)\right] = \mathbb{S}^{-1}\left[\frac{1}{v}e^{\,u^2}\right] = e^{\,u^2}, \tag{32}$$

$$q^{n+1}: w_{n+1}(u,t) = \mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[w_{n_{uu}}(u,t) - (1+4u^2)w_n(u,t)\right]\right]. \tag{33}$$

Putting $n = 0$ into Eq. (33), we get:

$$q^1: w_1(u,t) = \mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[w_{0_{uu}}(u,t) - (1 + 4u^2)w_0(u,t)\right]\right] = \mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[(2+4u^2)e^{\,u^2} - (1 + 4u^2)e^{\,u^2}\right]\right] = \mathbb{S}^{-1}\left[v^{r-1}e^{\,u^2}\right] = e^{\,u^2}\frac{t^{\,r}}{\Gamma(r+1)}.$$

Putting $n = 1$ into Eq. (33), we get:

$$q^2: w_2(u,t) = \mathbb{S}^{-1}\left[v^{r}\mathbb{S}\left[w_{1_{uu}}(u,t) - (1 + 4u^2)w_1(u,t)\right]\right] = e^{\,u^2}\frac{t^{\,2r}}{\Gamma(2r+1)}.$$

in the same way, we get:

$$q^3: w_3(u,t) = e^{\,u^2}\frac{t^{\,3r}}{\Gamma(3r+1)},$$

$$q^n: w_n(u,t) = e^{\,u^2}\frac{t^{\,nr}}{\Gamma(nr+1)}.$$

Therefore, the solution of Eq. (29) is given by:

$$\begin{aligned} w(u,t) &= w_0(u,t) + w_1(u,t) + w_2(u,t) + \cdots + w_n(u,t) + \cdots \\ &= e^{\,u^2} + e^{\,u^2}\frac{t^{\,r}}{\Gamma(r+1)} + e^{\,u^2}\frac{t^{\,2r}}{\Gamma(2r+1)} + e^{\,u^2}\frac{t^{\,3r}}{\Gamma(3r+1)} + \cdots + e^{\,u^2}\frac{t^{\,nr}}{\Gamma(nr+1)} + \cdots \\ &= e^{\,u^2}\left(1 + \frac{t^{\,r}}{\Gamma(r+1)} + \frac{t^{\,2r}}{\Gamma(2r+1)} + \frac{t^{\,3r}}{\Gamma(3r+1)} + \cdots + \frac{t^{\,nr}}{\Gamma(nr+1)} + \cdots\right) \\ &= e^{\,u^2}\sum_{z=0}^{\infty}\frac{t^{\,zr}}{\Gamma(zr+1)}. \end{aligned} \tag{34}$$

At $r = 1$, the exact solution is $w(u,t) = e^{\,u^2+t}$.

In Figure 4, we sketch the exact solution of Application 5.2, that is $w(u,t) = e^{\,u^2+t}$.

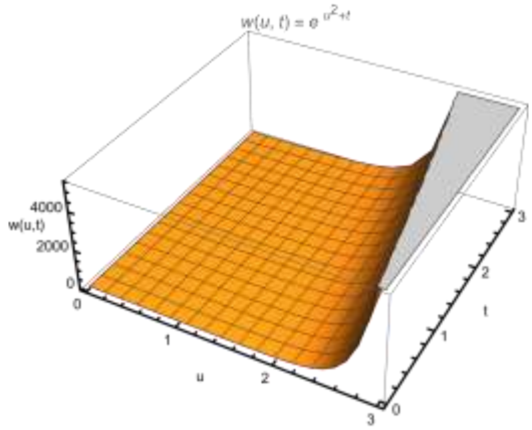
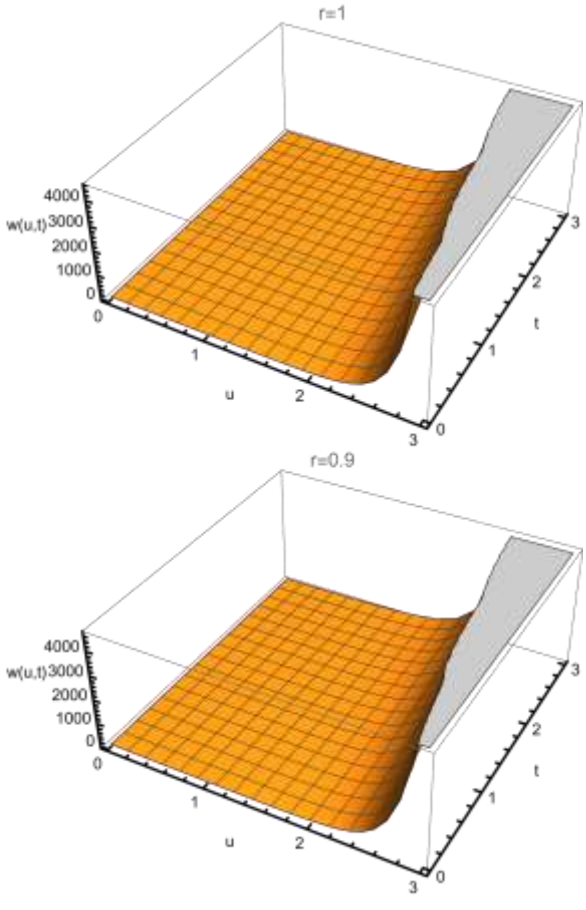


Fig. 4: The exact solution convection–reaction–diffusion Eq. (29)

In Figure 5, we plot the approximate solution in Eq. (34) with different values of $r = 1, 0.9, 0.8, 0.6$.



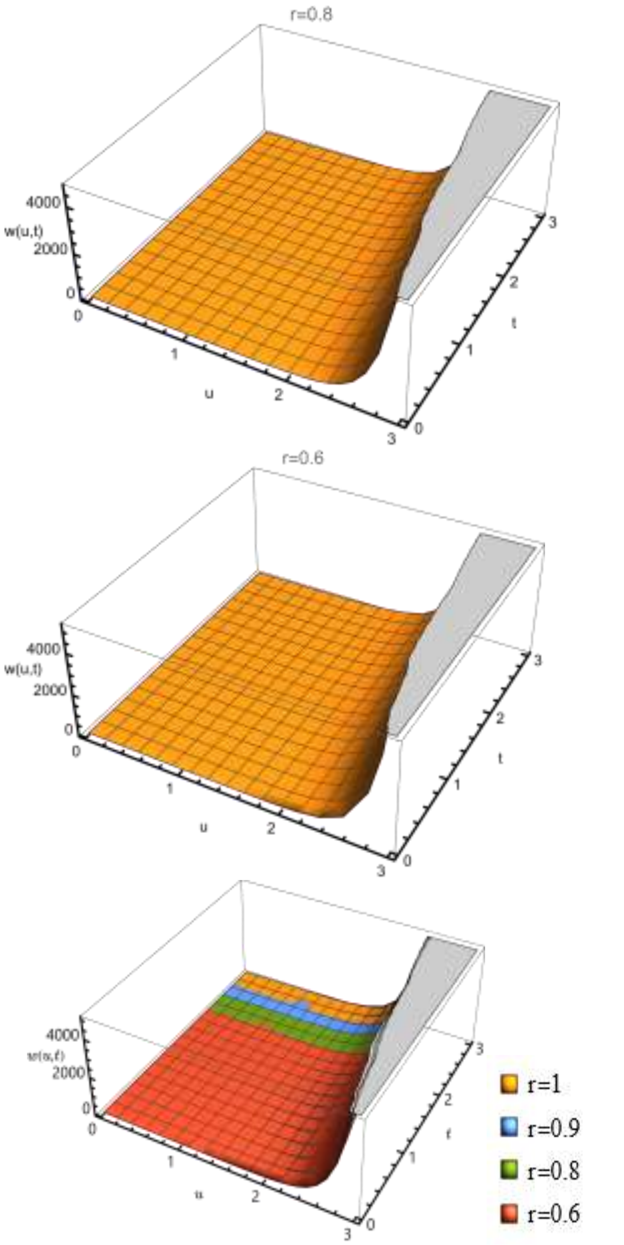


Fig. 5: The approximate solution in Eq. (34) with different values of r

Application 5.3. Consider the following convection–reaction–diffusion equation

$$\frac{\partial^r w(u,t)}{\partial t^r} = \frac{\partial^2 w(u,t)}{\partial u^2} + w(u,t) + w(u,t)\frac{\partial w(u,t)}{\partial u} - w^2(u,t), \tag{35}$$

subject to the conditions

$$w(u,0) = 1 + e^{\,u}. \tag{36}$$

Applying Sawi transform homotopy perturbation method for Eq. (35), we obtain

$$\begin{aligned}&\sum_{z=0}^{\infty} q^z W_z(u,v) \\ &= \frac{1}{v} w(u,0) \\ &+ q v^r \left(\mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^z w_z(u,t) \right)_{uu} \right.\right. \\ &\left.\left. + \sum_{z=0}^{\infty} q^z w_z(u,t) \right] \right) \\ &+ q v^r \left(\mathbb{S}\left[\sum_{z=0}^{\infty} q^z \mathcal{H}_z(u,t) \right] \right),\end{aligned} \tag{37}$$

Taking inverse Sawi transform to Eq. (37) ,we get

$$\begin{aligned}&\sum_{z=0}^{\infty} q^z w_z(u,t) \\ &= \mathbb{S}^{-1}\left[\frac{1}{v} w(u,0) \right] \\ &+ q\, \mathbb{S}^{-1}\left[v^r \mathbb{S}\left[\left(\sum_{z=0}^{\infty} q^z w_z(u,t) \right)_{uu} \right.\right. \\ &\left.\left. + \sum_{z=0}^{\infty} q^z w_z(u,t) \right] \right] \\ &+ q\, \mathbb{S}^{-1}\left[v^r \mathbb{S}\left[\sum_{z=0}^{\infty} q^z \mathcal{H}_z(u,t) \right] \right].\end{aligned}$$

Note that, the first few terms of $\mathcal{H}_z$ in this case is given by:

$$\begin{aligned}\mathcal{H}_0 &= w_0 w_{0_u} - (w_0)^2, \\ \mathcal{H}_1 &= w_0 w_{1_u} + w_1 w_{0_u} - 2 w_0 w_1, \\ \mathcal{H}_2 &= w_0 w_{2_u} + w_1 w_{1_u} + w_2 w_{0_u} - 2 w_0 w_2 \\ &\quad - (w_2)^2, \\ &\ \ \vdots\end{aligned} \tag{38}$$

The function of the Caputo derivative result is achieved by calculating the powers of q:

$$\begin{aligned} q^0 : w_0(u,t) &= \mathbb{S}^{-1}\left[\frac{1}{v} w(u,0) \right] \\ &= \mathbb{S}^{-1}\left[\frac{1}{v} (1 + e^{\,u}) \right] \\ &= 1 + e^{\,u},\end{aligned} \tag{39}$$

$$\begin{aligned} q^{n+1} : w_{n+1}(u,t) &\\ = \mathbb{S}^{-1}\Big[v^r \mathbb{S}\big[w_{n_{uu}}(u,t) &+ w_{n_u}(u,t) + \mathcal{H}_n \big] \Big].\end{aligned} \tag{40}$$

Putting $n = 0$ into Eq.(40) , we get

$$q^1 : w_1(u,t) = \mathbb{S}^{-1}\Big[v^r \mathbb{S}\big[w_{0_{uu}}(u,t) + w_{0_u}(u,t) + \mathcal{H}_0 \big] \Big]$$

$$= \mathbb{S}^{-1}\big[v^{r}\mathbb{S}[e^{u} + (1 + e^{u}) + (1 + e^{u})e^{u} - (1 + e^{u})^2]\big].$$

$$= \mathbb{S}^{-1}[v^{r-1}e^{u}] = e^{u}\frac{t^{r}}{\Gamma(r+1)}.$$

in the same way, we get:

$$q^2: w_2(u,t) = e^{u}\frac{t^{2r}}{\Gamma(2r+1)},$$

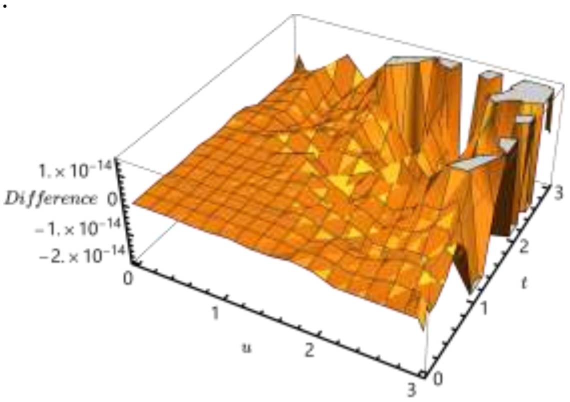
$$q^n: w_n(u,t) = e^{u}\frac{t^{nr}}{\Gamma(nr+1)}.$$

Therefore, the solution of Eq. (35) is given by:

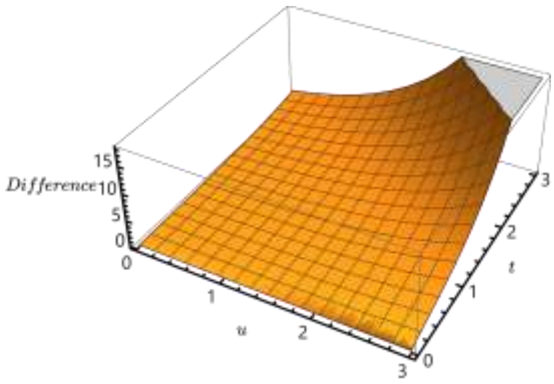
$$\begin{aligned} w(u,t) &= w_0(u,t) + w_1(u,t) + w_2(u,t) + \cdots + w_n(u,t) + \cdots \\ &= 1 + e^{u} + e^{u}\frac{t^{r}}{\Gamma(r+1)} + e^{u}\frac{t^{2r}}{\Gamma(2r+1)} + e^{u}\frac{t^{3r}}{\Gamma(3r+1)} + \cdots + e^{u}\frac{t^{nr}}{\Gamma(nr+1)} + \cdots \\ &= 1 + e^{u}\left(1 + \frac{t^{r}}{\Gamma(r+1)} + \frac{t^{2r}}{\Gamma(2r+1)} + \frac{t^{3r}}{\Gamma(3r+1)} + \cdots + \frac{t^{nr}}{\Gamma(nr+1)} + \cdots\right) = 1 + e^{u}\sum_{z=0}^{\infty}\frac{t^{zr}}{\Gamma(zr+1)}, \end{aligned}$$

at $r = 1$, then the exact solution is $w(u,t) = 1 + e^{u+t}$.

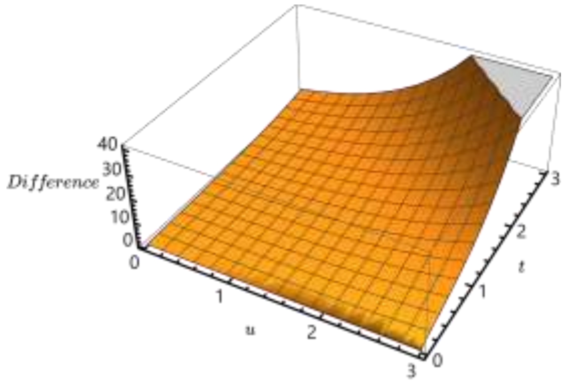
Here are the 3D plots Figure 6 showing the absolute error between the solution of Eq. (35) and the exact solution e^{u+t} for each specified value of r:



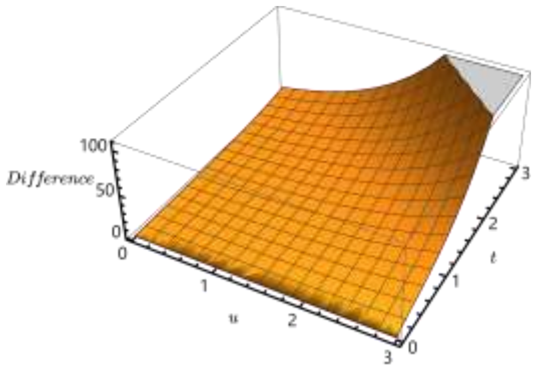
Difference for r=1



Difference for r=0.9



Difference for r=0.8



Difference for r=0.6

Fig. 6: The absolute error of Application 5.3

6 Conclusion

This paper provided a thorough analysis of the Sawi transform homotopy method perturbation, a novel and effective technique for solving fractional Caputo PDEs. In terms of computational efficiency and solution precision, the STHPM has demonstrated significant gains over traditional techniques like the Variational iteration method and the Adomian decomposition method. Through the integration of homotopy techniques and the Sawi transform, we effectively resolved a number of the most formidable challenges associated with the solution of nonlinear PDEs.

One benefit of this undertaking is that it is possible to decrease the processing time while maintaining the precision of the solutions. For this reason, researchers and professionals who need to solve fractional Caputo PDEs rapidly and precisely will find the STHPM to be an extremely useful instrument. There are numerous potential paths for further investigation in the future. We feel that the STHPM has the potential to transform how fractional Caputo PDEs are treated and solved, and we are hopeful about its future contributions to academia and industry.

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The authors have no conflicts of interest to declare.