# Conformal Self-Mappings of the Complex Plane with Arbitrary Number of Fixed Points

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*Abstract:* There are known conformal self-mappings of the fundamental domains of analytic functions via Möbius transformations. When two adjacent fundamental domains have a straight line or an arc of a circle as a common boundary, the Schwarz symmetry principle can be applied for one of those mappings and what we obtain is a conformal self-mapping of the union of those domains in which each one of the domains is mapped onto itself. Repeating this operation until the whole plane is exhausted, we obtain a conformal self-mapping of the complex plane in which every fundamental domain is conformally mapped onto itself. We prove in this paper that this is true for any analytic function. Since the self-mappings of fundamental domains have each one at least one fixed point, ultimately, for the self-mapping of the complex plane, we obtain at least as many fixed points as is the number of fundamental domains. When dealing with a rational function, this number is finite, otherwise we obtain infinitely many fixed points. Computer experimentation allows the illustration of these concepts for most of the familiar classes of analytic functions. There are known applications of the Möbius transformations in physics via the Lorentz group. Relating those application to the present work may contribute to the advancement of the knowledge in that field.

*Key-Words:* Conformal mappings, Fundamental domains, Möbius transformations, Dirichlet functions, Computer experimentation, Steiner net

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## 1 Introduction

Let $f(z)$ be a holomorphic function in $\overline{\mathbb{C}}$ with the exception of isolated singular points, which can be poles or essential singular points. It is known [1] that

$$\overline{\mathbb{C}} = \cup_{k=1}^{n\leq\infty}\overline{\Omega}_k,$$

where $\Omega_k$ are open connected sets, $\Omega_k \cap \Omega_j = \varnothing$, when $j \neq k$ and every $\Omega_k$ is conformally (hence bijectively) mapped by $f$ onto $\overline{\mathbb{C}} \setminus L_k$, where $L_k$ is a *slit*, or *cut*, i.e., a Jordan arc or a Jordan infinite curve. We will treat slits mostly as point sets. If $E \subset \overline{\mathbb{C}}$ is any point set we will denote by $f(E)$ the *image of $E$ by $f$*, i.e.,

$$f(E) = \{f(z)|z \in E\}$$

and by $f^{-1}(E)$ the *pre-image of $E$ by $f$*, i.e.,

$$f^{-1}(E) = \{z|f(z) \in E\}.$$

This convention cannot produce any confusion.

A slit exhibits two distinct *edges*, [2], and a point of the slit can be on one edge or the other. The function $f(z)$ is defined on every $\overline{\Omega}_k$, except for the essential singular points and it maps the boundary $\partial\Omega_k$ of $\Omega_k$ onto the slit $L_k$. However, the inverse function $f_{|\Omega_k}^{-1}$, which exists for every $k$ in view of bijectiveness of $f_{|\Omega_k}$ fails to have a continuous extension to $L_k$ since for sequences of points tending to the same point on $L_k$ from the sides of different edges the function has different limits. Yet, $f_{|\Omega_k}^{-1}$ can be extended to the two edges of $L_k$ and it maps those edges onto $\partial\Omega_k$. This fact is granted by the Riemann-Caratheodory Theorem of boundary correspondence in conformal mapping, [3] and [4]. Ahlfors has called the domains $\Omega_k$ fundamental regions of the function $f$ (see, e.g., [2]). They prove to be useful in revealing global mapping properties of analytic functions and in particular in the theory of distribution of zeroes of Dirichlet functions.

We have shown in [5] that to every Möbius transformation $M$ a conformal self-mapping of every such a domain $\Omega$ can be associated. Suppose that $\Omega$ is conformally mapped by $f$ onto the whole complex plane with a slit $L$. The function $M$ moves the slit $L$ into a slit $L'$, which is the image by $f$ of a slit $L_M$ of $\Omega$. On the other hand, $M^{-1}$ moves $L$ into a slit $L''$, which is the image by $f$ of another slit $L_{M^{-1}}$ of $\Omega$. We have proved in [5]:

**Proposition 1**. The function

$$\chi_M = f_{|\Omega}^{-1} \circ M \circ f$$

is a conformal mapping of $\Omega\setminus L_{M^{-1}}$ onto $\Omega\setminus L_M$ in which the boundary $\partial\Omega$ of $\Omega$ is carried into $L_M$ and $L_{M^{-1}}$ is carried into $\partial\Omega$.

This means that $\chi_M$ can be extended by continuity to $\partial\Omega$ and the image of $\partial\Omega$ by the extended function is $L_M$.

Also $\chi_M$ can be extended by continuity to the sides of $L_{M^{-1}}$ and the image by the extended function of $L_{M^{-1}}$ is $\partial\Omega$.

**Proposition 2**. The fixed points of the mapping $\chi_M$ and those of the Möbius transformation $M$ are related in the following way: $z = f(s)$ is a fixed point of $M$ if and only if $s$ is a fixed point of $\chi_M$.

It does not mean that the two functions have necessarily the same number of fixed points. Indeed, if $z \in L$ then there can be two points $s$ and $s'$ such that $f(s) = f(s') = z$, hence to a fixed point of $M$ it may correspond two fixed points of $\chi_M$. An example has been given in [5], where such a situation effectively occurs.

The purpose of this paper is to prove that for any fundamental domain $\Omega$ of $f$ and for any Möbius transformation $M$, the mapping $\chi_M$ can be extended to the whole plane in such a way that every fundamental domain of $f$ is mapped by the extended function onto itself. Then it can be concluded that the respective mapping has at least as many fixed points as the number of the fundamental domains of $f$.

## 2 The Case of Rational Functions

Global mapping properties of rational functions related to their fundamental domains have been studied in [6]. For the rational functions of the second degree we have shown that:

**Proposition 3**. Every rational function $R$ of the second degree can be written under the form $R = M_2 \circ T \circ M_1$, where $M_1$ and $M_2$ are Möbius transformations and $T(\zeta) = \zeta^2$.

The function $\zeta = M_1(z)$ transforms the $z$-plane into the $\zeta$-plane in such a way that a line or a circle $L$ from the $z$-plane goes to the real axis into the $\zeta$-plane. The function $\eta = \zeta^2$ transforms each one of the upper and lower half-planes from the $\zeta$-plane into the whole $\eta$-plane with the slit $L'$ which is the positive real half-axis. Finally, the function $w = M_2(\eta)$ transforms the $\eta$-plane with the slit $L'$ into the $w$-plane with a slit which is an arc of a circle or a half-line $L''$. Summing up, the rational function $w = R(z)$ transforms each one of the two domains determined by $L$ into the whole $w$-plane with the slit $L''$ which is an arc of a circle or a half-line.

**Example**. For the function

$$R(z) = \frac{(18-i)z^2 - 6(2-i)z + 2 - 9i}{(2+9i)z^2 - 6(2+i)z + 18 + i}, \quad (1)$$

we have

$$M_1(z) = \frac{3z-1}{3-z}, \qquad M_2(\eta) = \frac{2\eta - i}{2 + i\eta} \quad (2)$$

and $L$ is the real axis. We have $M_2(0) = -i/2$, $M_2(1) = 3/5 - 4i/5$ and $M_2(\infty) = -2i$. Thus, $w = R(z)$ conformally maps each one of the upper and the lower half-planes onto the whole $w$-plane with a slit alongside the arc of the circle determined by the points $-i/2$, $-2i$ and $3/5 - 4i/5$.

The pre-image by $R(z)$ of the Steiner net, [5], illustrates a conformal self-mapping of the complex plane in which the upper and the lower half-planes are each mapped onto themselves.

To find this pre-image we need an expression of the multi-valued function $R^{-1}(w)$, which is $M_1^{-1} \circ T^{-1} \circ M_2^{-1}$. We have

$$M_1^{-1}(\zeta) = \frac{3\zeta + 1}{\zeta + 3},$$
$$M_2^{-1}(w) = \frac{2w + i}{2 - iw},$$
$$T^{-1}(\eta) = \sqrt{\eta},$$

thus

$$R^{-1}(w) = \frac{3\sqrt{\frac{2w+i}{2-iw}} + 1}{\sqrt{\frac{2w+i}{2-iw}} + 3}.$$

The two branches of $T^{-1}(\eta)$ provide conformal mappings of the complex plane onto itself, Fig. 1, in which the upper, respectively lower half-plane are mapped each one by the respective branch onto itself. $\square$

Due to Proposition 1, such a mapping can be obtained for any second-degree rational function.

For higher degree rational functions different techniques are needed.

However, for selected rational functions of higher-degree a similar technique can be applied. For example, for the function

$$R(z) = \frac{(54+i)z^3 - 9(6+i)z^2 + 9(2+3i)z - (2+27i)}{(-2+27i)z^3 + 9(2-3i)z^2 - 9(6-i)z + 54 - i},$$

we have three branches of the multi-valued function $R^{-1}(w)$, namely,

$$R_k^{-1}(w) = \frac{3\omega_k \sqrt[3]{\frac{2w+i}{2-iw}} + 1}{\omega_k \sqrt[3]{\frac{2w+i}{2-iw}} + 3},$$

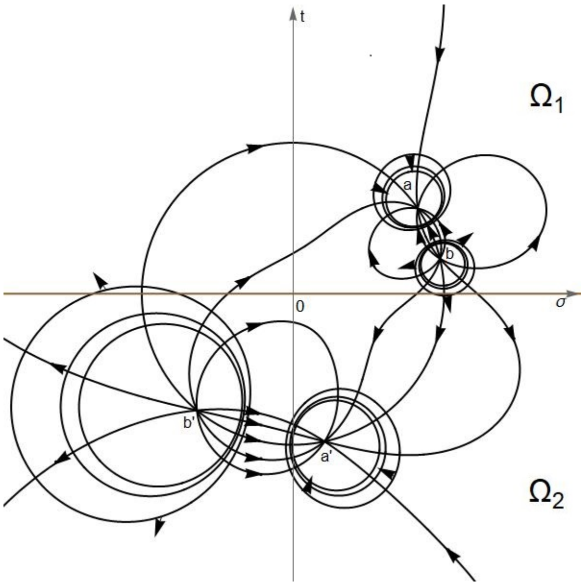


Fig. 1: The conformal self-mapping of the complex plane induced by a second-degree rational function.

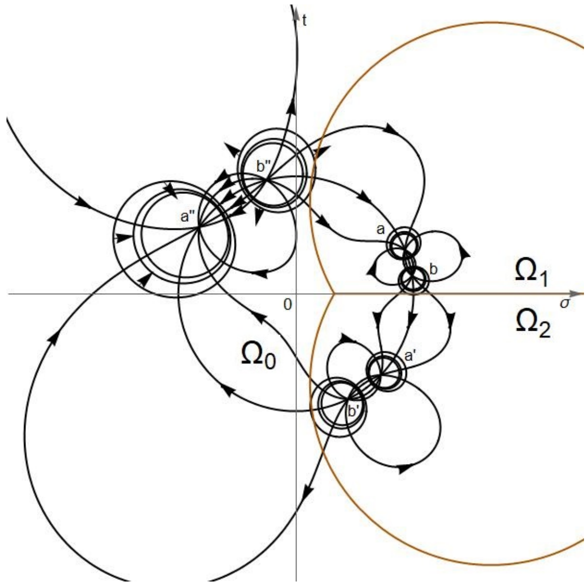


Fig. 2: The conformal self-mapping of the complex plane induced by a third-degree rational function.

$k = 0, 1, 2$, where $\omega_k$ are the roots of order three of the unity.

The function $T(\zeta) = \zeta^3$ maps conformally the three sectors determined by the rays

$$\zeta(t) = \omega_k t, \quad t \geq 0,$$

where $\omega_k$, $k = 0, 1, 2$, are the roots of order three of the unity, onto the whole $\eta$-plane with a slit alongside the positive real half-axis. The Möbius transformation $w = M_2(\eta)$ carries the real half-axis onto the slit $L''$ in the $w$-plane. On the other hand,

$$z = M_1^{-1}(\zeta) = \frac{3\zeta + 1}{\zeta + 3}$$

carries the three rays into the curves

$$z_k(t) = \frac{3\omega_k t + 1}{\omega_k t + 3}, \quad t \geq 0.$$

These curves are described in the next example, as well as the domains bounded by them, which are fundamental domains for the function $R(z)$. The conformal mapping of each one of these domains onto itself is described by the $\Omega$-Steiner nets shown in Fig. 2 below.

The case of Blaschke products is special, since the fundamental domains are circular and once we have the mapping for one fundamental domain, the others can be obtained by symmetries with respect to circles.

A second-degree rational function which is a Blaschke product is a function of the form:

$$B(z) = \frac{z - a}{1 - \overline{a}z} \frac{z - b}{1 - \overline{b}z} e^{i\theta}, \tag{3}$$

where $0 \leq |a| < 1$, $0 \leq |b| < 1$, $\theta \in \mathbb{R}$.

For illustration, let us take $a = 1/3$ and $b = -1/3$, $\theta = 0$. Then,

$$B(z) = \frac{9z^2 - 1}{9 - z^2}, \tag{4}$$

with the fundamental domains the left and the right-hand half-planes. The Möbius transformation

$$M(z) = \frac{2z - 1}{2 - z}$$

with the fixed points $-1$ and $1$ is illustrated by the Steiner net in [5]. It induces a conformal mapping of the complex plane onto itself in which the left and the right hand half planes are mapped each one onto itself. This transformation has four fixed points as seen in the Fig. 3 below. Indeed, for any $y \in \mathbb{R}$, we have

$$B(iy) = B(-iy) = -\frac{9y^2 + 1}{9 + y^2},$$

which shows that $B(z)$ maps the imaginary axis onto the interval $(-9, -\frac{1}{9})$, symmetric points with respect to the origin having the same image. Hence, each one of the left and the right half-planes is mapped conformally onto the whole complex plane with a slit alongside the real axis from $-9$ to $-\frac{1}{9}$.

We dealt in [6] and [7] with the Blaschke products of the form

$$B_a(z) = \left(\frac{\overline{a}}{|a|} \frac{z - a}{1 - \overline{a}z}\right)^n, \quad a = re^{i\alpha}, \quad 0 < r < 1, \alpha \in \mathbb{R}, n = 2, 3, .... \tag{5}$$

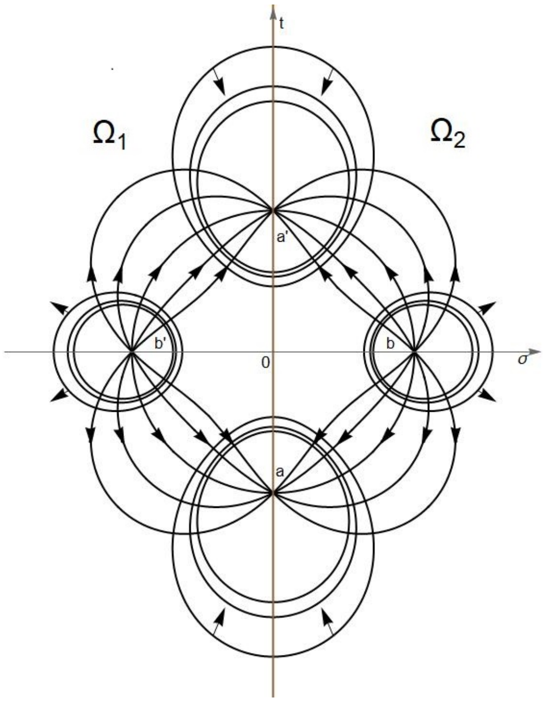


Fig. 3: The conformal self-mapping of the complex plane induced by a Blaschke product of the second-degree.

An elementary computation shows that the equation $B_a(z) = \lambda^n$, where $0 \leq \lambda \leq 1$ has the solutions

$$z_k(\lambda) = \frac{\omega_k \lambda + r}{\omega_k \lambda r + 1} e^{i\alpha}, \quad k = 0, 1, ..., n-1, \quad (6)$$

where $\omega_k$ are the roots of order $n$ of unity. The formula (6) is that of a Möbius transformation in $\lambda$ and therefore when $\lambda$ varies from 0 to 1, the point $z_k(\lambda)$ describes an arc of a circle $\gamma_k$ with the end points in

$$z_k(0) = a \quad \text{and} \quad z_k(1) = \frac{\omega_k + r}{\omega_k r + 1} e^{i\alpha}$$

on the unit circle. Two consecutive arcs $\gamma_k$ and $\gamma_{k+1}$ (where $\gamma_n = \gamma_0$) together with the arc of the unit circle between them bound a domain which is conformally mapped by $B_a(z)$ onto the unit disc with a slit alongside the real axis from 0 to 1. By the symmetry principle, the symmetric of this domain with respect to the unit circle is conformally mapped by $B_a(z)$ onto the exterior of the unit disc with a slit alongside the real axis from 1 to infinity. The conclusion is that $B_a(z)$ partitions the complex plane into $n$ point sets, the interior of which are fundamental domains of $B_a(z)$ and each one of them is conformally mapped by $B_a(z)$ onto the whole complex plane with a slit alongside the positive real half axis.

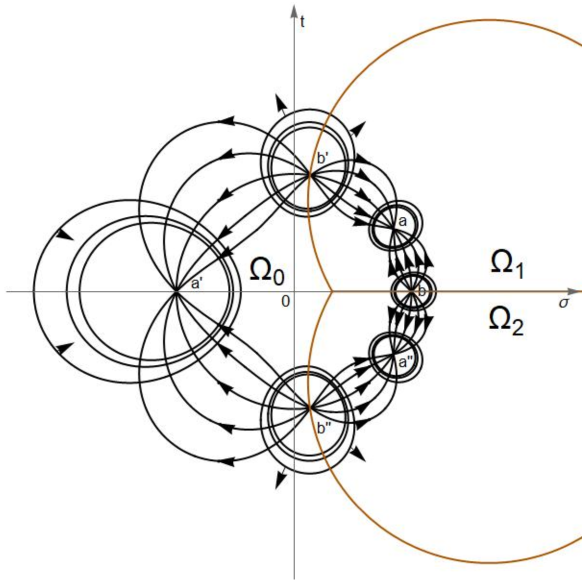


Fig. 4: The conformal self-mapping of the complex plane induced by a Blaschke product of the third degree.

Let us take $a = 1/3$ and $n = 3$. By using the same procedure as previously, we obtain, Fig. 4, a conformal self-mapping of the complex plane with 6 fixed points.

The boundaries of the fundamental domains of the Blaschke product

$$B(z) = \left(\frac{3z - 1}{3 - z}\right)^3,$$

can be obtained by solving for $z$ the equation

$$B(z) = t^3, \quad t \geq 0.$$

The solutions are

$$z_k(t) = \frac{1 + 3t\omega_k}{3 + t\omega_k}, \quad k = 0, 1, 2, \quad 0 \leq t \leq \infty,$$

which represent three arcs of circle $\gamma_k$, $k = 0, 1, 2$.

Since $\omega_0 = 1$, we have

$$z_0(t) = \frac{1 + 3t}{3 + t},$$

which is, for $t \geq 0$, the interval $\gamma_0 = (1/3, 3)$ of the real axis. It meets the unit circle for $t = 1$ at the point $z = 1$.

For $k = 1, 2$, the two arcs meet the unit circle when $|z_k(t)| = 1$, i.e., when

$$(1 + 3t\omega_k)(1 + 3t\overline{\omega}_k) = (3 + t\omega_k)(3 + t\overline{\omega}_k).$$

This equation is equivalent to $t^2 = 1$. Since $t \geq 0$, the right solution is $t = 1$. We have

$$z_k(1) = \frac{1 + 3\omega_k}{3 + \omega_k},$$

thus $\gamma_1$ and $\gamma_2$ are arcs of circle with the ends in $1/3$ and 3 and passing through $z_1(1)$ and respectively $z_2(1)$.

Thus, the fundamental domains of $B(z)$ are the domains: $\Omega_1$ bounded by $\gamma_0$ and $\gamma_1$, $\Omega_2$ bounded by $\gamma_0$ and $\gamma_2$, and the unbounded domain $\Omega_0$ bounded by $\gamma_1$ and $\gamma_2$.

Since $B(z_k(1)) = 1$ and $z = 1$ is a fixed point of the Möbius transformation

$$M(z) = \frac{2z - 1}{2 - z},$$

by the Proposition 2 the points $z_k(1)$ are fixed points of the transformations

$$\chi_k(z) = (B^{-1}_{|\Omega_k} \circ M \circ B)(z), \quad k = 0, 1, 2,$$

as shown in Fig. 4.

## 3 The Case of an Arbitrary Analytic Function

Let $f(z)$ be an arbitrary analytic function in $\overline{\mathbb{C}}$ with the exception of isolated singular points and let $\Omega_1$ and $\Omega_2$ be adjacent fundamental domains of $f$ which are conformally mapped by $f$ onto the complex plane with the same slit $L$. Any Möbius transformation $M(z)$ defines conformal mappings

$$\chi_k(s) = f^{-1}_{|\Omega_k} \circ M \circ f(s)$$

of $\Omega_k \backslash L'_k$ onto $\Omega_k \backslash L_k$ , $k = 1, 2$, where $L_k$ and $L'_k$ are slits in $\Omega_k$. Let $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ and let us extend by continuity every $f_{|\Omega_k}(s)$ to $\Gamma$ keeping the same notations for the extended functions.

**Theorem 1**. For every $s \in \Gamma$ we have

$$f_{|\Omega_1}(s) = f_{|\Omega_2}(s) = f(s),$$

hence

$$f_{|\Omega_1}(\Gamma) = f_{|\Omega_2}(\Gamma) = f(\Gamma).$$

Moreover, $f^{-1}_{|\Omega_1}(w)$ and $f^{-1}_{|\Omega_2}(w)$ exist for every $w \in f(\Gamma)$ and they are equal.

Then, $\chi_k(s)$ are extended to $\Gamma$, such that $\chi_1(s) = \chi_2(s)$ for every $s \in \Gamma$. The extended function to $(\Omega_1 \cup \Omega_2 \cup \Gamma) \backslash (L'_1 \cup L'_2)$ is a conformal mapping $\chi_{1,2}$ of this domain onto $(\Omega_1 \cup \Omega_2 \cup \Gamma) \backslash (L_1 \cup L_2)$ such that its restriction to every $\Omega_k \backslash L'_k$, $k = 1, 2$ is a conformal mapping of $\Omega_k \backslash L'_k$ onto $\Omega_k \backslash L_k$.

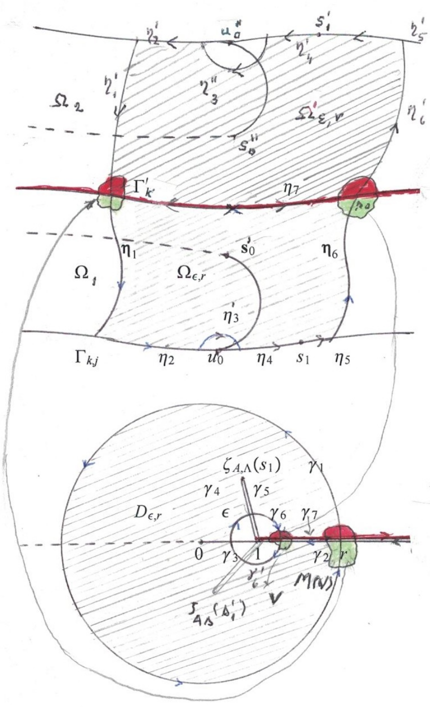

Fig. 5: Illustration of Theorem 1 for an arbitrary Dirichlet function $\zeta_{A,\Lambda}(s)$ and a real Möbius transformation $M$.

*Proof:* By the boundary correspondence theorem in a conformal mapping, for $k = 1, 2$, the functions $f_{|\Omega_k}$ can be extended by continuity to the boundaries $\partial\Omega_k$ of $\Omega_k$ and both extensions map $\partial\Omega_1 \cap \partial\Omega_2$ onto the same edge of $L$. Indeed, the function $f$ is locally injective at every point $s_0$ where $f'(s_0) \neq 0$ and $s_0$ is not a multiple pole or an essential singular point. In particular, this happens at every point $s_0 \in \Gamma$. Therefore, in a small neighborhood $V$ of $f(s_0)$ the function $f^{-1}(z)$ exists and it coincides with $f^{-1}_{|\Omega_k}$ where both are defined.

Consequently, $f^{-1}_{|\Omega_k}$ can be extended to $f(\Gamma)$ where they are equal, hence they are extensions of each other and therefore $\chi_k(s)$ are extensions of each other. The function $\chi_{1,2}(s)$ which coincides with $\chi_k(s)$ in $(\Omega_k \cup \Gamma) \backslash L'_k$, $k = 1, 2$ maps conformally $(\Omega_1 \cup \Omega_2 \cup \Gamma) \backslash (L'_1 \cup L'_2)$ onto $(\Omega_1 \cup \Omega_2 \cup \Gamma) \backslash (L_1 \cup L_2)$ in such a way that $\Omega_k \backslash L'_k$ is conformally mapped onto $\Omega_k \backslash L_k$ for $k = 1, 2$.

If it happens that $L'_k \subset \partial\Omega_k$ (as in [6]) then $\chi_k(s)$ is defined in $\Omega_k$. $\square$

In Fig. 5, two adjacent fundamental domains $\Omega_1$ and $\Omega_2$ with the common boundary $\Gamma'_k$ of an arbitrary Dirichlet function $\zeta_{A,\Lambda}(s)$ are exhibited. A big circle of radius $r$ centered at the origin is the image by

$\zeta_{A,\Lambda}(s)$ of an infinite curve $\eta_1$, while a small circle of radius $\epsilon$ centered at the point $z = 1$ is the image of another infinite curve $\eta_6$ and some arcs $\eta_3'$ and $\eta_3''$ around the points $u_0'$ and $u_0''$ for which we have $\zeta_{A,\Lambda}(u_0') = \zeta_{A,\Lambda}(u_0'') = 1$. The domain $D_{\epsilon,r}$ bounded by the two circles is the conformal image of the two domains $\Omega_{\epsilon,r}$ and $\Omega_{\epsilon,r}'$ included in the fundamental domains $\Omega_1$ and respectively $\Omega_2$ of the Dirichlet function $\zeta_{A,\Lambda}(s)$. When $\epsilon \to 0$ and $r \to +\infty$ the two domains become $\Omega_1$ and respectively $\Omega_2$, while $D_{\epsilon,r}$ becomes the whole complex plane. We have

$$\begin{aligned} &\zeta_{A,\Lambda}(s_0') = \zeta_{A,\Lambda}(s_0'') = 0, \\ &\zeta_{A,\Lambda}(u_0') = \zeta_{A,\Lambda}(u_0'') = 1, \\ &\zeta_{A,\Lambda}'(s_1) = \zeta_{A,\Lambda}'(s_1') = 0, \\ &\lim_{\sigma \to +\infty} \zeta_{A,\Lambda}(\sigma + it) = 1, \end{aligned}$$

where $\sigma + it$ belongs to $\Gamma_k'$ or to $\eta_5$ or to $\eta_5'$. The four points $s_0$, $\zeta_{A,\Lambda}(s_0)$, $M(\zeta_{A,\Lambda}(s_0))$ and $\chi_1(s_0) = \chi_2(s_0)$, as well as their neighborhoods are portrayed. This figure illustrates not only the affirmations of Theorem 1, but also the way a Dirichlet function is conformally mapping its fundamental domains onto the complex plane with some slits. Every slit is along the interval $(1, +\infty)$ of the real axis and along an interval from $z = 1$ to $z = \zeta_{A,\Lambda}(s_k)$ where $\zeta_{A,\Lambda}'(s_k) = 0$.

Regarding the sensitivity of the method, it is obvious that any change in a fundamental domain will trigger changes in all fundamental domains, perturbing the whole landscape of the Fig. 5. However, the branch points will remain the same.

We need to point out that the famous Riemann Zeta function is the particular Dirichlet function in which $\lambda_n = \log n$ and $a_n = 1$ for every n. It is known that the Riemann Zeta function has important applications in physics. The results above may draw a new light on this application.

**Theorem 2**. (**The Main Theorem**) To every function $f$, which is analytic in $\overline{\mathbb{C}}$ with the exception of isolated singular points, and to every Möbius transformation $M$, a conformal mapping of the complex plane, with the singular points of $f$ and some cuts removed, can be associated such that every fundamental domain of $f$ with a cut removed is conformally mapped into itself.

*Proof:* It is known that $\overline{\mathbb{C}} = \cup_{k=1}^{n\leq\infty}\overline{\Omega}_k$ where $\Omega_k$ are fundamental domains of $f$. Suppose that $\Omega_1$ and $\Omega_2$ are adjacent and construct $\chi_{1,2}$ as in Theorem 1. Now, suppose that $\Omega_3$ is adjacent to $\Omega_1 \cup \Omega_2$ and let $\chi_{1,2,3}$ a conformal mapping of $(\Omega_1 \cup \Omega_2 \cup \Omega_3) \setminus (L_1' \cup L_2' \cup L_3')$ obtained in the same manner as $\chi_{1,2}$. It maps conformally each one of $\Omega_k \setminus L_k'$, $k = 1, 2, 3$ onto $\Omega_k \setminus L_k$ . We continue in this way up to $n$, if $n$ is finite, or indefinitely if $n$ is infinite. What we obtain is a conformal mapping of the complex plane with the singular points of $f$ and some cuts removed in which every fundamental domain of $f$ is conformally mapped into itself. If $L_k' \subset \partial\Omega_k$ there is no cut to be removed from $\Omega_k$. $\square$

We illustrate next this theorem for some of the most known classes of analytic functions.

# 4 Illustrations For Some Classes of Analytic Functions

The exponential function has infinitely many fundamental domains which are horizontal strips of width $2\pi$. We have shown in [5] how each one of these strips can be conformally mapped onto itself through the intermediate of a Möbius transformation. We will use throughout in what follows the Möbius transformation

$$M(z) = \frac{2z - 1}{2 - z}. \tag{7}$$

Let

$$\Omega_k = \{z = x + iy | 2k\pi < y < 2(k + 1)\pi\},$$

for $k \in \mathbb{Z}$. The $\Omega_0$-Steiner net in [5], portrays the conformal self-mapping $\chi_M$ of the strip $\Omega_0$ defined by

$$\chi_M(s) = Log(M(e^s)),$$

where $Log$ is the principal branch of the multivalued function logarithm. To obtain a conformal self-mapping of an arbitrary fundamental domain of $e^z$ we need to use the corresponding branch of the logarithm. Yet, this can be obtained by making consecutively symmetries of the $\Omega_0$-Steiner net with respect to the lines $z = x + 2k\pi i$. Fig. 6 below illustrates a conformal self-mapping of the complex plane in which every strip $\Omega_k$ is mapped onto itself. We notice that this conformal mapping has infinitely many fixed points, namely $z = k\pi i$. For $k$ even they are repelling fixed points and for $k$ odd they are attracting.

For the cosine function the fundamental domains are vertical strips

$$\Omega_k = \{z = x + iy | k\pi < x < (k + 1)\pi\}, \quad k \in \mathbb{Z}.$$

The $\Omega_0$-Steiner net is shown in [5].

If we take the symmetric of this net with respect to the lines $z = k\pi + iy$ we obtain a net covering the whole complex plane and portraying a conformal self-mapping of the complex plane with the fixed points $z = k\pi$, $k \in \mathbb{Z}$, in which every fundamental domain $\Omega_k$ of the cosine function is conformally mapped onto itself. Fig. 7 illustrates this mapping.

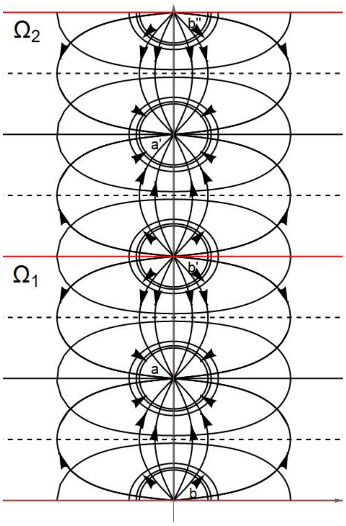


Fig. 6: The conformal self-mapping of the complex plane by the function $\chi(s)$ which coincides with $\chi_M(s)$ in every fundamental domain of the exponential function.

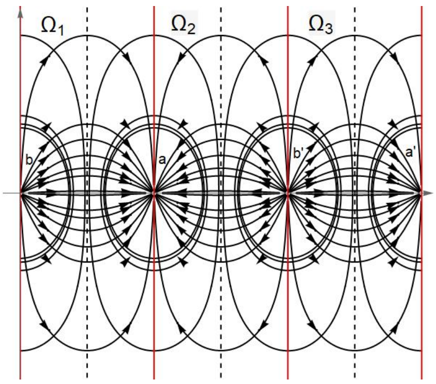


Fig. 7: The conformal self-mapping of the complex plane by the function $\chi(s)$ which coincides with $\chi_M(s)$ in every fundamental domain of the cosine function.

The Euler Gamma function is an extension to the whole complex plane of the arithmetic function

$$\Gamma(n) = (n-1)!.$$

For $\Re z > 1$ the extension is given by the formula

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt. \tag{8}$$

Integrating by parts, we find that $\Gamma(z)$ satisfies the functional equation $z\Gamma(z) = \Gamma(z+1)$, which allows its extension to the half-plane $\Re z \leq 1$. The function $\Gamma(z)$ has no zero and it has infinitely many poles which are $z = 0$ and $z = -n$, $n \in \mathbb{N}$. The pre-image of the real axis by $\Gamma(z)$ displays the fundamental domains of this function. These domains are conformally mapped by $\Gamma(z)$ onto the complex plane with slits alongside some intervals of the real axis. Fig. 8 below represents the pre-image by $\Gamma$ of the Steiner net of the Möbius transformation (7) which is a collection of $\Omega$-Steiner nets, each one illustrating a conformal self-mapping of the respective domain $\Omega$. These domains are bounded by the curves colored red for the pre-image of the positive real half-axis and blue for the pre-image of the negative real half-axis and they are mapped conformally by the function $w = \Gamma(z)$ onto the complex plane with slits alongside some intervals on the real axis. Each one of these domains contains one fixed point in the interior, which is repelling, and two attracting fixed points on the boundary. Both, attracting and repelling fixed points are at the intersection of the pre-images by $\Gamma(z)$ of the real axis in the $w$ and of the unit circle (colored green). These domains, put together, illustrate a conformal self-mapping of the complex plane in which every fundamental domain is mapped onto itself.

A Dirichlet function is obtained by performing analytic continuation to the whole complex plane of a Dirichlet series

$$\zeta_{A,\Lambda}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \tag{9}$$

where $\Lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers, $A = (a_n)$ is an arbitrary sequence of complex numbers and $s = \sigma + it$ is a complex variable. Any Dirichlet series can be normalized such that $a_1 = 1$ and $\lambda_1 = 0$ and we deal only with normalized Dirichlet series. For such a series we have

$$\lim_{\sigma\to+\infty} \zeta_{A,\Lambda}(\sigma + it) = 1, \tag{10}$$

uniformly with respect to $t$. When, for $a_n$ we have the values of a Dirichlet character $\chi$ and $\lambda_n = \log n$, [5],

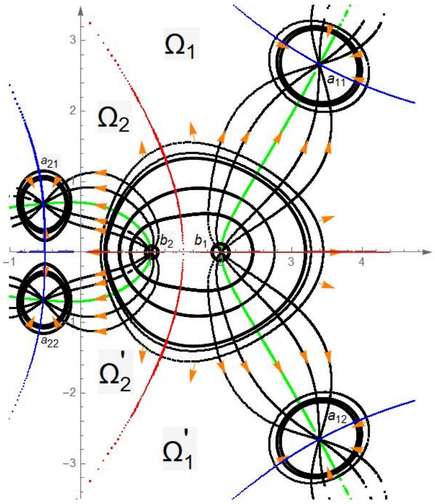


Fig. 8: The conformal self-mapping of the complex plane by the function $\chi$ which coincides with $\chi_M(s)$ in every fundamental domain of the function $\Gamma(z)$.

then the series (9) becomes

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{11}$$

and it is called Dirichlet L-series.

The fundamental domains of Dirichlet functions have been studied in [8]. They are bounded by the components of the pre-image by $\zeta_{A,\Lambda}(s)$ of the interval $(1, +\infty)$ of the real axis and by pre-images $\gamma_k$ of the segments from $z = 1$ to $z = \zeta_{A,\Lambda}(s_k)$, where $\zeta'_{A,\Lambda}(s_k) = 0$. For the Dirichlet L-functions, the location of these zeroes of the derivative has been studied in [9].

For any Dirichlet function (9) the fundamental domains are infinite strips which are mapped conformally by $\zeta_{A,\Lambda}(s)$ onto the whole complex plane with a slit alongside the interval $(1, +\infty)$ of the real axis followed by a slit alongside the segment from $z = 1$ to $z = \zeta_{A,\Lambda}(s_k)$. When $\zeta_{A,\Lambda}(s_k) = 0$, the point $s_k$ is a double zero of $\zeta_{A,\Lambda}(s)$ and the corresponding slit is the positive real half-axis. The existence of double zeroes of Dirichlet functions has been proved in [10] for linear combinations Dirichlet L-functions satisfying the same Riemann type of functional equation.

We have also shown in [11] that $\zeta_{A,\Lambda}(s)$ cannot have any zero of a higher order than two.

Since the interval $(1, \infty)$ of the real axis is included in the slit of every fundamental domain $\Omega_k$ of $\zeta_{A,\Lambda}(s)$, the pre-images by $\zeta_{A,\Lambda}(s)$ of the Apollonius circles around $z = 1$, of the Steiner net of the Möbius transformation (7), are orthogonal to the pre-image of that interval. Some of them may cut also under different angles the curve $\gamma_k$. The components of the pre-image of the orthogonal circles to the Apollonius circles, which pass through $-1$ and $1$, can be divided into two categories: the unbounded ones, which approach asymptotically $\partial\Omega_k$ when $\sigma \to +\infty$ and the bounded ones. Each one of them is mapped onto itself by the corresponding function $\chi_M(s)$. By Theorem 2, there is a conformal mapping $\chi(s)$ of the complex plane which coincides with $\chi_M(s)$ in every fundamental domain $\Omega_k$ of $\zeta_{A,\Lambda}(s)$. Fig. 9 illustrates this affirmation when $\zeta_{A,\Lambda}(s)$ is $L(7, 2, s)$.

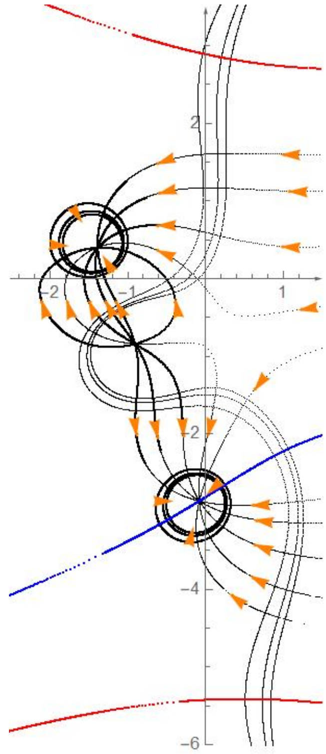

Fig. 9: The conformal self-mapping of the complex plane by the function $\chi$ which coincides with $\chi_M(s)$ in every fundamental domain of the Dirichlet L-function $L(7, 2, s)$.

Fig. 9 and Fig. 5 need to be seen together for a better understanding of the conformal self-mapping of the complex plane generated in Fig. 9 by the Dirichlet L-function $L(7, 2, s)$ and the Möbius transformation

$$M(z) = \frac{2z - 1}{2 - z}.$$

While Fig. 5 has been conceived by imagination, Fig. 9 is a computer-generated graphic. It shows that the fundamental domains of $L(7, 2, s)$ are indeed those illustrated in Fig. 5 and that their conformal self-mappings given by

$$\chi_M(s) = L^{-1}_{|\Omega} \circ M \circ L(7, 2, s)$$

in every fundamental domain, $\Omega$ of $L(7,2,s)$, produce together a conformal self-mapping of the complex plane with some slits. The pre-image by $L(7,2,s)$ of the orthogonal circles to the Apollonius circles of $M(s)$ shown in Fig. 9 prove that the $\eta$-curves from Fig. 5 are real. If we could add to Fig. 5 the fixed points of $\chi(s)$ and the $\Omega$-Apollonius circles, then the complete description of the conformal self-mapping of the complex plane with slits would result also in the case of the function illustrated by Fig. 5.

## 5 Conclusions

Up to now, the only known conformal self-mappings of the complex plane were the Möbius transformations. Moreover, it has been proved, [12], that these are the only possible such transformations. In the previous paper, [5], we dealt with conformal self-mappings of the fundamental domains of analytic functions. In this paper, we succeeded to extend this idea to the whole complex plane. We have proved that to any analytic function $f$ in $\overline{\mathbb{C}}$ with the exception of isolated singular points and to any Möbius transformation, a conformal self-mapping of the complex plane with some slits can be associated, such that it maps conformally every fundamental domain of $f$ onto itself. Computer experimentation has been used to illustrate this result for the most familiar classes of analytic functions.

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*References:*

[1] Andreian-Cazacu, C., and Ghisa, D., Global Mapping Properties of Analytic Functions, *Progress in Analysis and Applications, Proceedings of the 7th ISAAC Congress,* World Scientific Publishing Co. 2010, pp. 3-12.

[2] Ahlfors, L. V., *Complex Analysis*, International Series in Pure and Applied Mathematics, 1979.

[3] Nehari, Z., *Conformal Mappings*, International Series in pure and Applied Mathematics, 1951.

[4] Graham, I., and Kohr, G., *Geometric Function Theory in One and Higher Dimensions*, Marcel Decker Inc., New York, Basel, 2003.

[5] Albişoru, A.F., and Ghisa, D.. Conformal Self-mappings of the Fundamental Domains of Analytic Functions and Computer Experimentation, *WSEAS Transactions on Mathematics*, Volume 22, 2023, pp. 652-665.

[6] Ballantine, C., and Ghisa, D., Global Mapping Properties of Rational Functions, *Progress in Analysis and its Applications, Michael Ruzansky and Jens Wirth edits.*, London, UK, 2010, pp. 13-22.

[7] Ghisa, D., *Fundamental Domains and the Riemann Hypothesis*, Lambert Academic Publishing, Saarbrücken, 2012.

[8] Ghisa, D., Fundamental Domains of Dirichlet Functions, in *Geometry, Integrability and Quantization, Mladenov, Pulovand, and Yoshioka edits.*, Sofia, 2019, pp. 131-160.

[9] Barza, I., Ghisa, D. and Muscutar F., On the Location of the Zeros of the Derivative of Dirichlet L-functions, *Annals of the University of Bucharest*, No. 5 (LXIII), 2014, pp. 21-31.

[10] Cao-Huu, T., Ghisa, D., and Muscutar, F. A., Multiple Solutions of the Riemann Type of Functional Equations, *British Journal of Mathematics and Computer Science*, No. 17 (3), 2016, pp. 1-12.

[11] Ghisa, D., The Geometry of Mappings by General Dirichlet Series, *Advances in Pure Mathematics*, 7, 2017, pp. 1-20.

[12] Nevanlinna, R., *Analytic Functions*, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

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