Decomposition of Complete Multigraph into Wheel Graphs for Cyclic Triple System

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Abstract: - Let n be positive integer and W_n be a wheel graph of order n. In this paper, we construct a new decomposition of $8K_v$ into wheel graphs for $v \equiv 2 \pmod{12}$. Then, new cyclic triple system will be defined to arrange $v \times 2(v-1)$ triples satisfying certain criteria. In this development, the decomposition of $8K_v$ will be used to construct a cyclic 12-fold triple system of order $v \equiv 2 \pmod{12}$.

Key-Words: - Cyclic triple factorization, graph decomposition, near-four-factor, fuzzy set theory, group theory, graph theory.

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1 Introduction

A Throughout of this paper, all graphs and multisets considered have vertices in Z_v. Let G be a graph of order v, a near-k-factor of G is a spanning subgraph in which all vertices have a degree k with exception of one vertex (isolated vertex) which has a degree zero. An analysis of graph G involves a list of subgraphs $\mathcal{H} = \{H_1, \ldots, H_t\}$, in which the edge sets split the edge set of G as a whole. Another name for it is a (G, H)-design. A subgraph is called a $(G, H)-$design if every subgraph in H is isomorphic to a predefined subgraph H.

Let Π be a group of permutation on $V(G) = v$ leaving the multiset of subgraphs $\mathcal{H}$ invariant. If there is a permutation $\pi \in \Pi$ of order v, then $(G, \mathcal{H})$-design is called a cyclic. Thus, the permutation can be represented by $\pi = (0, 1, \ldots, v-1)$.

A complete multigraph λK_v is a graph where any two vertices are joined by λ distinct edges. The fundamental theorem for the existence of (K_v, H)-

design has been stated by, [1]. There have been several research papers relating to decomposing of complete multigraph λK_v into different subgraphs. For example, into crowns, [2], paths, [3] or cycles, [4].

Furthermore, $(\lambda K_v, K_k)$-design is known as a balanced incomplete block design denoted by (v,k,λ)-$BIBD$. On other words, (v,k,λ)-$BIBD$ is a pair $(V,\mathcal{B})$ where V is a finite set of v points and $\mathcal{B}$ is a list of k-subsets (called blocks) of V such that each pair of distinct points of V is contained in precisely λ blocks. A λ-fold triple system of order v, denoted by $TS(v,\lambda)$, is $(v,3,\lambda)$-$BIBD$. The $TS(v,\lambda)$, $(V,\mathcal{B})$, is called cyclic triple system, $CTS(v,\lambda)$, if $B=\{c_1,c_2,c_3\}\in\mathcal{B}$ then $B+1=\{c_1+1,c_2+1,c_3+1\}$ is also in $\mathcal{B}$. The ensemble of triplets generating all triplets within $CTS(v,\lambda)$ through the addition of one modulo v is termed starter triplets.

The orbit of triple B, represented by $orb(B)$, is the set that contains all unique triples in the collection $\{B+i \mid i\in Z_v\}$, where B is a triple. The length of the orbit, written as $orb(B)=k$, is the cardinality of this orbit, represented as $|orb(B)|$, the smallest positive integer, denoted by k in this case, for which $B+k=B$. B is said to be precisely defined if its orbit matches v; if not, it is deemed short. There is no block's short orbit when v is not equivalent to 0 $(mod\ 3)$, [5].

The existence of $CTS(v,\lambda)$ is an interesting open problem of combinatorics due to its vast applications. In, [6], they studied the existence of cyclic triple system over Z_v when $v\equiv 1,3 \pmod 6$. While Colbourn and Rosa have given the spectrum of $CTS(v,\lambda)$, [7]. Recently, in, [8], they introduced a new type of triple system called compatible factorization. They employed the near-one-factor to arrange $v\times\binom{v-1}{2}$ distinct triple into v rows according to certain conditions for $v\equiv 1,5 \pmod 6$. In, [9] they developed the compatible factorization to display $v\times\left(\frac{v-1}{2}-\frac{2}{3}\right)$ triples with minimum repetition for $v\equiv 3 \pmod 6$.

The primary aim of this paper is to devise a novel decomposition for the complete multigraph $8K_{12n+2}$ utilizing wheel graphs of distinct orders. Then we will employ this decomposition to define a new cyclic triple system to arrange $v\times 2(v-1)$ triples satisfying certain constraints.

2 Preliminaries and Definitions

Here, we introduce some key ideas in (v,k,λ) −BIBD and graph decomposition that are relevant to our conclusions. The primary goals of this research will be achieved by applying the partial difference approach, which is described in this section and has been successful in creating cyclic $(\lambda K_v, H)$-designs in many cases,[9].

Definition 1 A wheel graph of order $n\geq 4$, written as $W_n=c_0+(c_1,c_2,\dots,c_{n-1})$, is a graph that contains a cycle of order $n-1$, and each vertex in the cycle is joined to a new vertex, c_0, which is known as center, [10].

Definition 2 A starter of cyclic $(\lambda K_v,\mathcal{H})$-design is the collection of subgraphs of λK_v that generates all the subgraphs in $\mathcal{H}$, [11].

Definition 3 Let H be a subgraph of λK_v. The list of differences from H is the multiset, [12],

$$\Delta(H)=\{d(x,y)=mim\,\{|x-y|,v-|x-y|\},xy\in E(H)\}$$

In general, given a multiset $\delta=\{H_1,H_2,\dots,H_t\}$ of subgraphs of λK_v, the list of differences of the multiset δ is defined by:

$$\Delta(\delta)=\Delta(H_1)\cup\Delta(H_2)\cup\dots\cup\Delta(H_t).$$

Definition 4 Let H be a subgraph of λK_v, the stabilizer of H under Z_v is $stab(H)=\{z\in Z_v \mid z+H=H\}$ and is called trivial if $stab(H)=\{0\}$,[12].

As a particular result of, [13], we have the following lemma.

Theorem 5 Let v be even and δ be a multiset of subgraphs of λK_v and every subgraph of δ has trivial stabilizer. Then δ is a starter of cyclic($\lambda K_v,\mathcal{Y}$)-design if and only if $\Delta\delta$ covers each nonzero integer of $Z_{\frac{v}{2}}$ exactly λ and $\left(\frac{v}{2}\right)$occurs $\frac{\lambda}{2}$ times, [14].

Definition 6 Let B be a k-subset of Z_v. The list of difference of B is the multiset,[6],
$D(B)=\{min\{|a-b|,v-|a-b|\},\ a,b\in B, a\neq b\}$.

Generally, if $\mathcal{A}=\{B_1,B_2,\dots,B_t\}$ is a multiset of k-subsets of Z_v, then the list of differences of multiset $\mathcal{A}$ is defined as

$$D(\mathcal{A})=D(B_1)\cup D(B_2)\cup\dots\cup D(B_t).$$

Theorem 7 Let v be even and $\mathcal{A}$ be a multiset of 3-subsets of Z_v. An $\mathcal{A}$ is a starter of cyclic λ-fold triple system if and only if $D(\mathcal{A})$ covers each nonzero integer,[6], of $Z_{\frac{v}{2}}$ exactly λ times and the middle difference $\left(\frac{v}{2}\right)$ precisely $\frac{\lambda}{2}$ times.

Definition 8 Let H and F be two m-cycles of a graph G of order v. Then H and F are called parallel m cycles if they have the same difference set, [15].

Definition 9 Let H and F be two m-cycles of a graph G of order v. If the sum of each two corresponding vertices of them is v, then it is called adjoined m-cycles,[15].

Lemma 10 Any two adjoined m-cycles of a graph G are parallel m-cycles, [15].

3 Cyclic $(5^*, (4n)^*, (2n+3)^*)$-wheel System of $8K_{12n+2}$

In this section, we consider how to decompose the complete multigraph $8K_v$ into wheel graphs for $v = 12n + 2$. According to the Definition 1, the edges set of wheel graph $W_n = c_0 + (c_1, \ldots, c_{n-1})$ will be expressed below:

$$E(W_n) = E\left(K_{(1,n-1)}\right) \cup E(C_{n-1}) \text{ such that:}$$

$$E\left(K_{(1,n-1)}\right) = \{c_0 c_i \mid 1 \leq i \leq n-1\},$$

$$E(C_{n-1}) = \{c_i c_{i+1} \mid 1 \leq i \leq n-1\} \text{ where } c_n = c_1.$$

So, the list of difference from W_n is $D(W_n) = D(C_{n-1}) \underline{\cup} D\left(K_{(1,n-1)}\right)$. We will call $D(C_{n-1})$ and $D\left(K_{(1,n-1)}\right)$ the cycle differences $\left(CD(W_n)\right)$ and internal differences $\left(ID(W_n)\right)$, respectively, of W_n. As usual, any C_m is written as a permutation:

$$C_m = (c_0, c_1, \ldots, c_{m-1}).$$

To simplify determining a vertex set and computing the list of differences of m-cycle, we will write C_m of high order, when $m \geq 5$, as linking paths as follows:

$C_m = (c_0, c_1, \ldots, c_{m-1}) = (c_0, P_n, P_s)$, where n, s are positive integers and $1 + n + s = m$ in which P_n and P_s are paths and c_0 is a point such that:

$$P_n = [c_1, \ldots, c_n], P_s = [c_{n+1}, \ldots, c_{m-1}].$$

We represent a path of even order as follows:

$$P_{2n} = [a_1, b_1, a_2, b_2, \ldots, a_n, b_n] = [\cup_{i=1}^{n} a_i, b_i],$$

Therefore, the list of difference and vertex set of P_{2n} determined as:

$$D(P_{2n}) = \begin{cases} d(a_i, b_i), & 1 \leq i \leq n, \\ d(a_{i+1}, b_i), & 1 \leq i \leq n-1, \end{cases}$$

$$V(P_{2n}) = \{\cup_{i=1}^{n} a_i\} \cup \{\cup_{i=1}^{n} b_i\},$$

Moreover, the difference between P_n and P_s that located in the same cycle, denoted by $D(P_n, P_s)$, is defined as the difference between the last vertex in the path P_n and the first vertex in the path P_s. As a result, the vertex set and list of difference for an m-cycle $C_m = (c_0, P_n, P_s)$, $n, s \in \mathbb{N}$, will be written as:

$$V(C_m) = \{c_0\} \cup V(P_n) \cup V(P_s),$$

$$D(C_m) = D(P_n) \cup D(P_s) \cup D(c_0, P_n) \cup D(P_n, P_s) \cup D(P_s, c_0)$$

Definition 11 A $(m_1^*, m_2^*, \ldots, m_t^*)$-wheel system of λK_v is $(\lambda K_v, \mathcal{H})$-design where $\mathcal{H}$ is a collection of wheels in which the order of each wheel graph belong to $\{m_1, m_2, \ldots, m_t\}$.

We will denote of a cyclic $(m_1^*, m_2^*, \ldots, m_t^*)$-wheel system of λK_v by $CWS(\lambda K_v, \mathcal{W})$ where $\mathcal{W}$ is its a starter set. Following Tian and Wei,[13], we will use the notation $\mathcal{H} = \left\{H_{m_1}^{n_1}, H_{m_2}^{n_2}, \ldots, H_{m_r}^{n_r}\right\}$ to describe a set of subgraphs $\mathcal{H}$ meaning that there are n_1 subgraph of order m_1, n_2 subgraph of order m_2, etc. For more see, [16].

The following results will be used to prove the existence $(m_1^*, m_2^*, \ldots, m_t^*)$-wheel system of $8K_v$, [17].

Lemma 12 Let G be graph of order v. Let k be a positive even and $\mathcal{C}$ be a set of cycles of G. Then $\mathcal{C}$ is near-k-factor if and only if the vertex set of $\mathcal{C}$ covers every element of G exactly $\frac{k}{2}$ times except one vertex.

Proof. We will prove the first part of this lemma. The second part can be shown similarly. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be a set of cycles that forms a near-k-factor, then each vertex of G has a degree k except the isolated vertex. Let $x \in V(G)$ and x is not isolated vertex in $\mathcal{C}$. Then, the degree of x in G is

$$deg_G(x) = \sum_{i=1}^{m} deg_{C_i}(x).$$

Where $deg_G(x)$ and $deg_{C_i}(x)$ denote the degree of x in G and C_i respectively. Since a cycle graph is a 2-regular graph, then $deg_{C_i}(x) = 2$ or 0 according to whether x is a vertex of C_i . Suppose the number of cycles in $\mathcal{C}$ that contains x is n. Then, we have:

$$deg_G(x) = 2 + 2 + \cdots + 2 = 2 \times n.$$

Since $deg_G(x) = k$, then $n = \frac{k}{2}$.

Lemma 13 Let v and n be even integers and $\mathcal{W} = \{c_0 + C_{m_1}, c_0 + C_{m_2}, \ldots, c_0 + C_{m_t}\}$ be a set of wheels of λK_v. If the cycles $\{C_{m_1}, C_{m_2}, \ldots, C_{m_t}\}$ satisfy a near-n-factor, then the internal differences, (ID), of $\mathcal{W}$ covers each element of $Z^*_{\frac{v+2}{2}}$ exactly n times except the middle difference $\left(\frac{v}{2}\right)$, which occurs $\left(\frac{n}{2}\right)$ times.

Proof. Let $\mathcal{W} = \{c_0 + C_{m_1}, c_0 + C_{m_2}, \ldots, c_0 + C_{m_t}\}$ be a set of wheels of λK_v such that the set of cycles $\{C_{m_i}, 1 \le i \le t\}$ satisfies near-n-factor with isolated c_0. The internal differences of $\mathcal{W}$, (ID), is determined as follows:

$$D\left(K_{(1,m_i)}\right) = \{min\{|c_j - c_0|, v - |c_j - c_0|\}, c_j \in C_{m_i}, 1 \le i \le t, 1 \le j \le m_i\},$$

$$D\left(K_{(1,m_i)}\right) = \begin{cases} |c_j - c_0|, & |c_j - c_0| \le \frac{v}{2}, c_j \in C_{m_i}, 1 \le i \le t, 1 \le j \le m_i, \\ v - |c_j - c_0|, & |c_j - c_0| > \frac{v}{2}, c_j \in C_{m_i}, 1 \le i \le t, 1 \le j \le m_i. \end{cases}$$

Since the cycles $\{C_{m_i}, 1 \le i \le t\}$ form a near-n-factor, then the vertex set of set of cycles $\{C_{m_i}, 1 \le i \le t\}$ covers each element of Z_v exactly $\frac{n}{2}$ times except c_0 based on Lemma 12, [18].

Now if we label c_0 by "0", then every vertex of $\left\{1, 2, \ldots, \left(\frac{v}{2} - 1\right), \frac{v}{2}, \left(\frac{v}{2} + 1\right), \ldots, (v-2), (v-1)\right\}$ will appear as $c_j \in C_{m_i}$ exactly $\frac{n}{2}$ times. Therefore, (ID) can be written as:

$$D\left(K_{(1,m_i)}\right) = \begin{cases} c_j, & c_j \le \frac{v}{2}, \ c_j \in C_{m_i}, 1 \le i \le t, \\ v - c_j, & c_j > \frac{v}{2}, \ c_j \in C_{m_i}, 1 \le i \le t. \end{cases}$$

Thus, every element in the multiset of $\left\{1, 2, \ldots, \left(\frac{v}{2} - 1\right), \frac{v}{2}, \left(\frac{v}{2} - 1\right), \ldots, 2, 1\right\}$ will be shown $\frac{n}{2}$ times. Then $D\left(K_{(1,m_i)}\right)$ covers all the nonzero elements of $Z^*_{\frac{v+2}{2}}$ precisely n times except the middle difference $\frac{v}{2}$ occur $\frac{n}{2}$ times, [19].

Lemma 14 Let W_{m+1} be a wheel graph of λK_v. If the W_{m+1} is formed as $W_{m+1} = 0 + C_m$, then W_{m+1} has a trivial stabilizer.

Proof. Let $W_{m+1} = 0 + (c_1, c_2, \ldots, c_m)$ be $(m+1)$-wheel of λK_v, the stabilizer of W_{m+1} is represented as follows:

$$stab(W_{m+1}) = \{z \in Z_v \mid z + W_{m+1} = W_{m+1}\}$$

suppose $z \in stab(W_{m+1})$, then.

$$W_{m+1} + z = W_{m+1},$$

$$z + (c_1 + z, c_2 + z, \ldots, c_m + z) = 0 + (c_1, c_2, \ldots, c_m)$$

This implies that $z = 0$. Hence, $stab(W_{m+1}) = \{0\}$.

Now, we will present the existence cyclic $(m_1^*, m_2^*, \ldots, m_t^*)$-wheel system of $8K_v$ for $v = 12n + 2$.

Theorem 15 For $n > 1$, there exists a cyclic $(5^*, (4n)^*, (2n+3)^*)$-wheel system of $8K_{12n+2}$.

Proof. We construct the starter of cyclic $(5^*, (4n)^*, (2n+3)^*)$-wheel system of $8K_{12n+2}$ as follows:

Case 1. $n = 2$.

Consider that $\mathcal{W} = \{W_5^6, W_8^2, W_7^2\}$ is a wheel set of $8K_{26}$ such that:

$W_{5_1} = 0 + (1, 25, 14, 12)$, $W_{5_2} = 0 + (2, 24, 15, 11)$,

$W_{5_3} = 0 + (3, 23, 16, 10)$, $W_{5_4} = 0 + (4, 22, 17, 9)$,

$W_{5_5} = 0 + (5, 21, 18, 8)$, $W_{5_6} = 0 + (6, 7, 20, 19)$,

$W_8^* = 0 + (13, 2, 12, 3, 11, 4, 10)$,

$W_8^{**} = 0 + (13, 24, 14, 23, 15, 22, 16)$,

$W_7^{*} = 0 + (6, 1, 5, 17, 19, 18)$,

$W_7^{**} = 0 + (20, 25, 21, 9, 7, 8)$.

It is straightforward to check that $\mathcal{W} = \{W_5^6, W_8^2, W_7^2\}$ is the starter of cyclic $(5^*, (8)^*, (7)^*)$-wheel system of $8K_{26}$.

Case 2. $n \geq 3$ is odd.

Consider that $\mathcal{W} = \{W_5^{3n}, W_{4n}^2, W_{2n+3}^2\}$ is a set of wheel of $8K_{12n+2}$, where the list of wheels of order 5 is:

$$0 + (i, 12n + 2 - i, 6n + 1 + i, 6n + 1 - i), 1 \leq i \leq 3n,$$

$$i \neq \frac{n+1}{2}.$$

When $i = \frac{n+1}{2}$, let

$$W_{5_i} = 0 + \left(\frac{n+1}{2}, 12n + 2 - \frac{n+1}{2}, 6n + 1 - \frac{n+1}{2}, 6n + 1 + \frac{n+1}{2}\right).$$

Whereas, $W_{4n}^* = 0 + (4n + 2, P_{4n-2}^*)$ and $W_{4n}^{**} = 0 + (8n, P_{4n-2}^{**})$ are wheels of order $4n$ in which the paths $\{P_{4n-2}^*, P_{4n-2}^{**}\}$ are represented below:

$$P_{4n-2}^* = [6n + 1, 2, 6n, 3, \dots, 4n + 3, 2n]$$

$$= \left[\cup_{i=1}^{2n-1} 6n + 2 - i, i + 1\right],$$

$$P_{4n-2}^{**} = [6n + 1, 12n, 6n + 2, 12n - 1, \dots, 8n - 1, 10n + 2],$$

$$= \left[\cup_{i=1}^{2n-1} 6n + i, 12n + 1 - i\right].$$

Meanwhile, $W_{2n+3}^* = (8n + 1, P_3^*, P_{2n-2}^*)$ and $W_{2n+3}^{**} = 0 + (4n + 1, P_3^{**}, P_{2n-2}^{**})$ are considered the wheel graph such that the paths $\{P_3^*, P_{2n-2}^*, P_3^{**}, P_{2n-2}^{**}\}$ are written as:

$$P_3^* = [2n + 2, 1, 2n + 1],$$

$$P_{2n-2}^* = [4n, 2n + 3, 4n - 1, 2n + 4, \dots, 3n + 3, 3n, 3n + 2, 3n + 1]$$

$$= \left[\cup_{i=1}^{n-1} 4n + 1 - i, 2n + 2 + i\right],$$

$$P_3^{**} = [10n, 12n + 1, 10n + 1],$$

$$P_{2n-2}^{**} = [8n + 2, 10n - 1, 8n + 3, 10n - 2, \dots, 9n - 1, 9n + 2, 9n, 9n + 1],$$

$$= \left[\cup_{i=1}^{n-1} 8n + 1 + i, 10n - i\right].$$

In order to prove that $\mathcal{W} = \{W_5^{3n}, W_{4n}^2, W_{2n+3}^2\}$ is a starter set of cyclic $(5^*, (4n)^*, (2n + 3)^*)$-wheel system of $8K_{12n+2}$, the differences list of $\mathcal{W}$ will be determined as follows:

$$D(\mathcal{W}) = CD(W_i) \cup ID(W_i), \quad W_i \in \mathcal{W}$$

we begin with the cycle differences $CD(W_i)$ as follows:

$$CD(W_i) = \cup_{i=1}^{3n} D(C_{4_i}) \cup_{i=1}^{2} D(C_{(4n-1)_i}) \cup_{i=1}^{2} D(C_{(2n+2)_i}),$$

Such that

$$D(C_{4_i}) = D(c_{1,i}, c_{2,i}, c_{3,i}, c_{4,i}), \quad 1 \leq i \leq 3n$$

$$= d(c_{1,i}, c_{2,i}) \cup d(c_{2,i}, c_{3,i}) \cup d(c_{3,i}, c_{4,i}) \cup d(c_{4,i}, c_{1,i}),$$

$1 \leq i \leq 3n$, where

$$d(c_{1,i}, c_{2,i}) = \left\{min\left\{| c_{2,i} - c_{1,i}|, \ 12n + 2 - | c_{2,i} - c_{1,i}|\right\}\right\}$$

$$= 2i \quad, 1 \leq i \leq 3n, \quad i \neq \frac{n+1}{2},$$

$$= \{2, 4, \dots, 6n\} - \{n + 1\}.$$

$$d(c_{2,i}, c_{3,i}) = \left\{min\left\{| c_{3,i} - c_{2,i}|, 12n + 2 - | c_{3,i} - c_{2,i}|\right\}\right\}$$

$$= 6n + 1 - 2i, \ 1 \leq i \leq 3n, \ i \neq \frac{n+1}{2},$$

$$= \{6n - 1, 6n - 3, \dots, 3, 1\} - \{5n\},$$

$$d(c_{3,i}, c_{4,i}) = \left\{min\left\{| c_{4,i} - c_{3,i}|, 12n + 2 - | c_{4,i} - c_{3,i}|\right\}\right\}$$

$$= 2i, \ 1 \leq i \leq 3n, i \neq \frac{n+1}{2},$$

$$= \{2,4,\dots,6n-2,6n\} - \{n+1\},$$

$$d(c_{4,i}, c_{1,i}) = \{min\{| c_{1,i} - c_{4,i}|, 12n+2 - | c_{1,i} - c_{4,i}|\}\}$$

$$= 6n+1-2i, \quad 1 \le i \le 3n, \qquad i \ne \tfrac{n+1}{2},$$

$$= \{6n-1, 6n-3, \dots, 3, 1\} - \{5n\}.$$

when $i = \frac{n+1}{2}$, $D(C_{4_i}) = \{n+1, 6n+1, n+1, 6n+1\}$.

Since $\{C^*_{4n-1}, C^{**}_{4n-1}\}$ and $\{C^*_{2n+2}, C^{**}_{2n+2}\}$ are adjoined $(4n-1)$-cycles and $(2n+2)$-cycles respectively, then $D(C^*_{4n-1}) = D(C^{**}_{4n-1})$ and $D(C^*_{2n+2}), = D(C^{**}_{2n+2})$ based on Lemma 10. Hence, it is sufficient to determine the lists of $D(C^*_{4n-1})$ and $D(C^*_{2n+2})$ as follows:

$$D(C^*_{4n-1}) = D(4n+2, P^*_{4n-2}) \uplus D(P^*_{4n-2}) \uplus D(P^*_{4n-2}, 4n+2)$$

$D(4n+2, P^*_{4n-2}) = d(4n+2, 6n+1) = \{2n-1\}$.

$$D(P^*_{4n-2}) = D(P^*_{2(2n-1)})$$

$$= \begin{cases} d(a_i, b_i), & 1 \le i \le 2n-1, \\ d(a_{i+1}, b_i), & 1 \le i \le 2n-2, \end{cases}$$

$$= \begin{cases} 6n+1-2i, & 1 \le i \le 2n-1, \\ 6n-2i, & 1 \le i \le 2n-2, \end{cases}$$

$$= \begin{cases} \{6n-1, 6n-3, \dots, 2n+3\}, \\ \{6n-2, 6n-4, \dots, 2n+4\}. \end{cases}$$

$$D(P^*_{4n-2}, 4n+2) = d(2n, 4n+2) = \{2n+2\}.$$

For simplicity, $\Delta(C^*_{4n-1})$ can be written as:

$D(C^*_{4n-1}) = \{6n-1, 6n-2, \dots, 2n+3, 2n+2\} \cup \{2n-1\}$

Equally, $\Delta(C^*_{2n+2})$ is computed as:

$D(C^*_{2n+2}) = D(8n+1, P^*_3) \cup D(P^*_3) \cup D(P^*_3, P^*_{2n-2}) \cup D(P^*_{2n-2}) \cup D(P^*_{2n-2}, 8n+1)$.

Such that

$$D(8n+1, P^*_3) = d(8n+1, 2n+1) = \{6n\},$$

$$D(P^*_3) = \{2n, 2n+1\},$$

$$D(P^*_3, P^*_{2n-2}) = d(2n+2, 4n) = \{2n-2\}.$$

$$D(P^*_{2n-2}) = D(P^*_{2(n-1)})$$

$$= \begin{cases} d(a_i, b_i), & 1 \le i \le n-1, \\ d(a_{i+1}, b_i), & 1 \le i \le n-2, \end{cases}$$

$$= \begin{cases} 2n-2i-1, & 1 \le i \le n-1, \\ 2n-2i-2, & 1 \le i \le n-2, \end{cases}$$

$$= \begin{cases} \{2n-3, 2n-5, \dots, 3, 1\}, \\ \{2n-4, 2n-6, \dots, 4, 2\}. \end{cases}$$

$$D(P^*_{2n-2}, 8n+1) = d(9n+1, 8n+1) = \{n\}.$$

Furthermore, the differences list of C^*_{2n+2} can be expressed as:

$$D(C^*_{2n+2}) = \{1, 2, \dots, 2n-3, 2n-2\} \cup \{n, 2n, 2n+1, 6n\}.$$

In view of the former investigations, it can be noticed that the cycle differences of $\mathcal{W}$, $CD(W_i)$, $W_i \in \mathcal{W}$, covers each nonzero integer of Z_{6n+1} four times and the middle difference $\{6n+1\}$ occurs twice.

On the other hand, it is easy to prove that the vertex set of cycles associated with the set of wheel graphs $\mathcal{W}$ contains each element in Z^*_{12n+2} precisely twice, then it satisfy a near-four-factor by Lemma 12. Based on Lemma 13, the internal differences of $\mathcal{W}$ covers each element of Z^*_{6n+1} four times and the middle difference $\{6n+1\}$ occurs twice, [20]

Since wheel graph in $\mathcal{W}$ has a trivial stabilizer by Lemma 14, then the set of wheel graphs is the starter set of $(5^*, (4n)^*, (2n+3)^*)$-wheel system of $8K_v$ based on Theorem 5.

Case 3. $n \ge 4$ is even.

Consider that $\mathcal{W} = \{W_5^{3n}, W_{4n}^2, W_{2n+3}^2\}$ is a set of wheels of $8K_{12n+2}$. Where the wheels of order 5 and $(4n)$, $\{W_5^{3n}, W_{4n}^2\}$, are the same wheels that mentioned in Case 2 with slightly different in the list of wheels of order 5 as follows:

$$W_{5_i} = 0 + (i, 12n+2-i, 6n+1+i, 6n+1-i),$$

$$1 \le i \le 3n, \quad i \ne \frac{5n+4}{2}.$$

When $i = \frac{5n+4}{2}$, let

$$W_{4_i} = 0 + \left(\frac{5n+4}{2}, 6n+1-\frac{5n+4}{2}, 12n+2-\frac{5n+4}{2}, 6n+1+\frac{5n+4}{2}\right).$$

Meanwhile, the wheels of order $(2n+3)$ are $W^*_{2n+3} = (8n+1, P^*_3, P^*_{2n-2})$ and $W^{**}_{2n+3} = (4n+1, P^{**}_3, P^{**}_{2n-2})$ in which the paths $\{P^*_3, P^*_{2n-2}, P^{**}_3, P^{**}_{2n-2}\}$ are represented below:

$$P^*_3 = [2n+2, 1, 2n+1],$$

$$P^*_{2n-2} = [8n+1, 10n-1, 8n+2, 10n-2, \dots, 9n-1, 9n+1]$$

$$= \left[\cup_{i=1}^{n-1} 8n+i, 10n-i\right],$$

$$P^{**}_3 = [10n, 12n+1, 10n+1],$$

$$P^{**}_{2n-2} = [4n+1, 2n+3, 4n, 2n+4, \dots, 3n+3, 3n+1],$$

$$= \left[\cup_{i=1}^{n-1} 4n+2-i, 2n+2+i\right].$$

Similarly, by following the same strategy used Case 2, it is easy to check that the set of wheel graph $\mathcal{W} = \{W_5^{3n}, W_{4n}^2, W_{2n+3}^2\}$ is a starter of cyclic $(5^*, (4n)^*, (2n+3)^*)$-wheel system of $8K_{12n+2}$ when $n > 2$ is an even.

4 Cyclic Triple Factorization

We provide a new idea in this section called cyclic triple factorization, which is a kind of cyclic triple system. The decomposition of all Z_v triples into cyclic triple systems will be based on this novel method, [21].

Definition 16 A cyclic triple factorization with order v, labelled as $CTF(v)$, involves the arrangement of $v \times 2(v-1)$ triples into v rows while meeting the specified conditions:

(i) Object r appears precisely $2(v-1)$ times in each row r.
(ii) Each object except r appears four times in each row r.
(iii) The triples associated with row r contains no repetitions.

Note that condition (iii) in Definition 16 means that the triples in row r_i, for $0 \le i < v$ are distinct but not in whole $v \times 2(v-1)$ array. The construction of cyclic wheel system of $8K_v$ will be employed to prove the existence of a cyclic triple factorization of order $12n+2$ in the following theorem.

Theorem 17 For $n > 1$, there exists a near triple factorization of order $v = 12n+2$.

Proof. To construct CTF(12n + 2), we need to have 12n + 2 rows and 2 (12n + 1) columns based on Definition 16. Consider the starter set $\mathcal{W} = \{W_5^{3n}, W_{4n}^2, W_{2n+3}^2\}$ of $(5^*, (4n)^*, (2n+3)^*)$-wheel system of $8K_{12n+2}$ that constructed in Theorem 15. To construct $CTF(12n+2)$, we partition the wheel graphs of $\mathcal{W}$ into separated triangles (triples) by combining the centre of each wheel with every edge of its cycle. Hence, the number of triples of each row is equal to the number of edges of the cycles associated with the wheels in $\mathcal{W}$. Since the cycles set associated with the wheel graphs of $\mathcal{W}$ is $\{C_4^{2n}, C_{4n-1}^2, C_{2n+2}^2\}$, then the number of the columns is computed as the following formula:

$$4 \times 3n + 2 \times (4n-1) + 2 \times (2n+2) = 2(12n+1).$$

Therefore, the center vertex r in each row $r, 0 \le r < v$, will appears $2(2n+1)$ times in the generated triples while other vertices will appear four times since the cycles set satisfies a near-four-factor. On the other hand, all the triples in each row are distinct since there is no edge in $\{C_4^{2n}, C_{4n-1}^2, C_{2n+2}^2\}$ that it has the same endpoints. Then all conditions of $CTF(v)$ are satisfied for $v = 12n+2$.

Example 18 Let $G = 8K_{26}$ and $\mathcal{W} = \{W_5^6, W_8^2, W_7^2\}$ be a set of wheel graph of G such that:

$W_{5_1} = 0 + (1, 25, 14, 12)$, $W_{5_2} = 0 + (2, 24, 15, 11)$,

$W_{5_3} = 0 + (3, 23, 16, 10)$, $W_{5_4} = 0 + (4, 22, 17, 9)$,

$W_{5_5} = 0 + (5, 21, 18, 8), W_{5_6} = 0 + (6, 7, 20, 19),$

$W_8^* = 0 + (13, 2, 12, 3, 11, 4, 10),$

$W_8^{**} = 0 + (13, 24, 14, 23, 15, 22, 16),$

$W_7^* = 0 + (6, 1, 5, 17, 19, 18),$

$W_7^{**} = 0 + (20, 25, 21, 9, 7, 8).$

From Lemma 12, $\mathcal{W} = \{W_5^6, W_8^2, W_7^2\}$ is a starter set of cyclic $(5^*, (8)^*, (7)^*)$-wheel system of $8K_{26}$, $CWS(8K_{26}, \mathcal{W})$ that generates its wheels by adding one modular 26. To construct $CTF(26)$, we partition all wheel graphs of $CWS(8K_{26}, \mathcal{W})$ into separate triangles (triples). Figure 1 shows that how the wheel graph $0 + C_{4_1}$ can be partitioned into separate triples.

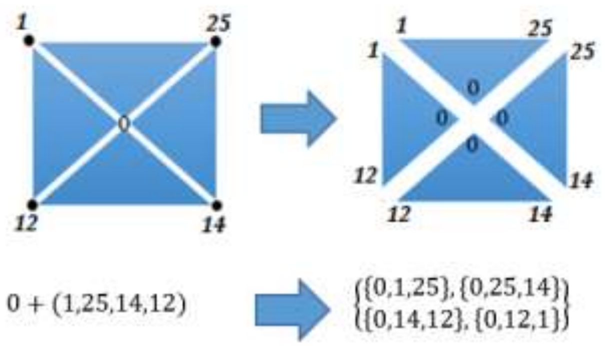

Fig. 1: Partition wheel graph into triples

Similarly, we can partition the remaining wheels of $CWS(8K_{26}, \mathcal{W})$ into triples in the same way. Clearly, it can be noticed that the center r of the wheels in each row r, $0 \leq r \leq 25$, of $CWS(8K_{26}, \mathcal{W})$ will appear in 50 triples, the number of edges of cycles $\{C_4^6, C_7^2, C_6^2\}$ that associated with $\mathcal{W}$, and other vertices appear four times since the cycles set satisfies near-four-factor. Table 1 (Appendix) shows the construction of $CTF(26)$.

In a λ-fold triple system, denoted as $CTS(v, \lambda)$, it is important to revisit the definition, wherein it is characterized as a pair (V, T). Here, V represents a set of v elements, and T constitutes a collection of 3-subsets of V, referred to as triples. Notably, each pair of distinct elements from V is precisely found together in λ triples within T. Therefore, no collection of triples may be regarded as a $CTS(v, \lambda)$. Consequently, it is reasonable to inquire if the λ-fold triple system, $CTS(v, \lambda)$, is formed via the creation of cyclic triple factorization. We must demonstrate that $CTF(v)$ has a balanced quality, namely that each pair of unique elements of v belongs to exactly λ triples, in order to demonstrate that $CTF(v)$ is $CTS(v, \lambda)$. The difference set approach will be used in this manner.

Definition 6 and Theorem 7 state that building an appropriate triples set $\mathcal{A}$ is equivalent to the presence of a λ-fold cyclic triple system, $CTS(v, \lambda)$. such that the list of differences $D(\mathcal{A})$ covers every nonzero element of $Z_{\frac{v+2}{2}}$ exactly λ times except the middle difference $\left(\frac{v}{2}\right)$, which occurs $\left(\frac{\lambda}{2}\right)$ times.

Theorem 19 For $n > 1$, there exists a 12-fold cyclic triple factorization of order $12n + 2$.

Proof. Let $\mathcal{W} = \{W_5^{3n}, W_{4n}^2, W_{2n+3}^2\}$ be the starter of $(5^*, (4n)^*, (2n+3)^*)$-wheel system of $8K_{12n+2}$ mentioned in Theorem 15. Then, the list of differences:

$$D(\mathcal{W}) = CD(W_i) \cup ID(W_i), W_i \in \mathcal{W}$$

covers each nonzero integer of Z_{6n+1} eight times and the middle difference four times in which the cycle differences $(CD(W_i))$ and the internal differences $(ID(W_i))$ have the same list of differences. Let $\mathcal{A}$ be the set of the generated triples from partition of the wheels in $\mathcal{W}$, then the triples of $\mathcal{A}$ will be formed by linking every two internal edges with an edge that connected them. As shown in Figure 1, each internal edge of $\mathcal{W}$ will appear twice in $\mathcal{A}$ while the edge set of cycles associated with $\mathcal{W}$ will occur once. Hence, the list of differences of $\mathcal{A}$, $D(\mathcal{A})$, contains $ID(W_i)$ twice and $CD(W_i)$ once. Therefore, $D(\mathcal{A})$ covers each nonzero integer of Z_{6n+1} twelve times and the middle difference $6n + 1$ six times. Based on Theorem 2.8, the set of triples $\mathcal{A}$ is the starter of cyclic 12-fold triple system of order v such that satisfies near triple factorization conditions.

5 Algorithm of Starter Triples of $CTF\ (12n + 2)$

In this section, we use the starter cycles of $CWS(8K_{12n+2}, \mathcal{W})$ to develop and formulate the algorithm of starter triples $\mathcal{A}$ of $CTF(12n + 2)$. The process of formulating an algorithm for the starter triples $\mathcal{A}$ will be split into three cases depending on n.

Case 1. $n = 2$.

See Example 4.3, and Table 1 (Appendix).

Case 2. $n > 1$ is odd.

The starter of $CTF(12n+2)$ is formed by partition the starter set $\mathcal{W} = \{W_5^{3n}, W_{4n}^2, W_{2n+3}^2\}$ of $(5^*, (4n)^*, (2n+3)^*)$-wheel system of $8K_{12n+2}$. Thus, we start with the generated triple from partition of wheels of order 5, $\{W_5^{3n}\}$, as follows:

$$S_1 = \left\{ \begin{matrix} \left\{0, \frac{n+1}{2}, \frac{23n+3}{2}\right\}, \left\{0, \frac{23n+3}{2}, \frac{11n+1}{2}\right\}, \\ \left\{0, \frac{11n+1}{2}, \frac{13n+3}{2}\right\}, \left\{0, \frac{13n+3}{2}, \frac{n+1}{2}\right\} \end{matrix} \right\},$$

$$S_2 = \left\{\{0, i, 12n+2-i\}, \quad 1 \le i \le 3n, i \neq \frac{n+1}{2}\right\},$$

$$S_3 = \left\{\{0, 12n+2-i, 6n+1+i\}, 1 \le i \le 3n, i \neq \frac{n+1}{2}\right\},$$

$$S_4 = \left\{\{0, 6n+1-i, 6n+1+i\},\ 1 \le i \le 3n, i \neq \frac{n+1}{2}\right\},$$

$$S_5 = \left\{\{0, i, 6n+1-i\}, \quad 1 \le i \le 3n, i \neq \frac{n+1}{2}\right\}.$$

Furthermore, the list of generated triples from wheels of order $4n$, $\{W_{4n}^2\}$, could be expressed as follows:

$$S_6 = \left\{\{0, i+1, 6n+2-i\}, \quad 1 \le i \le 2n-1\right\},$$

$$S_7 = \left\{\{0, i+1, 6n+1-i\}, \quad 1 \le i \le 2n-2\right\},$$

$$S_8 = \left\{\{0, 12n+1-i, 6n+i\}, \quad 1 \le i \le 2n-1\right\},$$

$$S_9 = \left\{\{0, 12n+1-i, 6n+1+i\}, \quad 1 \le i \le 2n-2\right\},$$

$$S_{10} = \left\{\{0, 2n, 4n+2\}, \{0, 4n+2, 6n+1\}, \{0, 10n+2, 8n\}, \{0, 8n, 6n+1\}\right\}.$$

Similarly, the produced triples from the wheels of order $(2n+3)$, $\{W_{2n+3}^2\}$ that could be represented in the following subsets:

$$S_{11} = \left\{\{0, 2n+2+i, 4n+1-i\}, \quad 1 \le i \le n-1\right\},$$

$$S_{12} = \left\{\{0, 2n+2+i, 4n-i\}, \quad 1 \le i \le n-2\right\},$$

$$S_{13} = \left\{\{0, 10n-i, 8n+1+i\}, \quad 1 \le i \le n-1\right\},$$

$S_{14} = \left\{\{0, 10n-i, 8n+2+i\}, \quad 1 \le i \le n-2\right\}$, $S_{15} = \left\{\{0, 3n+1, 8n+1\}, \{0, 8n+1, 2n+1\}, \{0, 1, 2n+1\}, \{0, 1, 2n+2\}, \{0, 2n+2, 4n\}, \{0, 9n+1, 4n+1\}, \{0, 4n+1, 10n+1\}, \{0, 12n+1, 10n+1\}, \{0, 12n+1, 10n\}, \{0, 10n, 8n+2\}\right\}$.

For simplicity, we will link the subsets together which have a relationship between their triples. As a result, the algorithm of the starter triples $\mathcal{A}$ of $CTF(12n+2)$, can be formulated as:

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$$

Such that:

$$\begin{cases} \{0, i, 12n+2-i\}, & 1 \le i \le 6n \\ \{0, 12n+2-i, 6n+1+i\}, & 1 \le i \le 3n, if\ i \notin \left\{\frac{n+1}{2}\right\} \\ \{0, 6n+1-i, i\}, & 1 \le i \le 3n, if\ i \notin \left\{\frac{n+1}{2}\right\} \\ \{0, 6n+2-i, i+1\}, & 1 \le i \le 3n, if\ \ i \notin \{2n, 2n+1\} \\ \{0, 6n+i, 12n+1-i\}, & 1 \le i \le 3n, if\ i \notin \{2n, 2n+1\} \\ \{0, 6n+1-i, i+1\}, & 1 \le i \le 3n-1, if\ i \notin \{2n-1, 2n\} \\ \{0, 6n+1+i, 12n+1-i\}, & 1 \le i \le 3n-1, if\ \ i \notin \{2n-1, 2n\} \end{cases}$$

$\mathcal{A}_2 = \left\{\left\{0, \frac{23n+3}{2}, \frac{11n+1}{2}\right\}, \left\{0, \frac{13n+3}{2}, \frac{n+1}{2}\right\}, \{0, 4n+2, 2n\}, \{0, 4n+2, 6n+1\}, \{0, 8n, 6n+1\},\right.$

$\{0, 3n+1, 8n+1\}, \{0, 8n+1, 2n+1\}, \{0, 2n+1, 1\}, \{0, 2n+1, 1\}, \{0, 1, 2n+2\},$

$\left.\{0, 9n+1, 4n+1\}, \{0, 4n+1, 10n+1\}, \{0, 10n+1, 12n+1\}, \{0, 12n+1, 10n\}\right\}$.

Case 3. $n > 2$ is even.

By following the same strategy of Case 2, the algorithm of starter triples $\mathcal{A}$ of a $CTF(12n+2)$, will be formulated as $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ such that

$$\mathcal{A}_1 = \begin{cases} \{0, i, 12n+2-i\}, & 1 \le i \le 6n, if\ i \notin \left\{\frac{5n+4}{2}, \frac{7n-2}{2}\right\} \\ \{0, 12n+2-i, 6n+1+i\} \cup \{0, 6n+1-i, i\}, & 1 \le i \le 3n, \\ \{0, 6n+i, 12n+1-i\}, & 1 \le i \le 3n-1, if\ i \notin \{2n, 2n+1\} \\ \{0, 6n+2-i, i+1\}, & 1 \le i \le 3n-1, if\ i \notin \{2n, 2n+1\} \\ \{0, 6n+1-i, i+1\} \cup \{0, 6n+1+i, 12n+1-i\}, & 1 \le i \le 2n-2 \\ \{0, 8n+i, 10n-i\} \cup \{0, 4n+2-i, 2n+2+i\}, & 1 \le i \le n-1 \end{cases}$$

$\mathcal{A}_2 = \left\{ \left\{0, \frac{7n-2}{2}, \frac{19n}{2}\right\}, \left\{0, \frac{17n+6}{2}, \frac{5n+4}{2}\right\}, \{0, 4n+2, 2n\}, \{0, 4n+2, 6n+1\}, \{0, 8n, 6n+1\}, \{0, 8n, 10n+2\}, \{0, 9n+1, 9n\} \{0, 9n, 2n+2\}, \{0, 2n+2, 1\}, \{0, 1, 2n+1\}, \{0, 2n+1, 8n+1\}, \{0, 3n+1, 3n+2\}, \{0, 3n+2, 10n\}, \{0, 10n, 12n+1\}, \{0, 12n+1, 10n+1\}, \{0, 10n+1, 4n+1\} \right\}$.

6 Conclusion

In this paper, we have investigated new decomposition of complete multigraph. Especially, we have decomposed of $8K_v$ into wheel graphs for $v \equiv 2 \pmod{12}$. We have also defined and proven the existence of cyclic triple factorization, $CTF(v)$, for $v \equiv 2 \pmod{12}$ along with the construction of $CTF(12n+2)$ has been demonstrated that is a cyclic 12-fold triple system. Then, the algorithms of the starter triples of $CTF(12n+2)$ have been formulated. We expect the construction of $CTF(12n+2)$ will be simple and can be extended it for all even cases, $v \equiv 0 \pmod{2}$.

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APPENDIX

Table 1. Case 1. $n = 2$.

$r = 0$	{0,1,25}	{0,25,14	{0,14,12	{0,12,1}	⋯	{0,9,7}	{0,7,8}	{0,8,20}
$r = 1$	{1,2,0}	{1,0,15}	{1,15,13	{1,13,2}	⋯	{1,10,8}	{1,8,9}	{1,9,21}
$r = 2$	{2,3,1}	{2,1,16}	{2,16,14	{2,14,3}	⋯	{2,11,9}	{2,9,10}	{2,10,22}
⋮	⋮	⋮	⋮	⋮	⋯	⋮	⋮	⋮
$r = 25$	{25,0,24}	{25,24,1	{25,13,1	{25,11,0	⋯	{25,8,6}	{25,6,7}	{25,7,19}

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