The Cauchy Problem for the General Telegraph Equation with Variable Coefficients under the Cauchy Conditions on a Curved Line in the Plane

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Abstract: The Riemann method is used to prove the global correctness theorem to Cauchy problem for a general telegraph equation with variable coefficients under Cauchy conditions on a curved line in the plane. The global correctness theorem consists of an explicit Riemann formula for a unique and stable classical solution and a Hadamard correctness criterion for this Cauchy problem. From the formulation of the Cauchy problem, the definition of its classical solutions and the established smoothness criterion of the right-hand side of the equation, its correctness criterion is derived. These results are obtained by Lomovtsev's new implicit characteristics method which uses only two differential characteristics equations and twelve inversion identities of six implicit mappings. If the right-hand side of general telegraph equation depends only on one of two independent variables, then it is necessary and sufficient that it be continuous with respect to this variable. If the right-hand side of this equation depends on two variables and is continuous, then in its integral smoothness requirements it is necessary and sufficient the continuity in one and continuous differentiability in the other variable. The correctness criterion represents the necessary and sufficient smoothness requirements of the right-hand side of the equation and the Cauchy data. From the established global correctness theorem, the well-known Riemann formulas for classical solutions and correctness criteria to Cauchy problems for the general and model telegraph equations in the upper half-plane are derived. In the works of other authors, there is no necessary (minimally sufficient) smoothness on the right-hand sides of the hyperbolic equations of real Cauchy problems for the set of classical (twice continuously differentiable) solutions.

Key–Words: Generalized Cauchy problem, Telegraph equation, Implicit characteristic, Cauchy conditions on the curve, Riemann formula, Correctness criterion, Smoothness requirement, Global correctness theorem

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1 Introduction

In this paper, by modification of the Riemann method the Cauchy problem for a linear general inhomogeneous telegraph equation with variable coefficients and Cauchy data on a smooth curved line in the plane has been explicitly solved in a set of classical solutions. In this paper a Hadamard correctness criterion has also been found for this Cauchy problem. The Hadamard correctness of this Cauchy problem means the existence, uniqueness and stability on the right-hand side of the equation and Cauchy data of its twice continuously differentiable solution. The explicit Riemann formula of its unique and stable classical solution is derived and its correctness criterion is established (Theorem 3). The correctness criterion of the Cauchy problem is the necessary and sufficient smoothness requirements to the right-hand side of equation and the Cauchy data for existence of a unique and stable classical (twice continuously differentiable) solution. Previously, the necessary and sufficient smoothness requirements for the right-hand sides of telegraph equations had been studied. The resulting Hadamard correctness theorem of the Cauchy problem is global. The concept of global correctness theorems of linear boundary value problems is introduced in the article, [1]. In it, with the help of Zorn's lemma, a theorem on the existence of global correctness theorems is proved: every correctly posed linear boundary value problem for a partial differential equation has a global theorem of its correct Hadamard solvability in the corresponding pair of locally convex topological vector spaces. Global theorems are called correctness theorems of boundary value problems with correctness criteria for the existence of a unique and stable classical (twice continuously differentiable) solutions, i.e. with necessary and sufficient conditions for their Hadamard correctness.

In this article, the well-known global correctness theorems with explicit formulas of classical solutions and Hadamard correctness criteria for Cauchy problems for the general telegraph equation (Theorem 8) and the model telegraph equation (Theorem 11) in the upper half-plane are derived from the proven global theorem 3. The Riemann formula of the classical solution and the Hadamard correctness criterion of the Cauchy problem for the general telegraph equation in the upper half-plane are known. The article, [2], contains an explicit formula for the classical solution and the Hadamard correctness criterion of the Cauchy problem for a model telegraph equation under the critical characteristic $g_2(s,\tau) = g_2(0,0)$ of the first quarter of the plane, where there is no influence of the boundary mode of the first kind. The latest results of Theorems 8 and 11 partially confirm the validity of Theorem 3. In this paper, the generalized Cauchy problem for the general telegraph equation with variable coefficients is solved and studied in a variety of classical solutions by a new implicit characteristics method, which uses only two characteristic differential equations and twelve inversion identities from the definition of six implicit mappings. The results obtained in this article are needed to derive ordinary and Riemann formulas of classical solutions of mixed problems for model and general telegraph equations with variable coefficients in rectilinear and curvilinear domains of the plane (see remark 14).

In the article, [3], il obtained formula allows us to explicitly express the classical solution, and at zero initial velocity for the initial displacement, sufficient smoothness requirements are found to ensure its existence, uniqueness and continuous dependence on the parameter $\lambda > 0$, the Cauchy problem for a homogeneous singular parameter Euler-Poisson-Darboux wave equation with Dirac potential. The solution to the Cauchy problem for the general Euler-Poisson-Darboux equation is unique only for non-negative values of the parameter k in the Bessel operator with respect to the time variable, [4]. The paper, [5], is devoted to the proof of the existence of a conformal scattering operator for a nonlinear cubic wave equation of defocusing on a nonstationary background. In it, the proof is based on solving a characteristic problem with an initial value by the Hermander method, which consists in reducing the characteristic initial problem to the standard Cauchy problem by slowing down the wave propagation velocity. Solutions of the Dalembert formula form of the Cauchy problem for linear homogeneous partial differential equations with constant coefficients of the third order in paper, [6], are obtained. Using the solutions obtained, some computer tests were carried out on three different roots. These tests indicate the dispersion dynamics of waves with some initial profile.

Using the Lomov regularization method, a regularized asymptotic solution to the linear singularly perturbed Cauchy problem in the presence of a spectral singularity in the form of a weak turning point for the limit operator was constructed in the article, [7]. The main singularities of this Cauchy problems are written out explicitly. In paper, [8], an asymptotic expansion in powers of a small parameter was obtained to solve the Cauchy problem for a first-order differential equation with a small parameter at the derivative. The conditions under which the boundary layer phenomenon occurs have been found. The existence and destruction of weak generalized solutions of two Cauchy problems for a wave equation with two nonlinearities is studied, [9]. The Cauchy problem for a third-order model partial differential equation with power-law nonlinearity is studied in article, [10]. For the linear part of the equation, analogues of Green's third formula for elliptic equations are constructed. An integral equation for classical solutions of the Cauchy problem is obtained. It is proved that every solution of the integral equation is a local-in-time weak solution of the Cauchy problem.

A global correctness theorem (with necessary and sufficient conditions for the coefficients of differential operators) of the Cauchy problem for linear complex systems of first-order differential equations in the scales of Banach spaces of complex-valued vector-exponential functions with an integral metric was obtained in journal, [11]. It turned out that the necessary and sufficient conditions for the correctness of his complex Cauchy problem in spaces of complex functions with integral metrics and in spaces of complex functions of vector-exponential type with supremum norms from the monograph, [12], coincide. It is proved the existence of local solution, global solution and three conditions about the blow-up of solution to the generalized damped Boussinesq equation, [13].

There are no global correctness theorems with Hadamard correctness criteria for initial data and right-hand sides of hyperbolic equations in real Cauchy problems for classical solutions in the works of other authors. In famous works, [3], [4], [5], [6], [14], [15], [16], [17], [18], [19], and some others there are theorems of existence, uniqueness and stability of classical solutions of real Cauchy problems with their D'alembert, Riemann and others formulas in some of them, but only if the right-hand sides of hyperbolic equations have an overestimated sufficient smoothness. In these works the right-hand sides of the wave equations do not have a necessary (minimum sufficient) smoothness. The minimum possible necessary and sufficient smoothness of the right-hand side of the general telegraph equation, respectively, is expressed by fundamentally new smoothness conditions (8) and (32), which are absent in the works of other authors.

Remark 1 *For example, in theorem 2.1 of the work, [20], with condition (32) on page 26 the replacement of integration variables on page 27 proves the necessity and sufficiency of only the continuity of the right-hand side f of the string oscillation equation for the doubly continuous differentiability of solution to the Cauchy problem when the function f depends only on the coordinates of the string points or time. If f depends on the coordinates of the string points x and time t, then in, [20], on page 52 there is an example of a continuous function $f(x,t) = 0$ for $x \in [0,+\infty)$, $t \in [0,1)$ and $f(x,t) = x(t-1)$ for $x \in [0,+\infty)$, $t \in [1,+\infty)$ that satisfies conditions (32) with a discontinuous time derivative $\partial f(x,t)/\partial t$.*

In our article, the same is also stated in Corollaries 5, 9, 12 and 6, 10, 13, respectively.

2 Statement of the generalized Cauchy problem

Solve the Cauchy problem for a general telegraph equation with real variable coefficients under Cauchy conditions on the curve l in the plane $\mathbb{R}^2$:

$$\mathcal{L}\,u(x,t) \equiv u_{tt}(x,t) - a^2(x,t)u_{xx}(x,t)+$$
$$+b(x,t)u_t(x,t) + c(x,t)u_x(x,t)+$$
$$+\,q(x,t)u(x,t) \;=\; f(x,t),\; (x,t) \in \mathbb{R}^2 \backslash l, \qquad (1)$$

$$u|_l = \varphi(x),\; u_{\vec{n}}|_l = \frac{\partial u}{\partial \vec{n}}\Big|_l = \psi(x),\; \vec{n}(x,t)\perp l, \qquad (2)$$

where the coefficients of the equation a, b, c, q are real functions and the input data of the problem f, φ, ψ are the given real functions of their independent variables x and t, $(\partial u/\partial \vec{n})|_l$ is the derivative of the normal to the curve l of the equation $t = \chi(x)$, $x \in \mathbb{R}$, $\mathbb{R} = (-\infty,+\infty)$. Without excluding the generality of the Cauchy problem in the plane $\mathbb{R}^2$, we study in detail this Cauchy problem (1), (2) only in part of the plane $G = \{(x,t) \in \mathbb{R}^2 : t \geq \chi(x),\, x \in \mathbb{R}\}$ (see after the remark 4). By the number of subscripts of functions, we denote the orders of their partial derivatives.

Let $C^k(\Omega)$ be the set of k times continuously differentiable functions on a subset $\Omega \subset \mathbb{R}^2$, $C(\Omega)$ be the set of continuous functions on a subset $\Omega \subset \mathbb{R}^2$. By the number of strokes over the functions of one variable, we denote the orders of their ordinary derivatives with respect to this variable.

Definition 1 *Classical solutions* of the Cauchy problem (1), (2) on G are called functions $u \in C^2(G)$, satisfying equation (1) on $\dot{G} = \{(x,t) \in \mathbb{R}^2 : t > \chi(x),\, x \in \mathbb{R}\}$ in the usual sense, and the Cauchy conditions (2) in the sense of the values of the limits of the functions $u(\dot{x},\dot{t})$ and $u_{\vec{n}}(\dot{x},\dot{t})$ at internal points $(\dot{x},\dot{t}) \in \dot{G}$ when $\dot{x} \to x$ and $\dot{t} \to t = \chi(x)$.

Equation (1) has characteristic differential equations

$$dx = (-1)^i a(x,t)dt,\; i = 1,\, 2, \qquad (3)$$

which in G correspond to two different families of implicit characteristics

$$g_i(x,t) = C_i,\; C_i \in \mathbb{R},\; i = 1,\, 2. \qquad (4)$$

If the coefficient $a(x,t) \geq a_0 > 0$, $(x,t) \in G$, then the characteristics $g_1(x,t) = C_1$, $C_1 \in \mathbb{R}$, are strictly decreasing, and the characteristics $g_2(x,t) = C_2$, $C_2 \in \mathbb{R}$, strictly increase with respect to the variable x on the set G of the plane Oxt, since by virtue of equations (3) the derivative $dx/dt = -a(x,t) \leq -a_0 < 0$ for $i = 1$ and $dx/dt = a(x,t) \geq a_0 > 0$ for $i = 2$. Therefore, implicit functions $y_i = g_i(x,t)$, $t \geq \chi(x)$, $x \in \mathbb{R}$, have explicit strictly monotone inverse functions $x = h_i\{y_i,t\}$, $t \geq \chi(x)$, and $t = h^{(i)}[x,y_i]$, $x \in \mathbb{R}$, $i = 1,\, 2$, for which, by definition of inverse functions, the following conversion identities from the article, [2], are fulfilled:

$$g_i(h_i\{y_i,t\},t) = y_i,\; t \geq \chi(x),$$
$$h_i\{g_i(x,t),t\} = x,\; x \in \mathbb{R},\; i = 1,\, 2, \qquad (5)$$

$$g_i(x,h^{(i)}[x,y_i]) = y_i\;\; x \in \mathbb{R},$$
$$h^{(i)}[x,g_i(x,t)] = t,\; t \geq \chi(x),\; i = 1,\, 2, \qquad (6)$$

$$h_i\{y_i,h^{(i)}[x,y_i]\} = x,\; x \in \mathbb{R},$$
$$h^{(i)}[h_i\{y_i,t\},y_i] = t,\; t \geq \chi(x),\; i = 1,\, 2. \qquad (7)$$

If the function $a \in C^2(G)$, then the implicit functions g_i, h_i, $h^{(i)} \in C^2(G)$ by x, t, y_i, $i = 1,\, 2$, [2].

The Cauchy problem (1), (2) is studied in a set of classical solutions by Lomovtsev's new implicit characteristics method, which uses only differential equations (3) and conversion identities (5)–(7).

Remark 2 *In the case of* $a(x,t) = a = const > 0$, $(x,t) \in G$, *they are functions:* $g_1(x,t) = x + at$, $g_2(x,t) = x - at$, $h_1\{y_1,t\} = y_1 - at$, $h_2\{y_2,t\} = y_2 + at$, $h^{(1)}[x,y_1] = (y_1 - x)/a$, $h^{(2)}[x,y_2] = (x - y_2)/a$, *[20].*

For unique solvability of the Cauchy problem, the curve l must be expressed by some, at least, continuously differentiable function on plane and satisfy natural requirement that characteristics $g_i(x,t) = C_i$, $C_i \in \mathbb{R}$, $i = 1, 2$, of equation (1) were not tangent to curve l and could intersect l no more than once.

3 The Riemann formula for the classical solution of the generalized Cauchy problem

If the carrier l of the Cauchy data φ, ψ is given by the equation $t = \chi(x)$, where $\chi \in C^2(\mathbb{R})$, then the Riemann formula for the classical solution of this Cauchy problem contains the following

Theorem 3 *Let the coefficients are* $a(x,t) \geq a_0 > 0$, $\in G$, $a \in C^2(G)$, b, c, $q \in C^1(G)$, *and each of the characteristics* $g_i(x,t) = C_i, i = 1,2$, *is not tangent to curve l of smoothness* $\chi \in C^2(\mathbb{R})$ *and intersects l at most once. There is a unique and stable on* f, φ, ψ *classical solution* $u \in C^2(G)$ *of the Cauchy problem (1), (2) if and only if the input data has smoothness* $f \in C(G)$, $\varphi \in C^2(\mathbb{R})$, $\psi \in C^1(\mathbb{R})$ *and*

$$H_i(x,t) \equiv \int\limits_{\chi(s_0)}^{t+\chi(x)} f(h_i\{g_i(x,t),\tau\},\tau)\, d\tau \in C^1(G)$$

$$\forall\, (s_0, \chi(s_0)) \in l,\; i = 1,\, 2. \tag{8}$$

The classical solution $u \in C^2(G)$ *to the Cauchy problem (1), (2) for* $(x,t) \in G$ *is the function*

$$u(x,t) = \frac{(a\,u\,v)\Big(s_1(x,t), \chi(s_1(x,t))\Big)}{2a(x,t)} +$$

$$+ \frac{(a\,u\,v)\Big(s_2(x,t), \chi(s_2(x,t))\Big)}{2a(x,t)} +$$

$$+ \frac{1}{2a(x,t)} \int\limits_{s_2(x,t)}^{s_1(x,t)} \left\{ \Big[\psi(s)\left(1 - \chi'^2(s)a^2(s,\chi(s))\right) + \right.$$

$$+ \chi'(s)\varphi'(s)\left(1 + a^2(s,\chi(s))\right)\Big] \frac{v(s,\chi(s))}{\sqrt{1+\chi'(s)^2}} -$$

$$- \varphi(s)\Big[v_\tau(s,\tau)|_{\tau=\chi(s)} - b(s,\chi(s))v(s,\chi(s)) +$$

$$+ \chi'(s)\Big((a^2(s,\tau)v(s,\tau))_s|_{\tau=\chi(s)} +$$

$$\left. + c(s,\chi(s))v(s,\chi(s))\Big)\Big]\right\} ds +$$

$$+ \frac{1}{2a(x,t)} \int\limits_{\triangle MPQ} f(s,\tau)v(s,\tau;\, x,t)\, ds\, d\tau. \tag{9}$$

It is here $u\Big(s_i(x,t), \chi(s_i(x,t))\Big) = \varphi(s_i(x,t))$, $i = 1, 2$, *due to Cauchy conditions (2),* $s_i \in C^2(G)$ *are solutions of equations* $g_i(s_i, \chi(s_i)) = g_i(x,t)$, $i = 1, 2$, *triangle* MPQ *is a curved characteristic triangle with vertex* $M(x,t)$ *and vertices* $P\Big(s_2(x,t), \chi(s_2(x,t))\Big)$, $Q\Big(s_1(x,t), \chi(s_1(x,t))\Big)$ *of its curved base* PQ *and* $v(s,\tau) = v(s,\tau;x,t)$ *is a corresponding Riemann function.*

Proof. First, we derive the formula for the formal solution of the Cauchy problem. Equation (1) for any functions $u \in C^2(G)$ multiply by any functions $v \in C^2(G)$ and using obvious equalities

$$u_{tt}\, v = (u_t\, v)_t - u_t\, v_t =$$

$$= (u_t\, v)_t - (u\, v_t)_t + u\, v_{tt},$$

$$a^2 u_{xx}\, v = (u_x\, a^2 v)_x - u_x\, (a^2 v)_x =$$

$$= (u_x\, a^2 v)_x - (u\,(a^2 v)_x)_x + u\,(a^2 v)_{xx},$$

$$b u_t\, v = (ubv)_t - u\,(bv)_t,$$

$$c u_x\, v = (ucv)_x - u\,(cv)_x$$

we come to the identity

$$(\mathcal{L}\, u(x,t))\, v(x,t) - u(x,t)\,(\mathcal{M}\, v(x,t)) =$$

$$= \frac{\partial H(u(x,t),v(x,t))}{\partial t} + \frac{\partial K(u(x,t),v(x,t))}{\partial x} \tag{10}$$

for all u, $v \in C^2(G)$. It is here

$$\mathcal{M}\, v = v_{tt} - (a^2 v)_{xx} - (bv)_t - (cv)_x + qv,$$

$$H(u,v) = u_t\, v - u\, v_t + buv =$$

$$= (uv)_t - u[2v_t - bv], \tag{11}$$

$$K(u,v) = -u_x\, a^2 v + u\,(a^2 v)_x + cuv =$$

$$= -(a^2 uv)_x + u[2(a^2 v)_x + cv]. \tag{12}$$

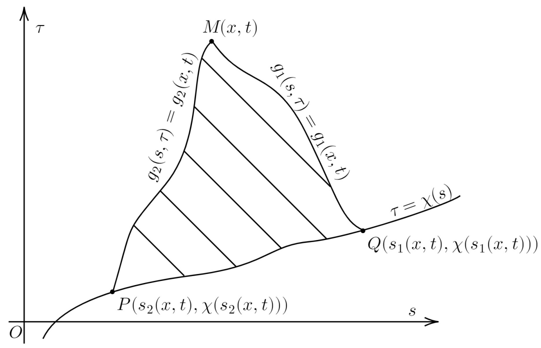

Figure 1: Curvilinear characteristic triangle $\triangle MPQ$ in $\mathbb{R}^2$.

The differential operator $\mathcal{M}$, which is the conjugate operator to the differential operator $\mathcal{L}$ in the sense of Schwartz distributions $\mathcal{D}'(\dot{G})$, [21], is usually called the formally conjugate operator to the operator $\mathcal{L}$. In view of the well-known Green formula, the double integral of the identity (10) over the curved characteristic triangle $\triangle MPQ$ in G with any vertex $M(x,t) \in G$ and the vertices of its base $P(s_2(x,t), \chi(s_2(x,t)))$ and $Q(s_1(x,t), \chi(s_1(x,t)))$ on the curve l is equal to

$$\int\limits_{\triangle MPQ} [(\mathcal{L}u(s,\tau))\, v(s,\tau) -$$

$$-u(s,\tau)\,(\mathcal{M}v(s,\tau))] ds\, d\tau =$$

$$= \int\limits_{\triangle MPQ} \Big[\frac{\partial H(u(s,\tau), v(s,\tau))}{\partial \tau} +$$

$$+ \frac{\partial K(u(s,\tau), v(s,\tau))}{\partial s}\Big] ds\, d\tau =$$

$$= \int\limits_{l^+} [K(u(s,\tau), v(s,\tau)) d\tau -$$

$$-H(u(s,\tau), v(s,\tau)) ds], \qquad (13)$$

where $l^+ = QM \cup MP \cup PQ$ is the contour of a curved triangle $\triangle MPQ$ with a positive bypass direction, due to the right orientation of the plane $Os\tau$ in Fig. 1.

Since the curve l is given by the equation $\tau = \chi(s) \in C^2(\mathbb{R})$ on the plane $Os\tau$, then the coordinates of the points $P(s_2, \tau_2)$ and $Q(s_1, \tau_1)$ are solutions of equation systems:

$$g_i(s_i, \tau_i) = g_i(x,t),$$

$$\tau_i = \chi(s_i),\; i = 1,\, 2.$$

Each of these two systems of equations has at most one solution, since, according to the assumptions of Theorem 3, the strictly monotone characteristics $g_i(x,t) = C_i$, $C_i \in \mathbb{R}$, $i = 1, 2$, intersect the curve l no more than once. Solutions of these equation systems are solutions of implicit equations $s_i = h_i\{g_i(x,t), \tau_i\} = h_i\{g_i(x,t), \chi(s_i)\}$, $i = 1$, 2. This means that the functions $s_i = s_i(x,t) \in C^2(G)$, $i = 1$, 2, depend only on the coordinates of the point $M(x,t)$. Therefore, the coordinates of the vertices Q and P of the base of the triangle $\triangle MPQ$ are expressed as $Q\Big(s_1(x,t), \chi(s_1(x,t))\Big)$ and $P\Big(s_2(x,t), \chi(s_2(x,t))\Big)$.

In the curvilinear integral (13) using expressions (11) and (12), the differential equation of the characteristic from (3) for $i = 1$ and the obvious equalities

$$(u\,v)_\tau a = (a\,u\,v)_\tau - a_\tau u\, v,$$

$$(a^2 u\, v)_s (1/a) = (a\,u\,v)_s - a^2 uv(1/a)_s =$$

$$= (a\,u\,v)_s + a_s u\, v,$$

we calculate the integral along the characteristic QM of the equation $g_1(s,\tau) = g_1(x,t)$:

$$\int\limits_Q^M [K(u(s,\tau), v(s,\tau)) d\tau - H(u(s,\tau), v(s,\tau)) ds] =$$

$$= \int\limits_Q^M [(a^2 u\, v)_s (1/a)\, ds + (u\,v)_\tau\, a\, d\tau] +$$

$$+ \int\limits_Q^M \Big(u[2(a^2 v)_s + cv] d\tau + u[2v_\tau - bv]\, ds\Big) =$$

$$= \int\limits_Q^M d(a\,u\,v) + \int\limits_Q^M \Big(u[2(a^2 v)_s +$$

$$+(c - a_\tau)v] d\tau + u\,[2v_\tau + (a_s - b)v]\, ds\Big) =$$

$$= \int\limits_Q^M d(a\,u\,v) + \int\limits_Q^M \Big(u[2(a^2 v)_s + (c - a_\tau)v] -$$

$$-au\,[2v_\tau + (a_s - b)v]\Big)\, d\tau =$$

$$= \int\limits_Q^M d(a\,u\,v) +$$

$$+\int_Q^M u\{4aa_s v + 2a^2 v_s + (c - a_\tau)v -$$

$$-2av_\tau - (a_s - b)av\}\, d\tau =$$

$$= \int_Q^M d(a\,u\,v) + \int_Q^M u\{-4av_\tau +$$

$$+[4aa_s + (c - a_\tau) - (a_s - b)a]v\}\, d\tau =$$

$$= (a\,u\,v)(M) - (a\,u\,v)(Q) -$$

$$\int_{\chi(s_1(x,t))}^t u\{4a\,v_\tau - [ab - 4a_\tau + c]v\}\, d\tau =$$

$$= (a\,u\,v)(x,t) - (a\,u\,v)\Big(s_1(x,t), \chi(s_1(x,t))\Big) -$$

$$-\int_{\chi(s_1(x,t))}^t u(s,\tau)\{4a(s,\tau)\,v_\tau(s,\tau) - [a(s,\tau)b(s,\tau) -$$

$$-4a_\tau(s,\tau) + c(s,\tau)]v(s,\tau)\}\, d\tau. \quad (14)$$

Here on the characteristic QM for $i = 1$, we used the well-known differential equation (3) of the implicit characteristics from (4) and for the functions $w \in C^1$(G) a new representation from

$$w_s(s,\tau) = (-1)^i w_\tau(s,\tau)/a(s,\tau),\ i = 1, 2. \quad (15)$$

Since on each of the characteristics QM and MP variables $s = s(\tau)$, $\tau = \tau(s)$ are simultaneously mutually dependent, that is, respectively for $i = 1$ and $i = 2$ variables $s = h_i\{g_i(x,t), \tau\}$ and $\tau = h^{(i)}[s, g_i(x,t)]$, according to the inversion formula (5)–(7), then these representations follow from the obvious formulas of the first partial derivatives

$$w_s\,(s, \tau(s)) =$$

$$= w_s\,(s,\tau)\,|_{\tau=\tau(s)} + w_\tau\,(s,\tau)\,|_{\tau=\tau(s)}\tau'(s) =$$

$$= w_s\,(s,\tau)\,|_{\tau=\tau(s)} + (-1)^i w_\tau\,(s,\tau)\,|_{\tau=\tau(s)}/a(s,\tau),$$

$$w_\tau\,(s(\tau), \tau) =$$

$$= w_\tau\,(s,\tau)\,|_{s=s(\tau)} + w_s\,(s,\tau)\,|_{s=s(\tau)}s'(\tau) =$$

$$= w_\tau\,(s,\tau)\,|_{s=s(\tau)} + (-1)^i w_s\,(s,\tau)\,|_{s=s(\tau)}a(s,\tau)$$

since $\tau'(s) = (-1)^i/a(s,\tau)$, $s'(\tau) = (-1)^i a(s,\tau)$, $i = 1, 2$, also due to the formulas (4).

In the previous equation from (14), to reduce a curved integral of the second type along the characteristic QM to an ordinary definite integral, we applied the parametric representation of the curve QM : $s = s_1(\tau) = h_1\{g_1(x,t), \tau\}$, $\tau' = \tau$, $\chi(s_1) \le \tau \le t$, and the differential equation from (3) for $i = 1$. Using the inversion identities (5) – (7), we conclude that if integration occurs over the variable s, then the variable $\tau = h^{(1)}[s, g_1(x,t)]$, and if by the variable τ, then the variable $s = h_1\{g_1(x,t),\, \tau\}$.

Using the characteristic differential equation from (3) for $i = 2$, we similarly calculate the integral (13) along the characteristic MP with the equation $g_2(s,\tau) = g_2(x,t)$:

$$\int_M^P [K(u(s,\tau), v(s,\tau))d\tau - H(u(s,\tau), v(s,\tau))ds] =$$

$$= -\int_M^P [(u\,v)_\tau\, a\, d\tau + (a^2 u\,v)_s (1/a)\, ds] +$$

$$+\int_M^P \Big(u[2v_\tau - bv]\, ds + u[2(a^2 v)_s + cv] d\tau \Big) =$$

$$= -\int_M^P d(a\,u\,v) + \int_M^P \Big(u\,[2v_\tau -$$

$$-(a_s + b)v]\, ds + u[2(a^2 v)_s + (c + a_\tau)v] d\tau \Big) =$$

$$= (a\,u\,v)(M) - (a\,u\,v)(P) -$$

$$-\int_{\chi(s_2(x,t))}^t u\{4a\,v_\tau - [ab - 4a_\tau - c]v\}\, d\tau =$$

$$= (a\,u\,v)(x,t) - (a\,u\,v)\Big(s_2(x,t), \chi(s_2(x,t))\Big) -$$

$$-\int_{\chi(s_2(x,t))}^t u\{4a(s,\tau)\,v_\tau(s,\tau) - [a(s,\tau)b(s,\tau) -$$

$$-4a_\tau(s,\tau) - c(s,\tau)]v(s,\tau)\}\, d\tau. \quad (16)$$

In the previous equation from (16), to reduce a curved integral of the second type along the characteristic MP to an ordinary definite integral, we applied the parametric representation of the curve $MP : s = s_2(\tau) = h_2\{g_2(x,t), \tau\}$, $\tau' = \tau$, $\chi(s_2(x,t)) \le \tau \le t$. In this previous equation, we also applied the differential equation from (3) and the above partial derivative representations from (15) for $i = 2$. In all expressions under the integral sign, all functions of two variables depend on s, τ. Moreover, using formulas (5)–(7), we conclude that if the integration occurs with the variable s, then the variable $\tau = h^{(2)}[s, g_2(x,t)]$, and if with τ, then $s = h_2\{g_2(x,t),\, \tau\}$.

Let the function $v(s,\tau)=v(s,\tau;x,t)$ with parameters (x,t) be a classical solution of a homogeneous formally conjugate differential equation

$$\mathcal{M}\,v(s,\tau)=0,\ (s,\tau)\in\Delta MPQ \qquad (17)$$

with conditions, respectively, on the characteristics of QM and MP

$$4a(s,\tau)v_\tau(s,\tau)-[a(s,\tau)b(s,\tau)-$$
$$-4a_\tau(s,\tau)+c(s,\tau)]v(s,\tau)=0,$$
$$4a(s,\tau)v_\tau(s,\tau)-[a(s,\tau)b(s,\tau)-$$
$$-4a_\tau(s,\tau)-c(s,\tau)]v(s,\tau)=0 \qquad (18)$$

in integrals (14) and (16) and the matching condition

$$v(M)=1 \qquad (19)$$

The conditions (18), (19) are obviously equivalent to the two agreed Goursat conditions

$$v(s,\tau)=\exp\left\{\int_t^\tau k_1(h_1\{g_1(x,t),\ \rho\},\rho)\,d\rho\right\},$$
$$g_1(s,\tau)=g_1(x,t),\ \tau\in[\chi(s_1(x,t)),t],$$
$$v(s,\tau)=\exp\left\{\int_t^\tau k_2(h_2\{g_2(x,t),\ \rho\},\rho)\,d\rho\right\},$$
$$g_2(s,\tau)=g_2(x,t),\ \tau\in[\chi(s_2(x,t)),t], \qquad (20)$$

where the functions $k_1(s,\tau)=\{a(s,\tau)b(s,\tau)-4a_\tau(s,\tau)+c(s,\tau)\}/4a(s,\tau)$ on the curve QM and $k_2(s,\tau)=\{a(s,\tau)b(s,\tau)-4a_\tau(s,\tau)-c(s,\tau)\}/4a(s,\tau)$ on the curve MP. It is well known that the Goursat problem (17), (20) with coefficients $a\in C^2(G)$, b, c, $q\in C^1(G)$ has a unique classical solution $v\in C^2(\Delta MPQ)$, which is naturally called the Riemann function of the Cauchy problem (1), (2) on G. In the general case, the Riemann function is uniquely found by the method of successive approximations, [19], p. 129–135.

According to the Cauchy conditions (2), the values of the solution at the base vertices of the triangle are known: $u\Big(s_i(x,t),\chi(s_i(x,t))\Big)=\varphi(s_i(x,t))$, $i=1,2$. In formulas (13) we assume $\mathcal{L}\,u(s,\tau)=f(s,\tau)$, $\mathcal{M}\,v(s,\tau)=0$ on the triangle ΔMPQ and in by virtue of the relations (17)–(20) and the equalities (14), (16) we obtain a formal solution of the Cauchy problem (1), (2) for all $(x,t)\in G$:

$$u(x,t)=\frac{(a\,u\,v)\Big(s_1(x,t),\chi(s_1(x,t))\Big)}{2a(x,t)}+$$
$$+\frac{(a\,u\,v)\Big(s_2(x,t),\chi(s_2(x,t))\Big)}{2a(x,t)}+$$
$$+\frac{1}{2a(x,t)}\int_P^Q[H(u(s,\tau),v(s,\tau))\,ds-$$
$$-K(u(s,\tau),v(s,\tau))d\tau]+$$
$$+\frac{1}{2a(x,t)}\int\limits_{\triangle MPQ} f(s,\tau)v(s,\tau;x,t)\,ds\,d\tau. \qquad (21)$$

If $\vec{e}$ is a tangent vector to the curve l, then from (2) the derivative $u_{\vec{e}}|_l=\partial u/\partial\vec{e}|_l=\varphi'(x)$ and the first partial derivatives from u are calculated by the formulas from, [19], p. 139:

$$u_s|_l=u_{\vec{e}}|_l\cos(\vec{e},s)+u_{\vec{n}}|_l\cos(\vec{n},s)=$$
$$=\frac{\varphi'(s)-\chi'(s)\psi(s)}{\sqrt{1+\chi'(s)^2}}, \qquad (22)$$
$$u_\tau|_l=u_{\vec{e}}|_l\cos(\vec{e},\tau)+u_{\vec{n}}|_l\cos(\vec{n},\tau)=$$
$$=\frac{\varphi'(s)\chi'(s)+\psi(s)}{\sqrt{1+\chi'(s)^2}} \qquad (23)$$

Substituting partial derivatives (22), (22) in (11), (12) and under a curved integral of the second type along the curve PQ in (21) and considering $d\tau=\chi'(s)ds$, we find

$$\int_P^Q[H(u(s,\tau),v(s,\tau))ds-K(u(s,\tau),v(s,\tau))d\tau]=$$
$$=\int_{s_2(x,t)}^{s_1(x,t)}\Big\{\Big[u_\tau(s,\tau)|_{\tau=\chi(s)}\,v(s,\chi(s))-$$
$$-u(s,\chi(s))\,v_\tau(s,\tau)|_{\tau=\chi(s)}+$$
$$+b(s,\chi(s))u(s,\chi(s))v(s,\chi(s))\Big]ds+$$
$$+\Big[u_s(s,\tau)|_{\tau=\chi(s)}\,a^2(s,\chi(s))v(s,\chi(s))-$$
$$-u(s,\chi(s))\,(a^2(s,\tau)v(s,\tau))_s|_{\tau=\chi(s)}-$$
$$-c(s,\chi(s))u(s,\chi(s))v(s,\chi(s))\Big]\chi'(s)ds\Big\}=$$
$$=\int_{s_2(x,t)}^{s_1(x,t)}\Big\{\frac{\varphi'(s)\chi'(s)+\psi(s)}{\sqrt{1+\chi'(s)^2}}\,v(s,\chi(s))-$$
$$-u(s,\chi(s))\,v_\tau(s,\tau)|_{\tau=\chi(s)}+$$

$$+b(s,\chi(s))u(s,\chi(s))v(s,\chi(s))+$$
$$+\frac{\varphi'(s)-\chi'(s)\psi(s)}{\sqrt{1+\chi'(s)^2}}\times$$
$$\times a^2(s,\chi(s))v(s,\chi(s))\chi'(s)-$$
$$-u(s,\chi(s))\left(a^2(s,\tau)v(s,\tau)\right)_s|_{\tau=\chi(s)}\chi'(s)-$$
$$-c(s,\chi(s))u(s,\chi(s))v(s,\chi(s))\chi'(s)\Big\}ds=$$
$$=\int\limits_{s_2(x,t)}^{s_1(x,t)}\Big\{\frac{\varphi'(s)\chi'(s)+\psi(s)}{\sqrt{1+\chi'(s)^2}}\,v(s,\chi(s))-$$
$$-\varphi(s)\,v_\tau(s,\tau)|_{\tau=\chi(s)}+$$
$$+b(s,\chi(s))\varphi(s)v(s,\chi(s))+$$
$$+\frac{\varphi'(s)-\chi'(s)\psi(s)}{\sqrt{1+\chi'(s)^2}}\times$$
$$\times a^2(s,\chi(s))v(s,\chi(s))\chi'(s)-$$
$$-\varphi(s)\left(a^2(s,\tau)v(s,\tau)\right)_s|_{\tau=\chi(s)}\chi'(s)-$$
$$-c(s,\chi(s))\varphi(s)v(s,\chi(s))\chi'(s)\Big\}ds=$$
$$=\int\limits_{s_2(x,t)}^{s_1(x,t)}\Big\{\Big[\psi(s)\Big(1-\chi'^2(s)a^2(s,\chi(s))\Big)+$$
$$+\chi'(s)\varphi'(s)\Big(1+a^2(s,\chi(s))\Big)\Big]\frac{v(s,\chi(s))}{\sqrt{1+\chi'(s)^2}}-$$
$$-\varphi(s)\Big[v_\tau(s,\tau)|_{\tau=\chi(s)}-b(s,\chi(s))v(s,\chi(s))+$$
$$+\chi'(s)\Big(\left(a^2(s,\tau)v(s,\tau)\right)_s|_{\tau=\chi(s)}+$$
$$+c(s,\chi(s))v(s,\chi(s))\Big)\Big]\Big\}ds=$$
$$=\int\limits_{s_2(x,t)}^{s_1(x,t)}\Big\{\Big[\psi(s)\Big(1-\chi'^2(s)a^2(s,\chi(s))\Big)+$$
$$+\chi'(s)\varphi'(s)\Big(1+a^2(s,\chi(s))\Big)\Big]\frac{v(s,\chi(s))}{\sqrt{1+\chi'^2(s)}}-$$
$$-\varphi(s)\Big[v_\tau(s,\tau)|_{\tau=\chi(s)}+$$
$$+\chi'(s)a^2(s,\chi(s))v_s(s,\tau)|_{\tau=\chi(s)}+$$
$$+\Big(\chi'(s)(c(s,\chi(s))+$$
$$+2a(s,\chi(s))a_s(s,\tau)|_{\tau=\chi(s)})-$$
$$-b(s,\chi(s))\Big)v(s,\chi(s))\Big]\Big\}ds. \quad (24)$$

We substitute the expression (24) into the formula (21). As a result, the final formula for the formal solution of the Cauchy problem (1), (2) on the set G takes the form (9).

Now we justify the necessity and sufficiency of the smoothness for the right-hand side of the equation indicated in Theorem 3 and the initial data for the doubly continuous differentiability of the function (9) on G. Directly from the telegraph equation (1), Cauchy conditions (2) and definition 1 of classical solutions $u\in C^2(G)$ of this Cauchy problem implies the need for smoothness of the input data $f\in C(G)$, $\varphi\in C^2(\mathbb{R})$, $\psi\in C^1(\mathbb{R})$. With the Riemann function $v\in C^2(G)$ smoothness of the initial data $\varphi\in C^2(\mathbb{R})$, $\psi\in C^1(\mathbb{R})$ is obviously sufficient for twice continuous differentiability on G of the first term and the curvilinear integral of the function (9) on G. In the plane of $Os\tau$ by replacing variables

$$\tilde{s}=s,\;\;\tilde{\tau}=\tau-\chi(s) \quad (25)$$

with a non-degenerate Jacobian $J(s,\tau)=\tilde{s}'_s\tilde{\tau}'_\tau-\tilde{s}'_\tau\tilde{\tau}'_s=1\neq 0$ characteristic triangle $\triangle MPQ$ with curved base PQ from Fig. 1 is reduced to the corresponding characteristic triangle $\triangle\widetilde{M}\widetilde{P}\widetilde{Q}$ with vertex $\widetilde{M}(x,t)$, vertices $\widetilde{P}(\tilde{h}_2\{\tilde{g}_2(x,t),0\},0)$, $\widetilde{Q}(\tilde{h}_1\{\tilde{g}_1(x,t),0\},0)$ of rectilinear base $\widetilde{P}\widetilde{Q}$ with the equation $\tilde{\tau}=0$, where the functions $\tilde{h}_i$, $\tilde{g}_i$ are found by replacing (25) from our functions h_i, g_i, $i=1,\,2$.

The double integral of the triangle $\triangle MPQ$ from (9) after replacing (25) becomes the corresponding double integral of the triangle $\triangle\widetilde{M}\widetilde{P}\widetilde{Q}$. These triangles are simultaneously twice continuously differentiable by x, t, since for $\chi\in C^2(\mathbb{R})$ replacement (25) is non-degenerate and twice continuously differentiable on G.

Moreover, this triangle $\triangle\widetilde{M}\widetilde{P}\widetilde{Q}$ even has the form of a triangle from Fig. 2, because the transformation (25) preserves the first variable s and has the type of rotation by the second variable τ.

Therefore, additional to the image $\tilde{f}\in C(\widetilde{G})$ of the right-hand side f after the change (25), the necessary and sufficient requirements

$$\tilde{H}_i(x,t)\equiv\int\limits_0^t\tilde{f}(\tilde{h}_i\{\tilde{g}_i(x,t),\tilde{\tau}\},\tilde{\tau})\,d\tilde{\tau}\in C^1(\widetilde{G}),$$
$$i=1,\,2, \quad (26)$$

after the reverse to (25) replacement of variables (see below (32))

$$s=\tilde{s},\;\tau=\tilde{\tau}+\chi(\tilde{s}) \quad (27)$$

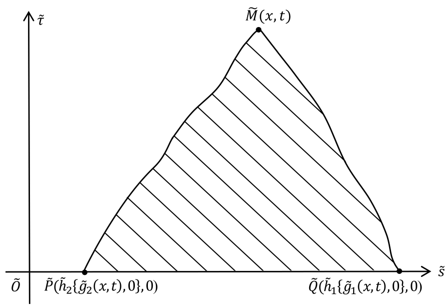

Figure 2: Characteristic triangle $\triangle\widetilde{M}\widetilde{P}\widetilde{Q}$ in $\mathbb{R}^2$.

become additional to $f \in C(G)$ necessary and sufficient smoothness requirements (8) on the double integral of (9) in Theorem 3.

If in the reverse substitution (27) the variable $\tilde{\tau} = 0$, then the variable $\tau = \chi(\tilde{s}) = \chi(s)$ for all points $(s, \chi(s)) \in l$ of the curve in the lower limits of integrals (8). If in (27) the variable $\tilde{\tau} = t$, then the variable $\tau = t + \chi(\tilde{s}) = t + \chi(x)$ in the upper limits of integrals (27), since $\tilde{s} = s = x$. Indeed, if in a triangle $\triangle MPQ$ the second coordinate is $\tau = t$, then it is obvious that the first coordinate must necessarily be $s = x$. This is also the second inversion identities (5) confirms when one variable $\tau = t$, then the other variable $s = h_i\{g_i(x,t), \tau\} = h_i\{g_i(x,t), t\} = x$ and vice versa. In integrals (26), the Jacobian of the inverse substitution to (25) is $\tilde{J}(\tilde{s}, \tilde{\tau}) = s'_{\tilde{s}}\tau'_{\tilde{\tau}} - s'_{\tilde{\tau}}\tau'_{\tilde{s}} = 1$ and the upper half-plane $\widetilde{G}$ is the image of the task set G of the Cauchy problem (1), (2) after replacement (25).

The uniqueness of the classical solution (9) of the Cauchy problem (1), (2) can be justified in the same way as in the textbook, [19], p. 139. Its stability with respect to f, φ, ψ ensures the established existence and uniqueness of the classical solution (9) by Banach's closed graph theorem or Banach's open mapping theorem. To conclude the proof of Theorem 3, we say that stability also follows from the formula (9).

Remark 4 *The Riemann formula for the classical solution of the Cauchy problem (1), (2) on G with the coefficient $a(x,t) \equiv 1$ is given in, [19], p. 139.*

For the lower half-plane $\mathbb{R}^2 \backslash G$ in (13) the orientation of the contour l^+ of the characteristic triangle $\triangle MPQ$, does not change if, as before, at the points P and Q respectively intersect the characteristics $g_2(s,\tau) = g_2(x,t)$ and $g_1(s,\tau) = g_1(x,t)$ with the curve l, since the points P and Q are swapped in $\mathbb{R}^2 \backslash G$. Thus, the unique and stable classical solution of the Cauchy problem (1), (2) on the lower half-plane $\mathbb{R}^2 \backslash G$ is also given by the formula (9).

Corollary 5 *If the right-hand side f of the equation (1) depends only on x or t and is continuous $f \in C(\mathbb{R})$, then Theorem 3 is true without smoothness requirements (8).*

Proof. It is easier to verify the continuous differentiability of the integrals from (26) on $\widetilde{G}$, which is equivalent to the continuous differentiability of the integrals from (8) on G. If the function $\tilde{f} = \tilde{f}(\tilde{\tau})$ does not depend on $\tilde{s}$, then the integrals from (26) are equal to the integral

$$\tilde{H}_i(t) \equiv \int_0^t \tilde{f}(\tilde{\tau})\, d\tilde{\tau}, \ i = 1, 2, \tag{28}$$

which is continuously differentiable with respect to t for continuous functions $\tilde{f}$, since the first derivative $\partial\tilde{H}_i(t)/\partial t = \tilde{f}(t)$ is continuous. If the function $\tilde{f} = \tilde{f}(\tilde{s})$ does not depend on $\tilde{\tau}$, then integrals from (26) after change of integration variables $\tilde{s}_i = \tilde{h}_i\{\tilde{g}_i(x,t), \tilde{\tau}\}$, $i = 1, 2$, are equal to integrals

$$\tilde{H}_i(x,t) \equiv \int_0^t \tilde{f}(\tilde{h}_i\{\tilde{g}_i(x,t), \tilde{\tau}\})\, d\tilde{\tau} =$$

$$= \int_{\tilde{h}_i\{\tilde{g}_i(x,t),0\}}^{t} \frac{\tilde{f}(\tilde{s}_i)}{(\partial\tilde{h}_i\{\tilde{g}_i(x,t), \tilde{\tau}\}/\partial\tilde{\tau})}\Big|_{\tilde{\tau}=\tilde{h}^{(i)}[\tilde{g}_i(x,t),\tilde{\tau}]} d\tilde{s}_i$$

$$\in C^1(\widetilde{G}), \ i = 1, 2. \tag{29}$$

Here the functions $\tilde{H}_i(x,t)$, $i = 1, 2$, are indeed continuously differentiable with respect to x and t for a continuous function $\tilde{f} = \tilde{f}(\tilde{s})$, because in the last integrals, it is explicitly independent of variables x and t. Corollary 5 is proved.

Corollary 6 *If the right-hand side f of the equation (1) depends on x and t, then in Theorem 3 the integrals (8) belong to the set $C^1(G)$ is equivalent to their belonging to the set $C^{(0,1)}(G)$ or $C^{(1,0)}(G)$. The sets $C^{(0,1)}(G)$ and $C^{(1,0)}(G)$ are sets of continuous or continuously differentiable with respect to x and continuously differentiable or continuous with respect to t functions on G.*

Proof. Firstly for smoother functions $\tilde{f} \in C^1(\widetilde{G})$, by changing the integration variables, we derive special equalities for the first partial derivatives of the integrals (26), which are equivalent to the integrals (8). Then the equalities of their first and last parts, which do not contain explicit derivatives of the functions $\tilde{f} \in C^1(\widetilde{G})$. are extended by passing to the limit along $\tilde{f}$ from smoother functions $\tilde{f} \in C^1(\widetilde{G})$ to continuous functions $\tilde{f} \in C(\widetilde{G})$, satisfying the integral smoothness requirements (26). Corollary 6 is proved.

We apply the derived Riemann formula of the classical solution and the correctness criterion of the Cauchy problem (1), (2) on G in global theorem 3 to its two important special cases, which partially confirm the validity of the results obtained.

4 The Cauchy problem for the general telegraph equation in the upper half-plane

A special case of the Cauchy problem (1), (2) in the plane is the following Cauchy problem in the upper half-plane:

$$u_{tt}(x,t) - a^2(x,t)u_{xx}(x,t) + b(x,t)u_t(x,t) +$$
$$+c(x,t)u_x(x,t) + q(x,t)u(x,t) = f(x,t),$$
$$(x,t) \in \dot{G}_\infty = \mathbb{R} \times (0,+\infty), \tag{30}$$
$$u|_{t=0} = \varphi(x),\ u_t|_{t=0} = \psi(x),$$
$$x \in \mathbb{R} = (-\infty,+\infty), \tag{31}$$

where the coefficients a, b, c, q are real functions and the input data f, φ, ψ are given real functions of their variables x and t. By the lower indices of the functions we indicate the variables of partial derivatives of these functions.

As above, the characteristic equations $dx - (-1)^i a(x,t)dt = 0$ give the characteristics $g_i(x,t) = C_i$, $i = 1, 2$. If $a(x,t) \geq a_0 > 0$, then they decrease strictly by t at $i = 1$ and increase at $i = 2$ with the growth of x, since $dx/dt = -a(x,t) \leq -a_0 < 0$ at $i = 1$ and $dx/dt = a(x,t) \geq a_0 > 0$ for $i = 2$. Therefore, the functions $y_i = g_i(x,t)$ have inverse functions $x = h_i\{y_i,t\}$, $t = h^{(i)}[x,y_i]$, $i = 1, 2$. Moreover, each of the characteristics $g_i(x,t) = C_i$, $i = 1, 2$, intersects the curve l of the equation $t = 0$ only once. If the coefficient is $a \in C^2(G_\infty)$, then the functions g_i, h_i, $h^{(i)}$, $i = 1, 2$, are twice continuously differentiable by their variables in G_∞, [2]. By definition of inverse functions, the conversion identities (5)–(7) are fulfilled.

Definition 7 *Classical solutions of the Cauchy problem (30), (31) on G_∞ are called functions $u \in C^2(G_\infty)$, $G_\infty = \mathbb{R} \times [0,+\infty)$, satisfying equation (30) on $\dot{G}_\infty = \mathbb{R} \times (0,+\infty)$ in the usual sense and the initial conditions (31) in the sense of the values of the limits of the functions $u(\dot{x},\dot{t})$ and $u_{\dot{t}}(\dot{x},\dot{t})$ at internal points $(\dot{x},\dot{t}) \in \dot{G}_\infty$ when $\dot{x} \to x$ and $\dot{t} \to 0$.*

Theorem 8 *Let in equation (30) the coefficients be $a(x,t) \geq a_0 > 0$, $(x,t) \in G_\infty$, $a \in C^2(G_\infty)$, b, c, $q \in C^1(G_\infty)$. The Cauchy problem (30), (31) on G_∞ has a unique and stable on f, φ, ψ classical solution $u \in C^2(G_\infty)$ if and only if when the right-hand side of the equation and the initial data are $f \in C(G_\infty)$, $\varphi \in C^2(\mathbb{R})$, $\psi \in C^1(\mathbb{R})$ and*

$$\int_0^t f(h_i\{g_i(x,t),\tau\},\tau)\,d\tau \in C^1(G_\infty), i = 1,2. \tag{32}$$

The classical solution $u \in C^2(G_\infty)$ of the Cauchy problem (30), (31) for all $(x,t) \in G_\infty$ is the function

$$u(x,t) = \frac{(auv)(h_2\{g_2(x,t),\,0\},0)}{2a(x,t)} +$$
$$+\frac{(auv)(h_1\{g_1(x,t),\,0\},0)}{2a(x,t)} +$$
$$+\frac{1}{2a(x,t)} \int_{h_2\{g_2(x,t),\,0\}}^{h_1\{g_1(x,t),\,0\}} [\psi(s)v(s,0) -$$
$$-\varphi(s)v_\tau(s,0) + b(s,0)\varphi(s)v(s,0)]\,ds +$$
$$+\frac{1}{2a(x,t)} \int_0^t d\tau \int_{h_2\{g_2(x,t),\tau\}}^{h_1\{g_1(x,t),\tau\}} f(s,\tau)v(s,\tau;x,t)ds. \tag{33}$$

By virtue of (31) $u(h_2\{g_2(x,t),\,0\},0) = \varphi(h_2\{g_2(x,t),\,0\})$, $u(h_1\{g_1(x,t),\,0\},0) = \varphi(h_1\{g_1(x,t),\,0\})$, Riemann function $v(s,\tau) = v(s,\tau;x,t)$ is solution of Goursat problem:

$$v_{\tau\tau}(s,\tau) - (a^2(s,\tau)v(s,\tau))_{ss} -$$
$$-(b(s,\tau)v(s,\tau))_\tau - (c(s,\tau)v(s,\tau))_s +$$
$$+q(s,\tau)v(s,\tau) = 0,\ (s,\tau) \in \Delta MPQ, \tag{34}$$
$$v(s,\tau) = \exp\left\{\int_t^\tau k_1(h_1\{g_1(x,t),\,\rho\},\rho)\,d\rho\right\},$$
$$g_1(s,\tau) = g_1(x,t),$$
$$v(s,\tau) = \exp\left\{\int_t^\tau k_2(h_2\{g_2(x,t),\,\rho\},\rho)\,d\rho\right\},$$

$$g_2(s,\tau)=g_2(x,t),\ \tau\in[0,t], \tag{35}$$

with functions $k_1(s,\tau)=\{a(s,\tau)b(s,\tau)-4a_\tau(s,\tau)+c(s,\tau)\}/4a(s,\tau)$ on the curve QM and $k_2(s,\tau)=\{a(s,\tau)b(s,\tau)-4a_\tau(s,\tau)-c(s,\tau)\}/4a(s,\tau)$ on the curve MP.

Proof. In the Cauchy problem (30), (31), the initial conditions are given at $t=0$. Therefore, in Theorem 3, the function $t=\chi(x)=0$ and hence its derivative $\chi'(x)=0$. Therefore, we have the functions $s_i(x,t)=h_i\{g_i(x,t),\,0\},i=1,2$, and hence the coordinates of the points $Q(h_1\{g_1(x,t),\,0\},0)$ and $P(h_2\{g_2(x,t),\,0\},0)$. In addition, the smoothness requirements for the right-hand side f of the equation (30) and the initial data (31), including integral requirements (8) for G, from Theorem 3 for $\chi(x)=0$ turn into equivalent smoothness requirements, including integral requirements (32), on G_∞ from Theorem 8. The Goursat problem (17), (20) on the triangle $\triangle MPQ$ with a curved base PQ at $\chi(x)=0$ becomes the Goursat problem (34), (35) on the triangle $\triangle MPQ$ with a rectilinear base PQ. Formulas (9) of the classical solution $u\in C^2(G)$ from Theorem 3 takes the form of formula (33) of the classical solution $u\in C^2(G_\infty)$ from Theorem 8, since the double repeated integral of (33) is equal to the double integral of (9) for $\chi(x)=0$. Theorem 8 is proved.

Corollaries 9, 10 follow from Corollaries 5, 6.

Corollary 9 *If the right-hand side f of the equation (30) depends only on x or t and is continuous $f\in C(\mathbb{R})$, then Theorem 8 is true without smoothness requirements (32).*

Corollary 10 *If the right-hand side f of the equation (30) depends on x and t, then in Theorem 8 the belonging of the integrals (32) to the set $C^1(G_\infty)$ is equivalent to their belonging to the set $C^{(0,1)}(G_\infty)$ or $C^{(1,0)}(G_\infty)$. The sets $C^{(0,1)}(G_\infty)$ and $C^{(1,0)}(G_\infty)$ are sets of continuous or continuously differentiable with respect to x and continuously differentiable or continuous with respect to t functions on G_∞.*

5 The Cauchy problem for the model telegraph equation in the upper half-plane.

In the half-plane $G_\infty=\mathbb{R}\times(0,+\infty)$, to solve the Cauchy problem for the model telegraph equation:

$$\hat{\mathcal{L}}\,u(x,t)\equiv u_{tt}(x,t)-a^2(x,t)u_{xx}(x,t)-$$

$$-a^{-1}(x,t)a_t(x,t)u_t(x,t)-$$

$$-a(x,t)a_x(x,t)u_x(x,t)=f(x,t),\ (x,t)\in\dot{G}_\infty \tag{36}$$

with the initial conditions

$$u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x),\ x\in\mathbb{R}. \tag{37}$$

The correctness criterion and the formula of the classical solution of this problem Cauchy gives

Theorem 11 *Let the coefficient be $a(x,t)\geq a_0>0,\ (x,t)\in G_\infty,\ a\in C^2(G_\infty)$. The Cauchy problem (36), (37) has a unique and stable on $f,\ \varphi,\ \psi$ classical solution $u\in C^2(G_\infty)$ if and only if $f\in C(G_\infty)$, $\varphi\in C^2(\mathbb{R}),\ \psi\in C^1(\mathbb{R})$ and (32) is true.*

The classical solution $u\in C^2(G_\infty)$ of the Cauchy problem (36), (37) on G_∞ is the function

$$u(x,t)=\frac{\varphi(h_2\{g_2(x,t),\,0\}+\varphi(h_1\{g_1(x,t),\,0\})}{2a(x,t)}+$$

$$+\frac{1}{2}\int\limits_{h_2\{g_2(x,t),\,0\}}^{h_1\{g_1(x,t),\,0\}}\frac{\psi(s)}{a(s,0)}\,ds+$$

$$+\frac{1}{2}\int\limits_0^t d\tau\int\limits_{h_2\{g_2(x,t),\tau\}}^{h_1\{g_1(x,t),\tau\}}\frac{f(s,\tau)}{a(s,\tau)}ds,\ (x,t)\in G_\infty. \tag{38}$$

Proof. Equation (36) of this Cauchy problem is a special case of the general telegraph equation (30) considered above for $b(x,t)=-a^{-1}(x,t)a_t(x,t)$, $c(x,t)=-a(x,t)a_x(x,t),\ q(x,t)=0$. Firsly we derive the formula (38) of the Cauchy problem (36), (37) of Theorem 11 from the formula (33) of the Cauchy problem (30), (31) of Theorem 8. For all $(x,t)\in G_\infty$ the Riemann function is well known

$$v(s,\tau)=v(s,\tau;x,t)=\frac{a(x,t)}{a(s,\tau)},\ (s,\tau)\in G_\infty, \tag{39}$$

which is a solution of the Goursat problem (34), (35) for the conjugate equation to the model telegraph equation (36). For $(x,t)\in G_\infty$, using the Riemann function (39) and the initial conditions (37), we calculate the first term of the formula (33)

$$\frac{(auv)(h_2\{g_2(x,t),\,0\},0)}{2a(x,t)}+$$

$$+\frac{(auv)(h_1\{g_1(x,t),\,0\},0)}{2a(x,t)}=$$

$$=\frac{(au)(h_2\{g_2(x,t),0\},0)a(x,t)}{2a(x,t)a(h_2\{g_2(x,t),0\},0)}+$$

$$+\frac{(au)(h_1\{g_1(x,t),0\},0)a(x,t)}{2a(x,t)a(h_1\{g_1(x,t),0\},0)}=$$

$$= \frac{u(h_2\{g_2(x,t),0\},0) + u(h_1\{g_1(x,t),0\},0)}{2} =$$

$$= \frac{\varphi(h_2\{g_2(x,t),\,0\}) + \varphi(h_1\{g_1(x,t),\,0\})}{2}. \quad (40)$$

For all $(s,\tau) \in G_\infty$, we similarly calculate the second term of the formula (33)

$$v_\tau(s,\tau)|_{\tau=0} - v(s,0)b(s,0) =$$

$$= \Big(\frac{a(x,t)}{a(s,\tau)}\Big)_\tau\Big|_{\tau=0} + \frac{a(x,t)}{a(s,0)}\,\frac{a_\tau(s,\tau)|_{\tau=0}}{a(s,0)} =$$

$$= a(x,t)\Big(-\frac{a_\tau(s,\tau)|_{\tau=0}}{a^2(s,0)} + \frac{a_t(s,\tau)|_{\tau=0}}{a^2(s,0)}\Big) = 0, (41)$$

$$\frac{\psi(s)v(s,0)}{a(x,t)} = \frac{\psi(s)a(x,t)}{a(x,t)a(s,0)} = \frac{\psi(s)}{a(s,0)}. \quad (42)$$

We calculate the third term of the formula (33)

$$\frac{1}{2a(x,t)} \int\limits_{\triangle MPQ} f(s,\tau)v(s,\tau:x,t)\,ds\,d\tau =$$

$$= \frac{1}{2} \int\limits_{\triangle MPQ} \frac{f(s,\tau)}{a(s,\tau)}\,ds\,d\tau =$$

$$= \frac{1}{2}\int\limits_0^t d\tau \int\limits_{h_2\{g_2(x,t),\tau\}}^{h_1\{g_1(x,t),\tau\}} \frac{f(s,\tau)}{a(s,\tau)}\,ds,\ (x,t) \in G_\infty. \ (43)$$

Since the smoothness on f, φ, ψ of the Cauchy problems (30), (31) and (36), (37) coincide, then from the equalities (40)–(43) we conclude that for the Cauchy problem (36), (37) the formula of the classical solution (33) is equal to the formula of the classical solution (38). Theorem 11 has been proved.

By Corollaries 9, 10, Corollaries 12, 13 hold.

Corollary 12 *If the right-hand side f of the equation (36) depends only on x or t and is continuous $f \in C(\mathbb{R})$, then Theorem 11 is true without integral smoothness requirements (32).*

Corollary 13 *If the right-hand side f of the equation (36) depends on x and t, then in Theorem 11, the belonging of integrals (32) to the set $C^1(G_\infty)$ is equivalent to their belonging to the sets $C^{(0,1)}(G_\infty)$ or $C^{(1,0)}(G_\infty)$.*

Remark 14 *The results of this work are needed to solve mixed problems for wave equations in curvilinear domains. The first and second mixed problems for model and general telegraph equations with variable coefficients have been solved by Lomovtsev F.E. using his new methods: implicit characteristics method from, [2], and generalization of Riemann method to these mixed problems on a half-line and a segment.*

6 Conclusion

The generalized Riemann formula (9) of the classical solution and the Hadamard correctness criterion of the generalized real Cauchy problem (1), (2) for a general linear inhomogeneous telegraph equation with variable coefficients under Cauchy conditions on a doubly continuously differentiable and noncharacteristic curve l of the plane are obtained. In global Theorem 3, the Riemann formula (9) of its unique and continuous on right-hand side f of the equation and Cauchy data φ, ψ of the classical solution $u \in C^2(G)$ is established by a modification of the Riemann method and new implicit characteristics method. From the formulation of the Cauchy problem and the definition of classical solutions, its Hadamard correctness criterion is found: $f \in C(G)$, $\varphi \in C^2(\mathbb{R})$, $\psi \in C^1(\mathbb{R})$ and new integral smoothness requirements (8). If the right-hand side f of the equation (1) depends on one of two variables x and t, then it is necessary and sufficient that it be continuous with respect to this variable. If the right-hand side f of the equation (1) depends on two variables and is continuous with respect to x and t, then in its integral smoothness requirements it is necessary and sufficient to have continuity in one and continuous differentiability in the other variable. From Theorem 3, the already known Riemann formulas of classical solutions and the correctness criteria of Cauchy problems in Theorem 8 for general and in Theorem 11 for model telegraph equations in the upper half-plane are derived. These Riemann formulas contain implicit functions of characteristics for telegraph equations. They partially confirm the correctness of the main results of this work in Theorem 3. In this work, the necessary and sufficient smoothness of the right-hand side of the telegraph equation with real variable coefficients under Cauchy conditions on a curved line of the plane for twice continuously differentiable solutions was proven for the first time.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Fedor Egorovich Lomovtsev derived the Riemann formula and the correctness criterion of the generalized Cauchy problem under Cauchy conditions on a curved line of a plane. He formulated and proved Theorem 2, Theorem 7 and Corollaries 4, 5, 8, 9. Andrey Leonidovich Kukharev derived Theorem 10 from Theorem 7 and translated this article into English.

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Conflict of Interest

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