Horocyclic Circles and Tubes around Complex Hypersurfaces in a Complex Hyperbolic Space

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Abstract: We show that every horocyclic circle of nonzero complex torsion on a complex hyperbolic space is expressed by a trajectory for a Sasakian magnetic field on some tube around totally geodesic complex hypersurface and that such an expression is unique up to the action of isometries on the complex hyperbolic space.

Key-Words: Circles; horocycle; Sasakian magnetic fields; real hypersurfaces; complex hyperbolic space; extrinsic shapes; congruent.

Received: March 16, 2023. Revised: November 2, 2023. Accepted: November 17, 2023. Published: December 7, 2023.

1 Introduction

For each circle of radius r in a Euclidean 3-space $\mathbb{R}^3$, if we take a suitable sphere S^2 of radius r, it can be seen as a geodesic on this sphere. We are interested in whether such a property holds for other symmetric spaces. For a complex projective space, if we restrict ourselves to circles whose velocity and acceleration vectors form a complex line, they can be seen as geodesics on some geodesic spheres. On the other hand, for a complex hyperbolic space, even if we restrict ourselves to such circles, if they are unbounded and do not lie on some horospheres, they can not be seen as trajectories for Sasakian magnetic fields, especially as geodesics, on any real hypersurfaces of type (A), [1]. Here, trajectories for Sasakian magnetic fields are natural generalizations of geodesics from the viewpoint of dynamical systems which are associated with almost contact metric structures on these real hypersurfaces. With these results it is natural to consider that unbounded circles are different from bounded circles. In this paper, we study circles lying on some horospheres and show that except the case that they lie on totally geodesic real hyperbolic plane they can be seen as trajectories for some Sasakian magnetic fields on some tubes around totally geodesic complex hypersurfaces.

2 Circles on a complex hyperbolic space

A smooth curve γ on a Riemannian manifold $\widetilde{M}$ parameterized by its arclength is said to be a *circle* if there exist a nonnegative constant k_γ and a field Y_γ of unit tangent vectors along γ satisfying the equations

$$\begin{cases} \widetilde{\nabla}_{\dot\gamma}\dot\gamma = k_\gamma Y_\gamma, \\ \widetilde{\nabla}_{\dot\gamma} Y_\gamma = -k_\gamma \dot\gamma, \end{cases} \tag{1}$$

which is equivalent to the equation

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma + k_\gamma^{\,2}\dot\gamma = 0.$$

The constant k_γ is called its geodesic curvature and $\{\dot\gamma, Y_\gamma\}$ its Frenet frame, [2]. When $k = 0$, it is a geodesic with an arbitrary parallel unit vector field. Hence, from the viewpoint of Frene-Serret formula, circles are simplest curves next to geodesics.

We say two smooth curves γ_1, γ_2 on $\widetilde{M}$ which are parameterized by their arclengths to be *congruent* to each other (in strong sense) if there is an isometry φ of $\widetilde{M}$ satisfying $\varphi \circ \gamma_1(t) = \gamma_2(t)$ for all t. In this paper, we study circles on a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c. By use of complex structure J, we set $\tau_\gamma = \langle \dot\gamma, JY_\gamma\rangle$ for a circle γ of positive geodesic curvature, and call it its *complex torsion*. Clearly it is constant along γ because J is parallel. Since $\mathbb{C}H^n(c)$ is a symmetric space of rank 1, we find that two circles are congruent to each other if and only if either they are geodesics or they satisfy $k_{\gamma_1} = k_{\gamma_2} > 0$ and $|\tau_{\gamma_1}| = |\tau_{\gamma_2}|$, [3]. Therefore, the moduli space $\mathcal{C}(\mathbb{C}H^n)$, which is the set of all congruence classes, of circles on $\mathbb{C}H^n(c)$ is set theoretically coincides with the band $[0,\infty) \times [0,1]/\sim$. Here we define $(k_1,\tau_1) \sim (k_2,\tau_2)$ if and only if either $k_1 = k_2 = 0$ or $k_1 = k_2 > 0$ and $\tau_1 = \tau_2$. When a circle γ on $\mathbb{C}H^n$ has complex torsion ± 1, the equations (1) turn to $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \mp k_\gamma J\dot\gamma$. We can interpret it as a trajectory for a Kähler magnetic field $\mp k_\gamma \mathbb{B}_J$ with the Kähler form $\mathbb{B}_J$.

We here recall some properties of circles on $\mathbb{C}H^n(c)$. Congruency of circles guarantees that every circle is a homogeneous curve, that is, it is an orbit of one parameter family of isometries of $\mathbb{C}H^n$. Since $\mathbb{C}H^n$ is a typical example of Hadamard manifolds, we can define its ideal boundary $\partial\mathbb{C}H^n$ as the set of asymptotic classes of geodesics, and have a compactification $\overline{\mathbb{C}H^n} = \mathbb{C}H^n \bigcup \partial\mathbb{C}H^n$, [4, 5]. When a curve γ which is unbounded in both directions, that is, both $\gamma\big([0,\infty)\big)$ and $\gamma\big(-\infty,0]\big)$ are unbounded sets, we set $\gamma(\infty) = \lim_{t\to\infty}\gamma(t)$, $\gamma(\infty) = \lim_{t\to-\infty}\gamma(t)$ if they exist in $\partial\mathbb{C}H^n$. These are called points at infinity of γ. We say a smooth curve γ parameterized by its arclength to be *horocyclic* if it satisfies the following conditions, [6]:

i) Both $\gamma(\infty)$ and $\gamma(-\infty)$ exist and coincide with each other;

ii) If it crosses with a geodesic β satisfying $\beta(\infty) = \gamma(\infty)$, then they cross orthogonally.

We set a function $\nu : [0,\infty) \to \mathbb{R}$ by

$$\nu(k) = \begin{cases} 0, & \text{if } k < \frac{\sqrt{|c|}}{2}, \\ \dfrac{(4k^2+c)^{3/2}}{(3\sqrt{3}|c|k)}, & \text{if } \frac{\sqrt{|c|}}{2} \le k \le \sqrt{|c|}, \\ 1, & \text{if } k > \sqrt{|c|}. \end{cases}$$

We take a horizontal lift $\hat\gamma$ of a circle γ on $\mathbb{C}H^n(c)$ through a Hopf fibration $\varpi : H_1^{2n+1}\ (\subset \mathbb{C}^{n+1}) \to \mathbb{C}H^n(c)$ of an anti-de Sitter space H_1^{2n+1} and regard it as a curve in $\mathbb{C}^{n+1}$. When its geodesic curvature is k_γ and complex torsion is τ_γ, by solving an ordinary differential equation on $\hat\gamma$, we can get the following feature of γ:

1) If $k_\gamma > \sqrt{|c|}$ or $\tau_\gamma < \nu(k_\gamma)$, it is bounded;

2) If it does not satisfies the condition in 1), it is unbounded in both directions and has points at infinity;

3) It is horocyclic if and only if $\sqrt{|c|}/2 \le k_\gamma \le \sqrt{|c|}$ and $|\tau_\gamma| = \nu(k_\gamma)$ hold;

4) When $\tau_\gamma = \pm 1$, it lies on a totally geodesic $\mathbb{C}H^1$, and when $\tau_\gamma = 0$, it lies on a totally geodesic $\mathbb{R}H^2$.

The following figure shows the images of the moduli spaces of unbounded and bounded circles on $\mathbb{C}H^n(c)$.

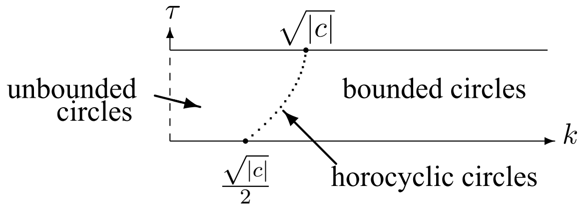

Figure 1: moduli space of circles on $\mathbb{C}H^n$

3 Horocyclic circles are expressed uniquely by trajectories

In this paper, we study whether circles in $\mathbb{C}H^n(c)$ can be seen as extrinsic shapes of some "nice" curves on a homogeneous real hypersurface. Let $M = T(r)$ be a tube of radius r around totally geodesic complex hypersurface $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n(c)$. On this hypersurface we have an almost contact metric structure $(\xi,\eta,\phi,\langle\ ,\ \rangle)$ induced by the complex structure J on $\mathbb{C}H^n$. By taking a unit normal vector field $\mathcal{N}$ of M in $\mathbb{C}H^n(c)$ and the induced metric $\langle\ ,\ \rangle$, we set the characteristic vector field ξ by $\xi = -J\mathcal{N}$, the 1-form η by $\eta(v) = \langle v,\xi\rangle$, the $(1,1)$-tensor field ϕ by $\phi(v) = Jv - \eta(v)\mathcal{N}$. The shape operator A_M of M satisfies $A_M\xi = \delta_M\xi$ and $A_M v = \lambda_M v$ for arbitrary tangent vector v orthogonal to ξ. Here, the principal curvatures are given as $\delta_M = \sqrt{|c|}\coth\sqrt{|c|}\,r$ and $\lambda_M = \big(\sqrt{|c|}/2\big)\tanh\big(\sqrt{|c|}\,r/2\big)$, [7, 8], for example.

If we denote by ∇ and $\widetilde{\nabla}$ the Riemannian connections on M and on $\mathbb{C}H^n(c)$, respectively, we have the formulae of Gauss and Weingarten:

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + \langle A_M X, Y\rangle\mathcal{N}, \\ \widetilde{\nabla}_X \mathcal{N} &= -A_M X, \end{aligned}$$

for arbitrary vector fields $X,\ Y$ tangent to M. Since the complex structure J is parallel with respect to $\widetilde{\nabla}$, these formulas guarantee

$$\big(\nabla_X\phi\big)Y = \eta(Y)A_M X - \langle A_M X, Y\rangle\xi, \quad (2)$$
$$\nabla_X\xi = \phi A_M X. \quad (3)$$

We define a 2-form $\mathbb{F}_\phi$ on M by $\mathbb{F}_\phi(v,w) = \langle v,\phi(w)\rangle$. By use of (3), we find that it is closed. We therefore call its constant multiple $\mathbb{F}_\kappa = \kappa\mathbb{F}_\phi$ ($\kappa\in\mathbb{R}$) a *Sasakian magnetic field*. A smooth curve σ parameterized by its arclength is said to be a *trajectory* for $\mathbb{F}_\kappa$ if it satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \kappa\phi\dot\gamma$. Since we find that trajectories for the null magnetic field $\mathbb{F}_0$ are geodesics and that every trajectory is determined by its initial unit tangent vector, we may say that trajectories are generalizations of geodesics and are simplest curve next to geodesics from the viewpoint of dynamical systems on the unit tangent bundle of M. For magnetic fields, [9], and, [10].

Given a smooth curve γ on $\mathbb{C}H^n(c)$ parameterized by its arclength, if we have a real hypersurface M and a smooth curve σ on M satisfying $\gamma(t) = \iota \circ \sigma(t)$ for all t with an isometric immersion $\iota : M \to \mathbb{C}H^n(c)$, we say (M, σ) is an *expression* of γ. We say two expressions (M_1, σ_1) and (M_2, σ_2) of γ to be *congruent* to each other if there is an isometry φ of $\mathbb{C}H^n(c)$ which satisfies $\varphi(M_1) = M_2$ and either it preserves γ or reverse γ. That is, either $\varphi \circ \gamma(t) = \gamma(t)$ for all t or $\varphi \circ \gamma(t) = \gamma(-t)$ for all t. In [1], the authors studied expressions of circles by geodesics on tubes.

Proposition 1 ([1]). *Every circles on $\mathbb{C}H^n(c)$ with geodesic curvature not smaller than $\sqrt{|c|}$ and complex torsion ± 1 is expressed by a geodesic on some tube around totally geodesic complex hypersurface.*

We are hence interested in expressions of circles with complex torsion $|\tau| < 1$. But being different from circles in a Euclidean 3-space, if we stick on expressions by geodesics on real hypersurfaces, our study does not go through any more (see Lemma 2 below). We therefore extend the family of curves on tubes and study expressions by trajectories for Sasakian magnetic fields.

Theorem 1. *Let γ be a circle on $\mathbb{C}H^n(c)$.*

(1) *When γ is horocyclic and $\tau_\gamma \neq 0$, it is expressed by a trajectory for a Sasakian magnetic field on some tube around totally geodesic complex hypersurface. If the complex torsion τ_γ of γ satisfies $0 < |\tau_\gamma| < 1$, such an expression is unique up to the congruence relation.*

(2) *When γ is horocyclic and $\tau_\gamma = 0$, it can not be expressed by trajectories for Sasakian magnetic fields on tubes around totally geodesic complex hypersurfaces.*

(3) *When γ is bounded, also it is expressed by a trajectory for a Sasakian magnetic field on some tube around totally geodesic complex hypersurface.*

For a trajectory σ for $\mathbb{F}_\kappa$ on a tube $M = T(r)$ of radius r around totally geodesic complex hypersurface, we set $\rho_\sigma = \langle \dot\sigma, \xi \rangle$ and call it its *structure torsion*. By (3) we find that the derivative of the structure torsion is given as

$$\begin{aligned} \rho_\sigma' &= \kappa\langle \phi\dot\sigma, \xi\rangle + \langle \dot\sigma, \phi A_M \dot\sigma\rangle \\ &= \langle \dot\sigma, \phi A_M \dot\sigma\rangle = -\langle A_M \phi\dot\sigma, \dot\sigma\rangle, \end{aligned}$$

because A_M is symmetric and ϕ is anti-symmetric. Since it is known that they satisfy $A_M\phi = \phi A_M$ for our tube $T(r)$, we find that ρ_σ is constant along σ. This is an important invariant for σ. Through a Hopf fibration $\varpi : H_1^{2n+1} \to \mathbb{C}H^n$, the inverse image of $T(r)$ is $H_1^{2n-1} \times S^1$. Considering the action of $\mathrm{U}(1, n)$ on $\mathbb{C}^{n+1}$, we have the following result on congruency of trajectories on $T(r)$.

Lemma 1. *Let σ_1, σ_2 be trajectories for Sasakian magnetic fields $\mathbb{F}_{\kappa_1}, \mathbb{F}_{\kappa_2}$ respectively on a tube $T(r)$ around totally geodesic $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n(c)$. Then γ_1 and γ_2 are congruent to each other if and only if one of the following conditions holds*:

i) $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| = 1$,

ii) $|\rho_{\sigma_1}| = |\rho_{\sigma_2}| < 1,\ |\kappa_1| = |\kappa_2|$ *and* $\kappa_1\rho_{\sigma_1} = \kappa_2\rho_{\sigma_2}$.

Since two circles are congruent to each other if and only if they have the same geodesic curvatures and the same absolute values of complex torsions, in order to show the existence of expressions of a circle of geodesic curvature k and complex torsion τ, we are enough to show that there exists a trajectory σ for a Sasakian magnetic field on some tube whose extrinsic shape is a circle of geodesic curvature k and complex torsion τ.

By the formulae of Gauss and Weingarten, considering a trajectory σ for $\mathbb{F}_\kappa$ on $M = T(r)$ as a curve in $\mathbb{C}H^n(c)$, we have

$$\begin{aligned} &\widetilde\nabla_{\dot\sigma}\dot\sigma = \kappa\phi\dot\sigma + \big\{\lambda_M(1-\rho_\sigma^2) + \delta_M\rho_\sigma^2\big\}\mathcal{N}, \\ &\widetilde\nabla\big[\kappa\phi\dot\sigma + \big\{\lambda_M(1-\rho_\sigma^2) + \delta_M\rho_\sigma^2\big\}\mathcal{N}\big] \\ &\qquad = -\big[\kappa^2(1-\rho_\sigma^2) + \big\{\lambda_M + (\delta_M - \lambda_M)\rho_\sigma^2\big\}^2\big]\dot\sigma \\ &\qquad\quad + \big\{\lambda_M - \kappa\rho_\sigma + (\delta_M - \lambda_M)\rho_\sigma^2\big\} \\ &\qquad\qquad \times \big\{\kappa + (\delta_M - \lambda_M)\rho_\sigma\big\}(\rho_\sigma\dot\sigma - \xi_\sigma). \end{aligned}$$

We therefore get conditions that the extrinsic shape of a trajectory on M is a circle on $\mathbb{C}H^n(c)$.

Lemma 2. *Let σ be a trajectory for $\mathbb{F}_\kappa$ on $T(r)$ in $\mathbb{C}H^n(c)$. Its extrinsic shape is a circle on $\mathbb{C}H^n(c)$ if and only if one of the following condition holds*:

i) $\rho_\sigma = \pm 1$,

ii) $\lambda_M - \kappa\rho_\sigma + (\delta_M - \lambda_M)\rho_\sigma^2 = 0$,

iii) $\kappa + (\delta_M - \lambda_M)\rho_\sigma = 0$.

Corresponding to these cases, the geodesic curvature k_σ and the complex torsion τ_σ of the extrinsic shape of σ are as follows:

i) $k_\sigma = \delta_M,\ \tau_\sigma = \mp 1$,

ii) $k_\sigma = |\kappa|,\ \tau_\sigma = \mathrm{sgn}(\kappa)$,

iii) $k_\sigma = \sqrt{\kappa^2 - 2\lambda_M \kappa\rho_\sigma + \lambda_M^2}$,
$\tau_\sigma = (2\kappa\rho_\sigma^2 - \kappa - \lambda_M\rho_\sigma)/k_\sigma$.

Here, $\mathrm{sgn}(\kappa)$ *denotes the signature of* κ*.*

Since Proposition 1 corresponds to the first and the second cases of Lemma 2, we study the third case. In this case, as we have $\kappa = -(\delta_M - \lambda_M)\rho_\sigma = -|c|\rho_\sigma/(4\lambda_M)$, we obtain

$$k_\sigma = \sqrt{\lambda_M^2 + \frac{|c|\rho_\sigma^2}{2} + \frac{c^2\rho_\sigma^2}{16\lambda_M^2}}, \tag{4}$$

$$\tau_\sigma = \frac{\rho_\sigma(|c| - 2|c|\rho_\sigma^2 - 4\lambda_M^2}{4k_\sigma\lambda_M}. \tag{5}$$

Since $|\rho_\sigma| \le 1$, by the equation (4) we see $\lambda_M \le k_\sigma \le \lambda_M + \big(|c|/(4\lambda_M)\big) = \delta_M$. Moreover, by using these two equalities we have

$$\tau_\sigma{}^2 = \frac{(k_\sigma{}^2 - \lambda_M^2)\big(32\lambda_M^2 k_\sigma{}^2 + 4c\lambda_M^2 - c^2\big)^2}{|c|(8\lambda_M^2 - c)^3 k_\sigma^2}. \tag{6}$$

We here study congruent expressions.

Lemma 3. *Let* γ *be a circle in* $\mathbb{C}H^n(c)$*. Suppose that its complex torsion satisfies* $|\tau_\gamma| < 1$*. If we have two expressions* (M_1, σ_1), (M_2, σ_2) *of* γ *by trajectories for Sasakian magnetic fields* $\mathbb{F}_{\kappa_1}, \mathbb{F}_{\kappa_2}$ *on tubes* M_1, M_2 *around totally geodesic complex hypersurfaces of the same radius, then they are congruent to each other.*

Proof. For the sake of simplicity, we consider M_1 and M_2 as subsets of $\mathbb{C}H^n(c)$ through isometric embeddings. Since the extrinsic shapes of σ_1 and σ_2 coincide with γ, by the equation (4) we find $|\rho_{\sigma_1}| = |\rho_{\sigma_2}|$. Hence, by the condition $\kappa_i + (\delta_{M_i} - \lambda_{M_i})\rho_{\sigma_i} = 0$, we have $|\kappa_1| = |\kappa_2|\ (\neq 0)$ and $\kappa_1\rho_{\sigma_1} = \kappa_2\rho_{\sigma_2}$.

Since M_1, M_2 are of the same radius, they are isometric to each other. Hence we have an isometry φ of $\mathbb{C}H^n(c)$ with $\varphi(M_1) = M_2$. Taking into account of principal curvatures of M_1 and M_2, we have $d\varphi(\mathcal{N}_{M_1}) = \mathcal{N}_{M_2}$. As φ is $\pm$-holomorphic, that is, $d\varphi \circ J = \pm J \circ d\varphi$, we have

$$\begin{aligned}\rho_{\varphi\circ\sigma_1} &= \langle d\varphi\circ\dot\sigma_1, \xi_{M_2}\rangle = \langle d\varphi\circ\dot\sigma_1, -Jd\varphi(\mathcal{N}_{M_2}\rangle \\ &= \pm\langle d\varphi\circ\dot\sigma_1, d\varphi(-J\mathcal{N}_{M_2}\rangle = \pm\langle\dot\sigma_1, \xi_{M_1}\rangle \\ &= \pm\rho_{\sigma_1}\end{aligned}$$

and

$$\begin{aligned}\nabla_{d\varphi\circ\dot\sigma_1} d\varphi\circ\dot\sigma_1 &= d\varphi\big(\nabla_{\dot\sigma_1}\dot\sigma_1\big) = \kappa_1 d\varphi\big(\phi\dot\sigma_1\big) \\ &= \kappa_1 d\varphi\big(J\dot\sigma_1 - \rho_{\sigma_1}\mathcal{N}_{M_1}\big) \\ &= \kappa_1\big(\pm Jd\varphi\circ\dot\sigma_1 - \rho_{\sigma_1}\mathcal{N}_{M_2}\big) \\ &= \pm\kappa_1\phi(d\varphi\circ\dot\sigma_1).\end{aligned}$$

Therefore, $\varphi\circ\sigma_1$ is a trajectory for $\mathbb{F}_{\pm\kappa_1}$ on M_2 whose structure torsion is $\pm\rho_{\sigma_1}$. Thus, we find that $\varphi\circ\sigma_1$ and σ_2 are congruent to each other. This means that there is an isometry ψ of M_2 satisfying $\sigma_2(t) = \psi\circ\varphi\circ\sigma_1(t)$ for all t. Since it is known that there is an isometry $\tilde\psi$ on $\mathbb{C}H^n(c)$ satisfying $\tilde\psi\big|_{M_2} = \psi$, by putting $\Phi = \tilde\psi\circ\varphi$, we have $\Phi(M_1) = M_2$ and $\Phi\circ\sigma_1(t) = \sigma_2(t)$ for all t. Hence (M_1, σ_1) and (M_2, σ_2) are congruent to each other. □

Proof of Theorem 1. In order to show the first and the second assertions, we solve the equation $\tau_\sigma{}^2 = \nu(k_\sigma)^2$ under the assumption $\sqrt{|c|}/2 \le k_\sigma \le \sqrt{|c|}$. Since this equation turns to

$$\begin{aligned}&(8k_\sigma{}^2 + |c|)^2(4\lambda_M^2 + c)^2 \\ &\quad\times(32\lambda_M^2 k_\sigma{}^2 + 5c\lambda_M^2 - ck_\sigma{}^2 - c^2) = 0\end{aligned}$$

and $4\lambda_M^2 + c < 0$, we find

$$k_\sigma{}^2 = \frac{|c|\big(5\tanh^2(\sqrt{|c|}\,r/2) + 4\big)}{4\big(8\tanh^2(\sqrt{|c|}\,r/2) + 1\big)}. \tag{7}$$

This equality shows that if we fix the radius of a tube we can express only one horocyclic circle on a complex hyperbolic space up to the action of isometries by some trajectory on this tube. Since the right-hand side of the equation (7) is monotone decreasing with respect to the radius r, and takes all values in the interval $\big(|c|/4, |c|\big)$, we find that for each horocyclic circle γ of complex torsion $0 < |\tau| < 1$ it is expressed by a trajectory on a tube whose radius is determined by k_γ. By Lemma 3, we get the first assertion. Also, we find that if a horocyclic circle γ has null complex torsion, it is not expressed by trajectories for Sasakian magnetic fields on tubes around complex hypersurfaces.

In order to show the third assertion, we study the behavior of τ_σ^2 with respect to k_σ. We denote by $f\big(k; T(r)\big)$ the continuous function defined by the right-hand side of the equation (6) by putting $k = k_\sigma$. We set

$$a(r) = \sqrt{-8c + 2c^2\lambda_{T(r)}^{-2}}/8, \quad b(r) = \delta_{T(r)}.$$

They satisfy $\lambda_{T(r)} < a(r) < b(r)$. Since we have

$$\begin{aligned}\frac{df}{dk} &= 2\lambda_M^2(8k^2 - c)(8k^2 - 4\lambda_M^2 + c) \\ &\quad\times(32\lambda_M^2 k^2 + 4c\lambda_M^2 - c^2) \\ &\quad\times\big\{|c|(8\lambda_M^2 - c)^3 k^3\big\}^{-1},\end{aligned}$$

we find that the function $f\big(k; T(r)\big)$ have the following properties:

1) It is monotone increasing in the interval $\big[a(r), b(r)\big]$,

2) $f\big(a(r); T(r)\big) = 0$ and $f\big(b(r); T(r)\big) = 1$.

This means that the moduli space of circles which are extrinsic shapes of trajectories satisfyiong the third condition in Lemma 2 is like Figure 2.

Since we have

$$\lim_{r\downarrow 0} a(r) = \lim_{r\downarrow 0} b(r) = \infty,$$
$$\lim_{r\to\infty} a(r) = \sqrt{|c|}/2, \quad \lim_{r\to\infty} b(r) = \sqrt{|c|},$$
$$\lim_{r\to\infty} f\big(k; T(r)\big) = (4k^2 + c)^3/\big(27c^2k^2\big),$$

comparing the last one with $\nu(k)^2$, we find that the union $\bigcup_{r>0} f\big([a(r), b(r)]; T(r)\big)$ covers the set

$$\big\{(k,\tau) \bigm| \sqrt{|c|}/2 < k \le \sqrt{|c|},\ 0 \le \tau < \nu(k)\big\} \cup \big(\sqrt{|c|}, \infty\big) \times [0, 1].$$

This means that every bounded circle on $\mathbb{C}H^n(c)$ is expressed by some trajectory on some tube $T(r)$.

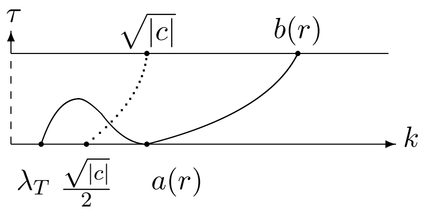

Figure 2: The graph of $f\big(k; T(r)\big)$

$\square$

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

The second author is partially supported by Grant-in-Aid for Scientific Research (C) (No. 20K03581), Japan Society for the Promotion of Science.

Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.