The Quenching Solutions of a Singular Parabolic Equation

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Abstract: - This article is dedicated to the study of the self-similar solutions of a nonlinear parabolic equation. More precisely, we consider the following uni-dimensional equation:

$$(E): \quad u_t(x,t) = (u^m)_{xx}(x,t) - |x|^q \, u^{-p}(x,t), \;\; x \in \mathbb{R}, \; t > 0,$$

where $m > 1$, $q > 1$ and $p > 0$.
Initially, we employed a fixed point theorem and an associated energy function to establish the existence of solutions. Subsequently, we derived some important results on the asymptotic behavior of solutions near the origin.

Key-Words: - Singular parabolic equation, Self-similar solutions, Quenching solutions, Existence of solutions, Asymptotic behavior.

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1 Introduction

In this article, we are studying the semi-linear parabolic equation (E) with a strictly positive initial datum

$$\begin{cases} u_t(x,t) = (u^m)_{xx}(x,t) - |x|^q \, u^{-p}(x,t), \\ (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x_0,t_0) > 0, \quad x_0 \in \mathbb{R}, \; t_0 > 0. \end{cases} \tag{1}$$

The case $m = 1$ has been widely studied in recent years, due to its multiple applications in micro-electro-mechanical system (MEMS), see for example works, [1], [2], [3], [4], [5], [6], [7]. The main aim of our paper is to generalize these results to the case $m > 1$. For additional details on the context and derivation of the MEMS model, we recommend the reader to see papers, [8], [9], [10].

Physically, the function u is the distance between an elastic membrane in the interior of a micro-electro-mechanical system, and the fixed bottom plate. So when u becomes zero, i.e., the membrane touches the bottom plate, the system then breaks down. This phenomenon is called "Quenching" or "Touchdown" or deactivating phenomenon.

Using standard parabolic equations techniques, the problem constituted by equation (E) and a strictly positive initial datum, admits a unique local solution. Such a solution can be vanish in a finite time, and therefore the term u^{-p} will not be defined. This is the "Quenching" phenomenon in Mathematics, for more details on this phenomenon, see the articles, [11], [12], [13], [14]. In other words, there exists a finite time T so that:

$$\begin{cases} u(x,t) > 0 \quad \text{in } \mathbb{R} \times (0,T), \\ \lim_{t \to T^-} \inf \left[\inf_{x \in \mathbb{R}} u(x,t) \right] = 0. \end{cases} \tag{2}$$

A point x_0 is said to be a quenching point if there exists a sequence of points $(x_n, t_n)_{n \in \mathbb{N}}$ such that $x_n \to x_0, t_n \to T$ and $u(x_n, t_n) \to 0$ when n tends to infinity.

Now, we will study the problem given by (1), using "self-similar" solutions. These solutions are obtained by looking for transformations that leave the equation invariant see, [15], [16], [17], [18].

Afterwards, inspired by the works, [1], [19], [20], the equation (E) possesses special self-similar solutions of the form:

$$u(x,t) = (T-t)^\alpha U(y), \quad y = (T-t)^\beta |x|, \tag{3}$$

where $x \in \mathbb{R}$, $0 \leq t \leq T$, and the two constants α and β are given by:

$$\alpha = \frac{q+2}{2(p+1) - q(m-1)}$$

and

$$\beta = -\frac{m+p}{2(p+1) - q(m-1)}.$$

So if $2(p+1) - q\,(m-1) \neq 0$, the function U must verify the following ordinary differential equation:

$$(U^m)''(y) + \beta y\, U'(y) + \alpha U(y) - y^q U^{-p}(y) = 0, \quad (4)$$

with $y > 0$.

Note that the equation (4) is not necessarily verified to the point $y = 0$. Also, if U is a solution of the equation (4) and it is well defined at the point $y = 0$, then $\lim_{t \to T^-} u(0,t) = u(0, T^-) = 0$. So for the point zero to be the only quenching point it is necessary and sufficient that $U(y) > 0$ for all $y > 0$ and that $\lim_{y \to +\infty} U(y) \neq 0$. Moreover, in this case the deactivating behavior is of the range $(T-t)^\alpha$. Indeed, to obtain information on the deactivating behavior of the solution of the parabolic equation (E), we need to study the ordinary differential equation (4).

The other sections of this article are divided as follows. In Section 2, we provide a result on the existence of global positive solutions to (4) using a fixed-point theorem and an associated energy functional, where α and β are two reals verifying the following condition:

$$\alpha > 0,\ \beta < 0, \text{ and } \Gamma := m\frac{\alpha}{\beta}\left(1 + m\frac{\alpha}{\beta}\right) > 0. \quad (5)$$

Then we assume that:

$$0 \leq \frac{p-m}{m} < q < \frac{2(p+1)}{m-1}. \quad (6)$$

In Section 3, we study the asymptotic behavior of solutions of equation (4) near the origin. First, we use the monotonicity argument of the transformed solution using the associated energy function. Then, we derive a result on the asymptotic behavior.

Finally, in Section 4, we summarize what we have been able to prove about this equation in the previous sections, and also we give some ideas that we will develop in our future work.

2 Existence of global solutions

In this section, we use a fixed-point theorem and an energy functional to demonstrate the existence of global solutions of equation (4).

Note that if $l = \Gamma^{-1/(m+p)}$, then the function:

$$U_0(y) = \Gamma^{-1/(m+p)} y^{-\alpha/\beta} \quad \text{for} \quad y > 0, \quad (7)$$

is a particular solution of equation (4) on $[0, +\infty]$ and satisfies $U_0(0) = 0$.

For this reason, we will restrict our study to the case $0 < l \neq \Gamma^{-1/(m+p)}$ and we will be interested in the solutions U of equation (4) which satisfy $U(0) = 0$ and behave near to infinity as $l\, y^{-\alpha/\beta}$ where l is a strictly positive constant. More precisely, we need to study the following problem:

$$(P)\begin{cases} (U^m)''(y) + \beta y U'(y) + \alpha U(y) - y^q U^{-p}(y) = 0, \\ \qquad\qquad \text{for} \quad y > 0, \\ U(0) = 0 \quad \text{and} \quad U'(0) = 0, \\ \lim_{y \to +\infty} y^{\alpha/\beta} U(y) = l > 0. \end{cases}$$

It must be said that the equation (4) is not verified at the point $y = 0$. We will prove that there is a positive solution of problem (P) that vanishes, and its derivative at the point $y = 0$. Due to this singular condition, the standard ODE theory cannot be applied.

Theorem 2.1. *Assume that* $0 < l \neq \Gamma^{-1/(m+p)}$. *Then the problem* (P) *has a positive solution* U *defined on* $[0, +\infty]$ *and having the following asymptotic behavior near to* $+\infty$*:*

$$U(y) = l y^{-\alpha/\beta} + \frac{1}{\beta\mu} f(l)\, y^{-\mu-\alpha/\beta} + O(y^{-2\mu-\alpha/\beta}), \quad \text{as } y \to +\infty. \quad (8)$$

where

$$f(y) = \Gamma y^m - y^{-p}, \quad \textit{for all} \quad y > 0, \quad (9)$$

and

$$\mu = 2 + \frac{\alpha(m-1)}{\beta} > 0. \quad (10)$$

The proof of this theorem is divided into three steps. The first step concerns the existence of a solution near to infinity. The second step involves to extend the solution on $[0, +\infty]$, and the last step consists to prove that the solution $U(y) > 0$ for all $y > 0$.

Proof. We will proceed in three steps.
Step1: There exists a constant $M > 0$, such that the problem (P) admits a solution on $[M, +\infty[$.

We Set

$$V(x) = y^{m\alpha/\beta} U^m(y), \quad (11)$$

where $y = e^x$, y> 0 and $x \in \mathbb{R}$.
Hence the function V verifies for all $x \in \mathbb{R}$ the following equation:

$$V''(x) - aV'(x) + \frac{\beta}{m} e^{\mu x} V^{\frac{1-m}{m}}(x) V'(x) + f(V^{1/m})(x) = 0, \quad (12)$$

where

$$a = 1 + 2m\frac{\alpha}{\beta} < 0. \quad (13)$$

Note that $f(y)$ can only vanish at the point $y = \Gamma^{-1/(m+p)}$, therefore $\Gamma^{-m/(m+p)}$ is the unique constant that is a solution to the equation (12).

Now we will express the equation (12) as the system below:

$$\frac{dV(x)}{dx} = W(x),$$
$$\frac{dW(x)}{dx} = (a - \frac{\beta}{m}e^{\mu x}V^{\frac{1-m}{m}}(x))W(x) - f(V^{1/m}(x)).$$

Because of the term $e^{\mu x}$ which poses a problem near infinity, we will introduce a new derivation variable, $t = e^{\mu x}$ which transforms the system above to:

$$\frac{dV(t)}{dt} = \frac{W(t)}{\mu t}, \quad (14)$$
$$\frac{dW(t)}{dt} = (\frac{a}{\mu t} - \frac{\beta}{m}V^{\frac{1-m}{m}}(t))W(t) - \frac{f(V^{1/m})(t)}{\mu t}. \quad (15)$$

Hence

$$(e^{-A(t)}W(t))' = -e^{-A(t)}\frac{f(V^{1/m})(t)}{\mu t}, \quad (16)$$

where $A(t)$ is a primitive of $\frac{a}{\mu t} - \frac{\beta}{m}V^{\frac{1-m}{m}}(t)$.

Assuming that $\lim_{t\to+\infty} W(t) = 0$ and integrating (16) over the interval (t, ∞), we obtain:

$$W(t) = \frac{1}{\mu}e^{A(t)}\int_t^{+\infty}\frac{1}{s}e^{-A(s)}f(V^{1/m})(s)ds. \quad (17)$$

From the expression of the function A, the integral (17) becomes:

$$W(t) = \frac{1}{\mu}\int_t^{+\infty}\frac{1}{s}(\frac{t}{s})^{a/\mu}f(V^{1/m})(s)\times \exp(-\int_s^t V^{\frac{1-m}{m}}(\tau)d\tau)ds. \quad (18)$$

Note that the condition at infinity of U in the problem (P) requires the function V to verify:

$$\lim_{t\to+\infty} V(t) = l^m. \quad (19)$$

Hence

$$V(t) = l^m - \int_t^{+\infty}\frac{W(s)}{\mu s}ds. \quad (20)$$

In what follows, we'll prove the existence of a function W near infinity that verifies (18) and (20) and the limit:

$$\lim_{t\to+\infty} t\left[W(t) + \frac{mf(l)}{\beta t}l^{m-1}\right] = 0.$$

For that, we define on the Banach space:

$$X = \left\{\varphi \in \mathcal{C}^0([M, +\infty[);\ \sup_{t\in[M,+\infty[}|t\,\varphi(t)| \le K\right\},$$

the functional $H : \forall t \in [M, +\infty[$ and $\forall \varphi \in X$,

$$H(\varphi)(t) = \frac{t}{\mu}\int_t^{+\infty}\frac{1}{s}(\frac{t}{s})^{a/\mu}f(\Psi(\varphi)(s))\times \exp(-\int_s^t \Psi^{1-m}(\varphi)(\tau)d\tau)ds + \frac{mf(l)}{\beta}l^{m-1}, \quad (21)$$

where

$$\Psi(\varphi)(t) = \left[l^m - \int_t^{+\infty}\frac{\varphi(s)}{\mu s^2}ds + \frac{mf(l)}{\mu\beta t}l^{m-1}\right]^{1/m}. \quad (22)$$

By cutting the integral of the formula (21) into two parts: the main part using a Taylor Expansion around l (with a remainder term), we obtain estimations on each part, then we prove that H is a contraction of X in X for certain constants.
In this case, according to a fixed-point argument, there will exist a function $\varphi \in X$ such that:

$$H(\varphi)(t) = \varphi(t), \quad \text{for all } t \ge M.$$

So the function $t \to \frac{\varphi(t)}{t} - \frac{mf(l)}{\beta t}l^{m-1}$ is a solution of the equation (18) in the set $[M, +\infty[$. Consequently the function:

$$V(t) = l^m + \frac{mf(l)}{\mu\beta t}l^{m-1} - \int_t^{+\infty}\frac{\varphi(s)}{\mu s^2}ds, \quad \forall t \ge M \quad (23)$$

is a solution of the equation (12). Or

$$\left|\int_t^{+\infty}\frac{\varphi(s)}{\mu s^2}ds\right| \le K\int_t^{+\infty}\frac{1}{\mu s^3}ds = \frac{K}{4\mu t^2} = O(\frac{1}{t^2}) \quad \text{as } t \to \infty. \quad (24)$$

Hence, for t sufficiently large

$$V^{\,1/m}(t) = l + \frac{f(l)}{\mu\beta t} + O(\frac{1}{t^2}). \quad (25)$$

In consequence, the function:

$$U(y) = y^{-\alpha/\beta} V^{\,1/m}(t), \quad (26)$$

with $t = e^{\mu x}$ is a solution to the problem (P) on $[M, +\infty[$. Moreover replacing $V^{1/m}(t)$ with its expression (25) in (26), we obtain:

$$U(y) = l y^{-\alpha/\beta} + \frac{f(l)}{\mu\beta e^{\mu x}} y^{-\alpha/\beta} + O(\frac{y^{-\alpha/\beta}}{(e^{\mu x})^2}), \quad \text{as} \quad y \to \infty.$$

Using the fact that $y = e^x$ from the expression (11), we obtain:

$$U(y) = l y^{-\alpha/\beta} + \frac{1}{\mu\beta} f(l) y^{-\mu-\alpha/\beta} + O(y^{-2\mu-\alpha/\beta}), \quad \text{as} \quad y \to \infty.$$

We've achieved the desired result, so the first step has been accomplished.

Step2: U can be extended to $[0, +\infty]$.

Let V be the function defined by the relation (11) on an interval of the type $[X, +\infty[$(with $X > 0$). Using a symmetrization, we put:

$$\omega(x) = V(-x), \qquad \forall x \in\,]-\infty, -X] \quad (27)$$

Thus ω verifies on $]-\infty, -X]$ the following equation:

$$\omega''(x) + a\omega'(x) - \frac{\beta}{m} e^{-\mu x} \omega^{\frac{1-m}{m}}(x)\omega'(x) + f(\omega^{1/m}) = 0. \quad (28)$$

We consider the energy functional:

$$E(\omega)(x) = \frac{1}{2}\omega'^{\,2}(x) + F(\omega(x)), \quad (29)$$

where

$$F(\omega(x)) = \int_{\Gamma^m}^{\omega(x)} f\,(s^{1/m}) ds \quad (30)$$

$$= \int_{\Gamma^m}^{\omega(x)} (\mu s - s^{-p/m}) ds > 0. \quad (31)$$

Hence

$$E'(\omega)(x) = \left[-a\, + \frac{\beta}{m}\, e^{\,-\mu x} \omega^{\frac{1-m}{m}}(x)\right] \omega'^{\,2}(x).$$

Since $\beta < 0$ and $a < 0$, then

$$E'(\omega)(x) < -2a\, E(\omega)(x) \quad \text{for all} \quad x \in\,]-\infty, -X]\,.$$

Therefore the energy $E(\omega)$ is finite for any finite x and increases as an exponential, so ω can be extended to $x = +\infty$. In addition, we have by (19), $\lim\limits_{x\to-\infty} \omega(x) = l^m > 0$. Which completes the proof of the second step .

Step3: $U(y) > 0$ for all $y > 0$.

From the expression of the energy functional E, the integral $F(\omega)(x)$ exists, and as $p \geq m$ then necessarily $\omega(x) > 0$, hence $U(y) > 0$ for all $y > 0$.

Finally, we deduce the existence of a positive solution U of the problem (P) defined on $[0, +\infty]$. This completes the proof of Theorem 2.1. □

3 Asymptotic behavior near the origin

In this section, we propose to study the asymptotic behavior of solutions of problem (P) near the origin, which is the same as studying the asymptotic behavior of solutions V of equation (12) near to $-\infty$.

Now, we introduce the following lemma.

Lemma 3.1. *Let V be a monotone solution of equation* (12)*. Then*

(i) $\lim\limits_{x\to-\infty} E(V)(x) = +\infty.$
where $E(V)$ is given by the relation (29).

(ii) V is decreasing near to $(-\infty)$, moreover

$$\lim_{x\to-\infty} V(x) = +\infty.$$

Proof. For (i), we suppose that $\lim\limits_{x\to-\infty} E(V)(x)$ exists and it is finite, i.e. $\lim\limits_{x\to-\infty} E(V)(x) = L \geq 0$. Then, since V is monotone, $\lim\limits_{x\to-\infty} V(x) := d \in [0, +\infty]$. As L is finite so $\lim\limits_{x\to-\infty} F(V)(x)$ and $\lim\limits_{x\to-\infty} |V'(x)|$ are finite, and from the expression (30) we deduce that d is finite. But V converges, then $\lim\limits_{x\to-\infty} V\,'(x) = 0$. Considering that $\mu > 0$ and that $d \neq \Gamma^{-m/(m+p)}$, as $x \to -\infty$ in the equation (12), then we obtain:

$$V''(-\infty) = -f(d^{\,1/m}) \neq 0.$$

Which is impossible since V' converges. This contradiction gives that $\lim\limits_{x\to-\infty} E(V)(x) = +\infty$.

For (ii), assuming that $\lim_{x\to-\infty} V(x) = d$ is finite. Then

$$\begin{aligned} +\infty &= \lim_{x\to-\infty} E(V)(x) \\ &= \lim_{x\to-\infty} \left[\frac{1}{2}V'^2(x) + F(V)(x)\right] \\ &= \lim_{x\to-\infty} F(V)(x) = F(d). \end{aligned}$$

Hence necessarily $d = 0$. Otherwise, by integrating the equation (12) over $]x, x_1[$, we obtain:

$$\begin{aligned} V'(x_1) - V'(x) - a\,[V(x_1) - V(x)] \\ + \frac{\beta}{m}\int_x^{x_1} e^{\mu s} V^{\frac{1-m}{m}}(s) V'(s)ds \\ = -\int_x^{x_1} f(V^{1/m})(s)ds. \end{aligned}$$

Note that

$$\frac{\beta}{m}\int_x^{x_1} e^{\mu s} V^{\frac{1-m}{m}}(s) V'(s)ds =$$

$$\beta e^{\mu x_1} V^{\frac{1}{m}}(x_1) - \beta e^{\mu x} V^{\frac{1}{m}}(x) - \beta\mu \int_x^{x_1} e^{\mu s} V^{\frac{1}{m}}(s)ds.$$

Since $\mu > 0$, V is bounded and monotone then $V'(-\infty) = 0$ and thus the first member is bounded whereas $p \geq m$ the second member will not be bounded, which is contradictory. In conclusion $\lim_{y\to-\infty} V(y) = +\infty$ and V is decreasing near to $(-\infty)$. The proof is complete. □

The following theorem gives the asymptotic behavior of the solution U of problem (P) near the origin which is equivalent to giving the asymptotic behavior of the solution V of equation (12) near to $-\infty$, through the change of variable $y = e^x$ and the expression of V given by (11). Let us assume that V is monotone. Then we have the following result.

Theorem 3.1. *Assume that* $0 < l \neq \Gamma^{-1/(m+p)}$. *Let* U *be a solution of problem* (P) *on* $[0, +\infty[$. *Then*

$$\lim_{y\to0^+} \frac{y\,U'(y)}{U(y)} \in \{0, 1/m\}. \qquad (32)$$

Proof. Using expression (11) and the change of variable $y = e^x$, then proving (32) is equivalent to prove:

$$\lim_{x\to-\infty} \frac{V'(x)}{V(x)} \in \left\{m\frac{\alpha}{\beta}, 1 + m\frac{\alpha}{\beta}\right\}.$$

According to the previous Lemma, we have $\lim_{x\to-\infty} \frac{V'(x)}{V(x)} = L$ and $\lim_{x\to-\infty} V'(x) = +\infty$.

Dividing the equation (12) by V', we obtain:

$$\begin{aligned} \frac{V''(y)}{V'(y)} - a + \frac{\beta}{m} e^{\mu y} V^{\frac{1-m}{m}}(y) \\ +\mu\frac{V'(y)}{V(y)} - \frac{V^{-p/m}(y)}{V'(y)} = 0. \end{aligned}$$

Applying the Hopital's rule, we obtain:

$$L^2 - aL + \mu = 0,$$

i.e.

$$L^2 - (1 + 2m\frac{\alpha}{\beta})L + m\frac{\alpha}{\beta}(1 + m\frac{\alpha}{\beta}) = 0. \qquad (33)$$

Resolving this equation, we obtain:

$$L = m\frac{\alpha}{\beta} \quad \text{or eslse} \quad L = 1 + m\frac{\alpha}{\beta}.$$

Hence

$$\lim_{y\to0^+} \frac{y\,(U^m)'(y)}{U^m(y)} \in \{0, 1\}.$$

Thus we proved the required result. □

4 Conclusion

In the present paper, we have proven the global existence of positive solutions U of problem (P). Under some assumptions, These solutions behave like $y^{-\alpha/\beta}$ near $+\infty$. We have given also their asymptotic behavior near the origin.

The results obtained can be more developed by finding the exact limit of $\frac{y\,U'(y)}{U(y)}$ at the origin and also treating the case when the function $y^{\alpha/\beta}U(y)$ is not monotone. This will be part of our future work.

References:

[1] Guo J. S., Morita Y., Yotsutani S., Self similar solutions for a quenching problem spatially dependent non-linearity, *J. Nonlinear Analysis*, Vol.147, No.3, 2016, pp. 45-62.

[2] Guo J. S., On the quenching behavior of the solution of a semilinear parabolic equation, *J.Math. Anal. Appl.*, Vol.151, No.7, 1990, pp. 58-79.

[3] Guo Y., On the partial differential equations of electrostatic MEMS devices III: refined touchdown behavior, *J. Differential Equations*, Vol.244, No.7, 2008, pp. 2277-2309.

[4] Guo Z., Wei J., Asymptotic behavior of touchdown solutions and global bifurcations for an elliptic problem with a singular nonlinearity, *Comm. Pure Appl. Anal.*, Vol.7, 2008, pp. 765-786.

[5] Brezis H., Cazenave T., Martel Y., Ramiandridoa A., Blow up for $u_t - \Delta u = g(u)$ revisited, *Adv. Diff. Eqns.*, 1996, pp. 73-90.

[6] Ye D., Zhou F., On a general family of nonautonomous elliptic and parabolic equations, *Calc. Var.*, Vol.37, 2010, pp. 259-274.

[7] Filippas S., Kohn R. V., Refined asymptotics for the blow up of $u_t - \Delta u = u^p$, *Comm. Pure Appl. Math.*, Vol.45, 1992, pp. 821-869.

[8] Pelesko J. A., Triolo A. A., Nonlocal problems in MEMS device control, *J. Engrg. Math.*, Vol.41, 2001, pp. 345-366.

[9] Flores G., Mercado G., Pelesko J. A., Smyth N., Analysis of the dynamics and touchdown in a model of electrostatic MEMS, *SIAM J. Appl. Math.*, Vol.67, No.2, 2007, pp. 434-446.

[10] Acker A., Walter W., The Quenching Problem for Nonlinear Parabolic Differential Equations, *Lecture Notes in Mathematics, SpringerVerlag, New York*, Vol.564, No.1, 1976, pp. 1-12.

[11] Giga Y., Kohn R. V., Asymptotically self-similar blow-up of semilinear heat equations, *Comm. Pure Appl. Math.*, Vol.38, No.3, 1985, pp. 297-319.

[12] Guo J. S., Souplet. Ph., No touchdown at zero points of the permittivity profile for the MEMS problem, *SIAM J. Math. Anal.*, Vol.47, 2015, pp. 614-625.

[13] Levine H. A., Quenching, nonquenching, and beyond quenching for solutions of some parabolic equations, *Annali di Mat. Pura et Appl.*, Vol.155, 1990, pp. 243-260.

[14] Merle F., Zaag H., Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$, *Duke Math. J.*, Vol.86, 1997, pp. 143-195.

[15] Bouzelmate A., Gmira A., Singular solutions of an in homogeneous elliptic equation, *J. Nonlinear Functional Analysis and Appliations*, Vol.26, No.2, 2021, pp. 237-272.

[16] Bouzelmate A., EL Baghouri H., Behavior of Entire Solutions of a Nonlinear Elliptic Equation with An Inhomogeneous Singular Term, *Wseas Transactions on Mathematics*, Vol.22, No.62, 2023, pp. 232-244.

[17] Adimurthi K., Yadava S.L., An elementary proof of the uniqueness of positive radial solutions of a quasilinear Dirichlet Problem, *Arch. Rat. Mech. Anal.*, Vol.127, 1994, pp. 219-229.

[18] Ladyzenskaya O. A., Solonikov V. A., Ural'ceva N. N., Linear and Quasilinear Equations of Parabolic Type (2nd ed.), *Translations of Mathematical Monographs*, American Mathematical Society, 1968.

[19] Filippas S., Tertikas A., On similarity solutions of a heat equation with a nonhomogeneous nonlinearity, *J. Differential Equations*, Vol.165, No.2, 2000, pp. 468-492.

[20] Abdelhedi B., Zaag H., Refined blow-up asymptotics for a perturbed nonlinear heat equation with a gradient and a non-local term, *J. Differential Equations*, Vol.272, No., 2021, pp. 1-23.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Arij Bouzelmate and Abdelilah Gmira proposed the subject of the article to their Ph.D. student Fatima Sennouni. This paper is an extension of the work carried out by Arij Bouzelmate and Abdelilah Gmira. It brings together the techniques of Nonlinear Analysis. All the results were carried out by the three authors Arij Bouzelmate, Abdelilah Gmira and Fatima Sennouni.

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