Introduction to Tolerance Semigraph

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Abstract: - Tolerance semigraph comes from intersection semigraph which varies with a tolerance value in such a way that it generalizes interval semigraph. In this paper, we introduced the concept of tolerance semigraphs. We also present some related definitions, examples, and results of tolerance semigraphs. We also described a real-life application of tolerance semigraph with a suitable example, which is related to the scheduling problem and Greedy Algorithm.

Key-Words: - Semigraph, Intersection Semigraph, Interval Semigraph, Tolerance Semigraph, Interval Graph, Intersection Graph, Pendant Dendroids.

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1 Introduction

A graph $G = (V, E)$ is called an intersection graph, [1], for a finite family F of a non-empty set if there is a one-to-one correspondence between F and V such that two sets in F have a non-empty intersection if and only if their corresponding vertices in V are adjacent. We call F an intersection model of G. For an intersection model F, we use $G(F)$ to denote the intersection for F.

An undirected graph $G = (V, E)$ is said to be an interval graph, [1], if the vertex set V can be put into one-to-one correspondence with a set I of intervals on the real line such that two vertices are adjacent in G if and only if their corresponding intervals have non-empty intersection. That is, there is a bijective mapping $f: V \to I$. The set I is called an interval representation of G.

An undirected graph $G = (V, E)$ is called a tolerance graph, [3], if there exists a collection $\mathcal{I} = \{I_x | x \in V\}$ of closed intervals on the real line and a set $t = \{t_x | x \in V\}$ of positive numbers satisfying $xy \in E \Leftrightarrow |I_x \cap I_y| \geq \min\{t_x, t_y\}$ where $|I|$ denotes the length of interval I and the pair $< \mathcal{I}, t >$ is called a tolerance representation of G.

The notion of semigraph, [2], [7], is a generalization of that of a graph. While generalizing a structure, one naturally looks for one in which every concept in the structure has a natural generalization. Semigraph is such a natural generalization of the graph and it resembles a graph when drawn in a plane.

In a graph, one edge contains two vertices i.e. the graph is represented in a situation where these involve only two objects in a relation. But when there is a relation contained among the objects more than two it is not possible to explain in terms of graph. So, to handle this type of relation situation we need a generalized graph and this generalized structure is Semigraph. In semigraph, we see that all properties enjoyed by vertices are also enjoyed by edges.

The problem of characterizing tolerance graphs, [3], [4], [5], [6], was first proposed by M.C. Golumbic. Since the concept of semigraph was first introduced in mathematics in the year 2000 by E. Sampathkumar. But, within this long period, the concept of tolerance semigraph was still not introduced though almost all of the concepts were generalized on semigraph.

Here, we established tolerance semigraph and some related results.

2 Preliminary

Definitions, [2], [7].

Semigraph: A semigraph G is a pair (V, X) where V is a non-empty set whose elements are called vertices of G and X is a set of n-tuples, called edges of G, of distinct vertices, for various $n \geq 2$, satisfying the following conditions.

(a) Any two edges have at most one vertex in common.

(b) Two edges $(u_1,u_2,\ldots,u_n)$ and $(v_1,v_2,\ldots,v_m)$ are considered to be equal if and only if

(i) $m = n$ and

(ii) Either $u_i = v_i$ for $1 \leq i \leq n$, or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

An edge e is represented by a simple open Jordan curve which is drawn as a straight line whose endpoints are called the end vertices of the edge e and the m-vertices of the edge e each of which is not an m-vertex of any other edge of the semigraph G are denoted by small circles placed on the curve in between the end vertices, in the order specified by e. The end vertices of edges that are not m-vertices are specially represented by thick dots. If an m-vertex of an edge e is an end vertex of an edge e' i.e. an (m, e) vertex, we draw a small tangent to the circle at the end of the edge e' .

Example 2.1: Let $G = (V, X)$ be a semigraph. Then the edges of the semigraph in Figure 1 are $(v_0, v_1, v_2), (v_2, v_6, v_7, v_8), (v_1, v_3, v_4), (v_4, v_5)$, (v_5, v_6)

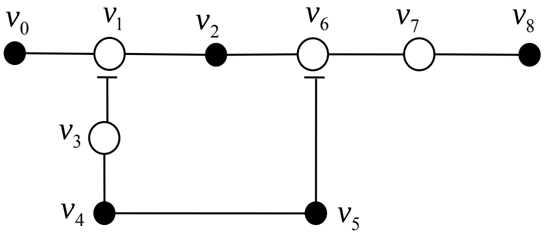

Fig. 1: Example of Semigraph

Subedges : A subedge of an edge $E = (v_1, v_2, \ldots, v_n)$ is a k-tuple $E' = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ or $1 \leq i_k < i_{k-1} < \cdots < i_1 \leq n$. We say that the subedge E' is induced by the set of vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$.

Example 2.2: Let $G = (V, X)$ be a semigraph. Then the sub-edges of the semigraph from Figure 1 are $(v_0, v_2), (v_2, v_7, v_8), (v_3, v_4), (v_4, v_5), (v_5, v_6)$.

Partial edge: A partial edge of E is a $(j - i + 1)$-tuple $E(v_i, v_j) = (v_i, v_{i+1}, \ldots, v_j)$, where $1 \leq i \leq n$. Thus a subedge E' of an edge E is a partial edge if and only if, any two consecutive vertices in E' are also consecutive vertices of E.

Example 2.3: Let $G = (V, X)$ be a semigraph. Then the partial edges of the semigraph from Figure 1 are $(v_6, v_7, v_8), (v_0, v_1), (v_1, v_2)$.

Pendant Dendroids: A pendant dendroid is a dendroid in which every edge is a pendant.

Intersection Semigraph: Let $G = (V, X)$ be a semigraph and F be a set of collections of partial edges of the semigraph G, i.e. $F = \{S_i = p_i | \ p_i$ is a partial edge of some edge in $G\}$. Then $\tau(F)$ is said to edge intersection semigraph in G where the edge set of $\tau(F)$ consists of the edge of the types:

(a) $E^* = (p_{i_1}, p_{i_2}, \ldots, p_{i_r}), r \geq 2$, where $p_{i_j}, p_{i_{j+1}}, 1 \leq j \leq r-1$ are the consecutive partial edges of the same s- edge $E \in X$.

(b) $E^* = (p_i, p_j)$ where p_i and p_j are not the partial edges of the same s-edge and they have a vertex in common and also $\tau(F) \cong G$.

Interval Semigraph: Let $G = (V, X)$ be a semigraph and F be a set of collection of partial edges of some edge E of the semigraph G, i.e. $F = \{S_i = p_i | \ p_i$ is a partial edge of some edge E of $G\}$. Then p_i corresponds to an interval $[a, b]$ in $\mathbb{R}$, $\forall i$ provided that if p_i and p_j are adjacent then our corresponding interval is $[a, b] \cap [c, d] \neq \emptyset$.

3 Tolerance Semigraph

Definition 3.1: Let $G = (V, X)$ be a semigraph and $p_i \in F$ where p_i's are the partial edges and F is a set of collections of partial edges of some edge E of the semigraph $G = (V, X)$. Then a semigraph $G = (V, X)$ is said to be a tolerance semigraph if for edge partial edge $p \in X$ can be assigned to a closed interval I_p and a tolerance $t_p \in \mathbb{R}^+$ so that p_i and p_j are adjacent if and only if $|I_p \cap I_q| \geq \min\{t_p, t_q\}$.This type of collection $(I, \mathcal{T})$ of intervals and tolerances is called a tolerance representation of the semigraph where $\mathfrak{I} = \{I_p | p \in F\}$and $\mathcal{T} = \{t_p | p \in F\}$.

Example 3.1: Let $G = (V, X)$ be a semigraph with partial edges $p_1 = \{(a, b), (b, c), (c, d)\}$, $p_2 =$

$\{(e,b)\}$, $p_3 = \{(f,c)\}$, $p_4 = \{(b,g)\}$, $p_5 = \{(c,h)\}$ as shown in Figure 2 and let F be a set which is a collection of the partial edges. Now, according to the definition, there exist corresponding intervals for each partial edge. Let $I_{p_1}, I_{p_2}, I_{p_3}, I_{p_4}, I_{p_5}$ be the corresponding intervals of the partial edges p_1, p_2, p_3, p_4, p_5 where $I_{p_1} = [1,7], I_{p_2} = [0,3], I_{p_3} = [6,9], I_{p_4} = [2,8], I_{p_5} = [5,9]$ as shown in Figure 3 .

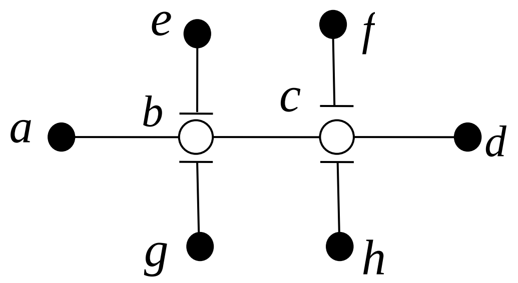

Fig. 2: Semigraph $G = (V, X)$

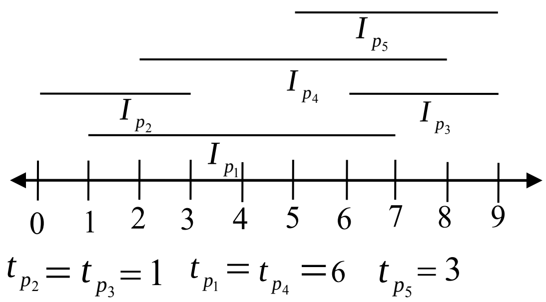

Fig. 3: Tolerance representation on the real line

Theorem 3.1: If $G = (V, X)$ is an interval semigraph then G is a tolerance semigraph with constant tolerances.

Proof: Let $G = (V, X)$ be an interval semigraph with a representation in which the interval S_p is assigned to a partial edge p. Let k be any positive number less than $\min\{|S_p \cap S_q| : |S_p \cap S_q| > 0 \ and \ p, q \in X\}$. Then the intervals $\{S_e | e \in X(G)\}$ related to the tolerances $t_e = k$ for all $e \in X$ gives a tolerance representation of the semigraph G with constant tolerances.

Definition 3.2: A semigraph $G = (V, X)$ has a representation with $t_p \leq |I_p|$, for all $p \in X$ where p is the partial edge then G is called a bounded tolerance semigraph.

Regular representation of a tolerance semigraph: A tolerance semigraph is called a regular representation if it satisfies one or more of the following properties-

I. If all the tolerance values of the interval semigraph are different from each other.

II. If no two distinct intervals share an endpoint.

III. Any interval is set to infinity if any tolerance is greater than its corresponding interval.

IV. The tolerances are strictly positive.

V. The intersection of all the intervals should be a non-empty intersection.

Then a tolerance a tolerance representation that satisfies these properties is called regular representation.

Theorem 3.2: If G is a tolerance representation with constant tolerances then G is a bounded tolerance graph with constant tolerances.

Proof: Let $G = (V, X)$ be a tolerance semigraph and $(\mathcal{J}, \mathcal{T})$be a tolerance representation of the tolerance semigraph $G = (V, X)$with $t_p = k$ for all $p \in X$, where p is the partial edge of the tolerance semigraph $G = (V, X)$.
Now, for the partial edge $p \in X$ with $|I_p \geq k|$ for all $p \in X$,
then define $I'_p = I_p$.
If $|I_p| < k$ then p is a pendent dendroid of G and we define $I_a{}'$ to be an interval of breadth k on a real line that cannot intersect any other intervals $I_b{}'$ and this defines a bounded tolerance semigraph representation of $G = (V, X)$ with constant tolerant values.

Theorem 3.3: If $G = (V, X)$ is a bounded tolerance semigraph with constant tolerances then $G = (V, X)$ is an interval semigraph.

Proof: Let $(\mathcal{J}, \mathcal{T})$ be a bounded tolerance semigraph representation of the semigraph $G = (V, X)$ with $t_p = k$ for all $\in X$.

Now, if we denote the interval I_p by $[L(p), R(p)]$ then the partial edges $p \in X$ with

$[R(p) - L(p)] \geq k$ and define $I'_p = [L(p) + \frac{k}{2}, R(p) - \frac{k}{2}]$.

Otherwise, if $[R(p) - L(p)] < k$ then the partial edge p is a pendant dendroid of the semigraph $G = (V, X)$, and the interval I'_p be a point that lies on the real line and that interval doesn't intersect any other intervals on the real line and the intervals $\{I'_p | p \in X\}$ gives an interval semigraph representation of the semigraph $G = (V, X)$.

4 Application

Use of Common Classroom Problems in Teaching at the College Level

Tolerance semigraphs are among the most helpful mathematical structures for modeling real-world problems. In interval semigraph, we see the objects are conflicted because of overlapping in time or like any other. Generally, there are lots of applications available in the areas of scheduling, biology, data storage, chemistry, etc. As an example, here we consider a classroom scheduling problem that arises at a College. In a College, professors are each assigned to a single classroom for their entire teaching interval. But sometimes a conflict arises because, at the same time and same classroom, there are assigned meetings and teaching the classroom. In that time, tolerance arises naturally. We face one of the main problems that are within different courses, we use one common classroom for meetings as well as for teaching in the College. When this type of situation or coincidence arises in the scheduling of classrooms, we must be assigned to make minimal possible classrooms for various courses so that one can tolerate a certain amount of overlap time with their classrooms.

This model of the problem is solvable by using a Greedy Algorithm on the concept of Tolerance Semigraph. Here we give a tolerance to each interval of time and upon these tolerances we apply the Greedy Algorithm.

5 Conclusion

This paper studied the tolerance semigraph where tolerance semigraph is a special class of intersection semigraph. There are lots of tolerance graph theoretic concepts still waiting for investigation in the structure of tolerance semigraph. So, the study of tolerance semigraph is expected to bring significant results in the near future.

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Conflict of Interest

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