On the Fundamental Spinor Matrices of Real Quaternions

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Abstract: - In this study, the real quaternions and spinors are studied. The motivation of this study is to express the Hamilton matrices of real quaternions more shortly and elegantly, namely spinors. Therefore, firstly, two transformations between real quaternions and spinors are defined. These transformations are defined for two different spinor matrices corresponding to the left and right Hamilton matrices since the quaternion product is not commutative. Thus, the fundamental spinor matrix corresponding to the fundamental matrix of real quaternions is obtained and some properties are given for these spinor matrices. Finally, the eigenvalues and eigenvectors of the fundamental spinor matrix are obtained.

Key-Words: - Spinors, Quaternions, Hamilton matrices, Spinor metrices

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1 Introduction

Quaternions were obtained by generalizing complex numbers to develop a new number system. The studies, [1], [2], [3], introduced a new multiplication operation with this into vector algebra and obtained the quaternions that may be possible in its division. Moreover, real quaternions have no commutative property. Quaternion product is a combination of the scalar multiplication, the Euclidean inner product, and the vector product. Thus, the set of real quaternions is a two-dimensional vector space on the set of complex numbers and a four-dimensional vector space on the set of real numbers. Today, the matrices are used in many branches of science and have a great importance. For this reason, the relationship of matrices with quaternions has attracted much attention. Since the quaternion product is not commutative, the product of the right and the product of the left with another quaternion are not equal. This is paired matrices with $4x4$ the type quaternions with two separate multiplications, thanks to a transformation. Thus, the left product represents the left matrix representation of a quaternion, and the right product represents the right matrix representation of a quaternion. These matrix representations are called Hamilton matrices corresponding to the Hamilton operators of the real quaternions. On the other hand, a lot of studies have been done about matrices whose elements are composed of quaternions. The studies on quaternions date back to 1936. The study, [4], gave the concept of similarity for matrices whose elements are real quaternions. Later, the study, [5], made a study on the eigenvalue and diagonalization of quaternion matrices. In addition, the study, [6], showed that every quaternion matrix, including a square, has a characteristic root, and in addition, similar matrices have the same characteristic root. The study, [7], gave one of the most important studies on quaternion matrices.

With the introduction of Hamilton in 1843, quaternions have found many uses until today. Quaternions, which provide great convenience in engineering fields apart from geometry and algebra, are also of great importance in the mathematics of today's technology. It is also used in computer graphics, physics, mechanics, kinematics, computer games, animations, and digital imaging. Quaternions are of great importance in geometry, especially in the representation of the rotation of objects in 3-dimensional space. Quaternions also have many uses in physics. The use of complex numbers in mechanical and electrical applications, especially in circuit analysis, limits the applications since they are two-dimensional. In 3-dimensional applications, in cases where vectors are insufficient in some applications, quaternions add a fourth dimension to applications, providing great convenience. The most important of the fields where quaternions find applications in physics is Einstein's special and general relativity theories. Quaternions are used to describe electron spin in quantum mechanics. Quantum operators operating on a spinor can be represented by quaternions considering the relationship of $2x2$ matrices with quaternions. With the help of this approach, B. L. Van der Waerdan developed the mathematical formula of spinors in 1930. Pauli in 1927 and Dirac in 1938 demonstrated spinor equations to describe the electron spin physically.

The introduction of spinors is one of the most difficult topics in quantum mechanics. Even if spin-1/2 is considered, some fundamental aspects of spinors, such as the effects of rotation on spinors, turn out to be difficult to explain. According to physicists, spinors are multilinear transformations. Thanks to this feature, spinors are mathematical entities somewhat like tensors and allow a more general treatment of the notion of invariance under rotation and Lorentz boosts. For mathematicians, spinors are vectorial objects and their multilinear features do not play any role. In addition, spinors have one index. In discussing vectors and tensors, there are two ways, in which we can proceed: the geometrical and analytical. To use the geometrical approach, we describe each kind of quantity in terms of its magnitudes and directions. In the analytical treatment, we use components. The study, [8], while investigating linear representations of simple groups developed the most general mathematical form of spinors. In geometric terms, the study, [9], introduced spinors. The study, [9], developed the spinor theory geometrically by giving only the geometric definition of spinors. Then, the spinor formulation of curves is given by the study, [10], considering Frenet vectors of curves in three-dimensional Euclidean space. That study is an important study for the relationship between curve theory and spinors in differential geometry. Pauli matrices, which are a basis $SU(2)$, and spinor algebra, which have two complex components, provide a nice representation of rotations in three-dimensional real space. In this context, the study, [11], established a new relationship between quaternions and spinors and expressed quaternion kinematics with spinors. In the study, [11], a one-to-one and linear relationship was established between spinors and real quaternions, and spinor formulation and thus spinor kinematics of spins represented by real quaternions were obtained. On the other hand, the study, [12], investigated quaternions and spinors in quantum mechanics by establishing a relationship between $SO(3)$ and $SU(2)$. The study, [13], obtained a main expression of quaternions with matrices by $2x2$ by considering the isomorphism between real quaternions and spinors. Moreover, the properties of the fundamental real matrix associated with a quaternion were investigated and a frequently considered quaternion equation was examined, from which the nth power of a quaternion can be determined, [14]. Therefore, the spinor representations in Euclidean 3-space were studied using different frames such as Darboux, Bishop, q-frame, [15], [16], [17]. The study, [18], obtained the Frenet spinor equations of Lie groups in Euclidean space with a bi-invariant metric and gave some special situations for these Lie groups with three-dimensional. Later, studies on spinors in this field of differential geometry focused on special curves. The study, [19], revealed the spinor representations of the involute-evolute curves and the relationship between these spinors. Then, Bertrand curves were represented by spinors in the complex plane, and the study, [20], proved the relationship between spinors corresponding to these Bertrand curves. After that, the successor curve couple corresponded to two different spinors, and geometric interpretations were derived, [21]. In addition to that, the spinor representations of some curve pairs selected in Minkowski space were obtained, [22], [23], [24], [25]

The motivation of this study is to obtain a new and easier matrix representation of the Hamilton matrices corresponding to the Hamilton operators of the real quaternions. For this, first, considering the isomorphism between spinors and quaternions, the spinor matrices corresponding to the right and left Hamilton matrices of real quaternions have been created. Since the quaternion product is not commutative, two separate spinor matrices corresponding to these Hamilton matrices have been formed. These matrices have been called the left and right Hamilton spinor matrices. Moreover, some properties of these right and left Hamilton spinor matrices have been given. Consequently, considering the left Hamilton spinor matrix as the fundamental spinor matrix some theorems and results about the eigenvalues and eigenvectors of the fundamental spinor matrix have been obtained.

2 Preliminaries

2.1 Quaternions and Spinors

In this section, we give some propositions and theorems about the real quaternions and spinors. Let $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$ be real quaternions in the set of real quaternions H where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k} \in \mathbb{R}^3$ such that $\boldsymbol{ii} = \boldsymbol{jj} = \boldsymbol{kk} = -1$, $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}$, $\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}$, $\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}$. In this case, the set of real quaternions H can be expressed

$$H = \{q \mid q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3,\ q_0, q_1, q_2, q_3 \in R,\ \boldsymbol{i},\boldsymbol{j},\boldsymbol{k} \in R^3\}.$$

Now, we assume that the real quaternion $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$ is $q = S_q + \boldsymbol{V}_q$ where the scalar part of q is $S_q = q_0$ and the vector part of q is $\boldsymbol{V}_q = \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$, [26]. Let any two real quaternions be $p = S_p + \boldsymbol{V}_p$ and $q = S_q + \boldsymbol{V}_q \in H$. Therefore, the addition of these real quaternions is expressed

$$p \oplus q = \left(S_p + \boldsymbol{V}_p\right) \oplus \left(S_q + \boldsymbol{V}_q\right) = \left(S_p + S_q\right) + \left(\boldsymbol{V}_p + \boldsymbol{V}_q\right)$$

where $\oplus : H \times H \to H$. Moreover, the scalar product of the real quaternion $q = S_q + \boldsymbol{V}_q \in H$ is written by $\lambda \odot q = \lambda \odot \left(S_q + \boldsymbol{V}_q\right) = \lambda S_q + \lambda \boldsymbol{V}_q$ where $\lambda \in R$ and $\odot : R \times H \to H$. Consequently, with the aid of these operations, we say that the system $\{H, \oplus, R, +, ., \odot\}$ is a real vector space, briefly, this space can be denoted by H where $H = S_p\{\boldsymbol{1},\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ and $\dim(H) = 4$, [26]. Assume that any two real quaternions are $p = S_p + \boldsymbol{V}_p,\ q = S_q + \boldsymbol{V}_q \in H$. In this case, this quaternion product is $pq = S_p S_q - \langle \boldsymbol{V}_p, \boldsymbol{V}_q \rangle + S_p \boldsymbol{V}_q + S_q \boldsymbol{V}_p + \boldsymbol{V}_p \wedge \boldsymbol{V}_q$ where "$\langle , \rangle$" and "$\wedge$" are Euclidean scalar and vector products in E^3. We know that the product of two quaternions is a quaternion, the product of quaternions has the properties of associative and distributive. But the quaternion product is not commutative. Thanks to these properties, the system $\{H, \oplus, R, +, ., \odot\}$ is an associative algebra, [26]. Let any real quaternion be $q = S_q + \boldsymbol{V}_q \in H$. Therefore, the conjugate of the quaternion q is expressed as $q^* = S_q - \boldsymbol{V}_q$. On the other hand, the norm of the real quaternion q is defined by $N(q) = \sqrt{qq^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$. In this case, if the norm of the quaternion $q \in H$ is $N(q) = 1$ then, this quaternion is called the unit quaternion. In addition that, the inverse of the real quaternion q is $q^{-1} = q^* / N^2(q)$ where $q \neq 0$, [26].

On the other hand, assume that the real quaternion $q \in H$. Therefore, we can write this quaternion as $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 = \left(q_0 + \boldsymbol{i}q_1\right) + \left(q_2 + \boldsymbol{i}q_3\right)\boldsymbol{j}$ and we say that $H = \left\{q = z_1 + z_2\boldsymbol{j} \mid z_1 = q_0 + \boldsymbol{i}q_1,\ z_2 = q_2 + \boldsymbol{i}q_3 \in C\right\}$ is isomorphic with C^2 since the real quaternion q matches with the complex number $z_1 + z_2\boldsymbol{j} \in C^2$, [26], [27]. In addition to that, the real quaternion can be written as $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 = \left(q_0 + \mathrm{i}q_1\right) + \left(q_2 - \boldsymbol{i}q_3\right)\boldsymbol{j} = z_1 + \boldsymbol{j}\overline{z_2}$. In this case, H has the basis $\{\boldsymbol{1},\boldsymbol{j}\}$. Therefore, the transformation $T_q(p) = pq$ is linear and the 2×2 matrix corresponding to this linear transformation is

$$\begin{bmatrix} q_0 + \boldsymbol{i}q_1 & q_2 + \boldsymbol{i}q_3 \\ -q_2 + \boldsymbol{i}q_3 & q_0 - \boldsymbol{i}q_1 \end{bmatrix},$$

[26]. The algebra of the real quaternions H contains infinite sub-algebras derived from bases such as $\{\boldsymbol{1},\boldsymbol{i}\}$, $\{\boldsymbol{1},\boldsymbol{j}\}$, $\{\boldsymbol{1},\boldsymbol{k}\}$,......, [28]. Therefore, the set of quaternions H can be written according to many bases. Moreover, the study, [28], wrote the real quaternion q as $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 = \left(q_0 + \boldsymbol{j}q_2\right) + \boldsymbol{i}\left(q_1 + \boldsymbol{j}q_3\right)$ with the aid of the basis $\{\boldsymbol{1},\boldsymbol{i}\}$ and obtained the complex matrix $\begin{bmatrix} q_0 + \boldsymbol{j}q_1 & -q_1 + \boldsymbol{j}q_3 \\ q_1 + \boldsymbol{j}q_3 & q_0 - \boldsymbol{j}q_2 \end{bmatrix}$.

Assume that $C^{\boldsymbol{3}}$ is the complex vector space and the vector $\boldsymbol{x} = (x_1, x_2, x_3) \in C^{\boldsymbol{3}}$ is isotropic vector where $x_1^2 + x_2^2 + x_3^2 = 0$. The set of the isotopic vectors in the complex vector space $C^{\boldsymbol{3}}$ corresponds to a surface with two dimensional in $C^{\boldsymbol{3}}$. If this surface with two dimensional is parameterized by the complex numbers γ_1 and γ_2 then, the equations $x_1 = \gamma_1^2 - \gamma_2^2$, $x_2 = \mathbf{i}(\gamma_1^2 + \gamma_2^2)$, $x_3 = -2\gamma_1^2\gamma_2^2$ are provided where $\mathbf{i}$ is the complex unit, $\mathbf{i}^2 = -1$ and $\gamma_1, \gamma_2 \in C$. It can be easily seen that every isotopic vector in the complex vector space $C^{\boldsymbol{3}}$ corresponds to two vectors in C^2 such that (γ_1,γ_2) and $(-\gamma_1,-\gamma_2)$. On the contrary, these both vectors are given in this way in C^2 correspond to a single isotropic vector $\boldsymbol{x}$. Cartan expressed that the complex vector with two dimensional such that $\gamma=\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}$ is called as spinor, [9]. Moreover, the study, [9], gave that the conjugate of the spinor $\gamma=(\gamma_1,\gamma_2)$ is $\tilde{\gamma}=\mathbf{i}C\bar{\gamma}$. $SO(3)$, the group of rotations about the origin in a three-dimensional real vector space R^3, is homomorphic to $SU(2)$, the group of unitary matrices with two dimensional 2×2. The elements of group $SO(3)$ move vectors in vector space R^3, while elements of group $SU(2)$ move vectors with two complex components, i.e. spinors, [10], [29], [30]. With the aid of this homomorphism, the study, [31], paired an isotropic vector with a spinor $\gamma=\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix}$. Now, we assume that σ is a vector whose Cartesian components are the complex symmetric 2×2 matrices such that $\sigma_1=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$, $\sigma_2=\begin{bmatrix}\mathbf{i}&0\\0&\mathbf{i}\end{bmatrix}$, $\sigma_3=\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$. These matrices are produced in Pauli matrices $P_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $P_2=\begin{bmatrix}0&-\mathbf{i}\\ \mathbf{i}&0\end{bmatrix}$, $P_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ with the help of the matrix $C=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$, [31]. Therefore, the isotropic vector $\boldsymbol{a}+\mathbf{i}\boldsymbol{b}\in C^3$ can be written by any spinor γ such that $\boldsymbol{a}+\mathbf{i}\boldsymbol{b}=\gamma^t\sigma\gamma$. In addition to that, the real vector $\boldsymbol{c}\in R^3$, orthogonal with the vectors $\boldsymbol{a},\boldsymbol{b}\in R^3$, can be expressed $\boldsymbol{c}=-\check{\gamma}\sigma\gamma$ in terms of the spinor γ where "$\check{\ }$" is the mate of the spinor such that $\check{\gamma}=-C\gamma$, [31].

The study, [11], gave the relation between real quaternions and spinors with the transformation $f:H\to S$ such that

$$q\to f(q)=f(q_0+\mathbf{i}q_1+\boldsymbol{j}q_2+\boldsymbol{k}q_3)\cong\varphi=\begin{bmatrix}q_3+\mathbf{i}q_0\\ q_1+\mathbf{i}q_2\end{bmatrix}.$$

In addition, the study, [11], obtained that the bijective transformation f provides the following equations

i) $f(q+p)=f(q)+f(p)$,

ii) $f(\lambda q)=\lambda f(q),\quad \lambda\in R$

for $p,q\in H$. Therefore, the transformation f is linear.

2.2 The Hamilton Operators of Real Quaternions

We assume that any real quaternion is $q=q_0+\boldsymbol{i}q_1+\boldsymbol{j}q_2+\boldsymbol{k}q_3\in H$ and the linear transformation h^+ is as

$$\begin{aligned}h^+&:H\to H\\ p&\to h^+(p)=qp\end{aligned}. \tag{1}$$

In this case, we can write the matrix corresponding to basis $\{\boldsymbol{1},\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$

$$H^+(q)=\begin{bmatrix}q_0&-q_1&-q_2&-q_3\\ q_1&q_0&-q_3&q_2\\ q_2&q_3&q_0&-q_1\\ q_3&-q_2&q_1&q_0\end{bmatrix}$$

with the aid of the transformation h^+. Similarly, with the aid of the linear transformation

$$\begin{aligned}h^-(q)&:H\to H\\ p&\to h^-(p)=pq.\end{aligned} \tag{2}$$

Therefore, the matrix corresponding to the basis $\{\boldsymbol{1},\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ can be obtained as

$$H^-(q)=\begin{bmatrix}q_0&-q_1&-q_2&-q_3\\ q_1&q_0&q_3&-q_2\\ q_2&-q_3&q_0&q_1\\ q_3&q_2&-q_1&q_0\end{bmatrix}$$

according to the transformation h^-. Here, the matrices $H^+(q)$ and $H^-(q)$ are called the left and right Hamilton matrices corresponding to the Hamilton operators h^+ and h^- of the real quaternion q, [27], [32]. On the other hand, we suppose two real quaternion $p=p_0+\boldsymbol{i}p_1+\boldsymbol{j}p_2+\boldsymbol{k}p_3$ and $q=q_0+\boldsymbol{i}q_1+\boldsymbol{j}q_2+\boldsymbol{k}q_3$. In this case, the matrix form of the product qp of these real quaternions is

$$qp\cong\begin{bmatrix}q_0p_0-q_1p_1-q_2p_2-q_3p_3\\ q_1p_0+q_0p_1-q_3p_2+q_2p_3\\ q_2p_0+q_3p_1+q_0p_2-q_1p_3\\ q_3p_0-q_2p_1+q_1p_2+q_0p_3\end{bmatrix}=\begin{bmatrix}q_0&-q_1&-q_2&-q_3\\ q_1&q_0&-q_3&q_2\\ q_2&q_3&q_0&-q_1\\ q_3&-q_2&q_1&q_0\end{bmatrix}\begin{bmatrix}p_0\\p_1\\p_2\\p_3\end{bmatrix}=H^+(q)P.$$

Similarly, the matrix form of the product pq of these real quaternions is

$$pq \cong \begin{bmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\ q_1p_0 + q_0p_1 + q_3p_2 - q_2p_3 \\ q_2p_0 - q_3p_1 - q_0p_2 + q_1p_3 \\ q_3p_0 + q_2p_1 - q_1p_2 + q_0p_3 \end{bmatrix} = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = H^-(q)P, \text{ [27].}$$

Moreover, the representation $H^+(q)$ of a quaternion q is well-known in algebra. Since $H^+(q)$ plays a crucial role in our subsequent considerations, we call it as the fundamental matrix.

Theorem 2.1: Let any real quaternion be $q \in H$. In this case, the Hamilton matrices H^+ and H^- produced from the operators h^+ and h^- are orthogonal, [27].

Theorem 2.2: The product of Hamilton matrices H^+ and H^- are commutative. Therefore, there is equality $H^+(q)H^-(p) = H^-(p)H^+(q)$, [27].

Theorem 2.3: Let any two real quaternions be $p, q \in H$ and $\lambda \in R$. Hence, the Hamilton matrices H^+ and H^- provide the following properties;

i) $p = q \Leftrightarrow H^+(p) = H^+(q) \Leftrightarrow H^-(p) = H^-(q)$

ii) $H^+(p+q) = H^+(p) + H^+(q),$

$H^-(p+q) = H^-(p) + H^-(q)$

iii) $H^+(pq) = H^+(p)H^+(q),\ H^-(pq) = H^-(q)H^-(p)$

iv) $H^+(\lambda q) = \lambda H^+(q),\ H^-(\lambda p) = \lambda H^-(q)$

v) $\det\left[H^+(q)\right] = N^2(q),\ \det\left[H^-(q)\right] = N^2(q)$

vi) $H^+(q^{-1}) = \left(H^+(q)\right)^{-1},$

$H^-(q^{-1}) = \left(H^-(q)\right)^{-1},\ N(q) \neq 0$

3 Main Theorems and Proofs

In this section, we establish a relationship between quaternions and spinors and we give spinors corresponding to real quaternions. Then, we express the spinor matrices corresponding to Hamilton matrices of real quaternions. In addition to that, we calculate the eigenvalues and eigenvectors of these spinor matrices, which we call Hamilton spinor matrices, after giving some properties of these matrices. Consequently, we obtain some conclusions.

3.1 Spinor Representation of Real Quaternions

Let $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \in H$ be an arbitrary real quaternion where $q_0, q_1, q_2, q_3 \in \mathbb{R}$, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} \in \mathbb{R}^3$, and H is real quaternion space. In this case, we can write the real quaternion q with regard to the basis $\{\boldsymbol{k}, \boldsymbol{i}\}$ with left multiplication as

$$q = \boldsymbol{k}(q_3 + \mathbf{i}q_0) + \boldsymbol{i}(q_1 + \mathbf{i}q_2) \tag{3}$$

where $\mathbf{i} = \boldsymbol{ji} = -\boldsymbol{k}$ and since $\mathbf{i}^2 = -1$ we can consider that $\mathbf{i}$ is imaginer unit. Therefore, the quaternion q can be expressed in terms of two complex numbers $\varphi_1 = q_3 + \mathbf{i}q_0,\ \varphi_2 = q_1 + \mathbf{i}q_2 \in C$. So, we can give the following definition.

Definition 3.1: Let $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$ be a real quaternion in H . In this case, the set of real quaternions is defined by

$$H = \left\{ q = \boldsymbol{k}\varphi_1 + \boldsymbol{i}\varphi_2 \mid \varphi_1 = q_3 + \mathbf{i}q_0,\ \varphi_2 = q_1 + \mathbf{i}q_2 \in C \right\} \tag{4}$$

where ‘‘ $\bar{\ }$ ‘‘ is a complex conjugate.
As a result of Definition 3.1, according to the expression given in equation (4) the following transformation can be given.

Let the real quaternion be q written by in terms of the spinor φ. Therefore, we can give the transformation between quaternions and spinors as $f: H \to S$

$$q \to f(q) = f(q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3) \cong \varphi = \begin{bmatrix} q_3 + \mathbf{i}q_0 \\ q_1 + \mathbf{i}q_2 \end{bmatrix} \tag{5}$$

with aid of the equation (3), [11]. As is known, the vector space H is isomorphic to the space C^2, [33]. Therefore, the transformation f defined in equation (5) is isomorphism.

Now, we give the following definition about the conjugate of spinors based on the relationship between quaternions and spinors in the equation the transformation f.

Definition 3.2: Suppose that the quaternion $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \in H$ matches the spinor $\varphi = \begin{bmatrix} q_3 + \mathbf{i}q_0 \\ q_1 + \mathbf{i}q_2 \end{bmatrix} \in S$. In this case, the four different definitions of conjugate for spinors;

i) The complex conjugate $\bar{\varphi}$ of the spinor $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \in S$ corresponding to the real quaternion $q \in H$ is defined by

$$\bar{\varphi} = \begin{bmatrix} \overline{\varphi_1} \\ \overline{\varphi_2} \end{bmatrix} = \begin{bmatrix} q_3 - \mathbf{i}q_0 \\ q_1 - \mathbf{i}q_2 \end{bmatrix}.$$

ii) The spinor φ^* corresponding to the quaternionic conjugate $q^* = q_0 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3$ of the real quaternion $q \in H$ is defined by

$$\varphi^* = \begin{bmatrix} -\overline{\varphi_1} \\ -\varphi_2 \end{bmatrix} = -\begin{bmatrix} q_3 - \mathbf{i}q_0 \\ q_1 + \mathbf{i}q_2 \end{bmatrix}.$$

iii) The spinor conjugate φ of the spinor $\varphi \in S$ by given, [9], is defined by

$$\varphi = \mathbf{i}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \overline{\varphi_1} \\ \overline{\varphi_2} \end{bmatrix} = \mathbf{i}\begin{bmatrix} \overline{\varphi_2} \\ -\overline{\varphi_1} \end{bmatrix} = \begin{bmatrix} q_2 + \mathbf{i}q_1 \\ -q_0 - \mathbf{i}q_3 \end{bmatrix}. \quad (6)$$

iv) The mate of the spinor $\varphi \in S$ by given, [31], is defined by

$$\varphi = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \overline{\varphi_1} \\ \overline{\varphi_2} \end{bmatrix} = \begin{bmatrix} -\overline{\varphi_2} \\ \overline{\varphi_1} \end{bmatrix} = \begin{bmatrix} -q_1 + \mathbf{i}q_2 \\ q_3 - \mathbf{i}q_0 \end{bmatrix}.$$

Therefore, the following corollaries can be given.

Corollary 3.1: Let q be an arbitrary real quaternion and φ be the spinor corresponding to this quaternion q. There is the relationship

$$\check{\varphi} = \mathbf{i}\varphi$$

between the spinor conjugate and the mate of the spinor φ.

Corollary 3.2: Consider that the spinor $\varphi \in S$ corresponds to the real quaternion $q \in H$. In this case, the spinor equation of the norm of q is given that

$$N(\varphi) = \check{\varphi}^t C^t \varphi = \mathbf{i}\varphi^t C^t \varphi$$

where $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Proof: Suppose that the spinor $\varphi \in S$ corresponds to the real quaternion $q \in H$. Therefore, we write

$$\left(\check{\varphi}\right)^t C^t \varphi = \begin{bmatrix} -\overline{\varphi_2} & \overline{\varphi_1} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$
$$= \varphi_1\overline{\varphi_1} + \varphi_2\overline{\varphi_2} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = N(\varphi)$$

and

$$\mathbf{i}\left(\varphi\right)^t C^t \varphi = \mathbf{i}\left(\mathbf{i}C\overline{\varphi}\right)^t C^t \varphi = -\overline{\varphi}^t C^t C^t \varphi$$
$$= q_0^2 + q_1^2 + q_2^2 + q_3^2 = N(\varphi)$$

where $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

3.2 Hamilton-Spinor Matrices

The quaternion product is not commutative in the quaternion algebra therefore, the Hamilton matrices corresponding to the right and left quaternion products are different. In this case, in this section, we obtain the spinor matrices for the right and left Hamilton matrices, separately. Moreover, we call these spinor matrices the left and right Hamilton spinor matrices. Then, we give some properties and theorems for these spinor matrices.

We know that the set of quaternions can be written as in equation (4). In this case, the operator in equation (1) is linear transformation therefore, this linear transformation corresponds to a matrix. Therefore, the following theorem can be given.

Theorem 3.3: Let $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \in H$ be an arbitrary real quaternion written as $q = \boldsymbol{k}\varphi_1 + \boldsymbol{i}\varphi_2$ where $\varphi_1 = q_3 + \mathbf{i}q_0$, $\varphi_2 = q_1 + \mathbf{i}q_2 \in C$. Therefore, the left Hamilton-spinor matrix corresponding to the left Hamilton matrix of quaternion $q \in H$ is given by

$$\varphi_L = \begin{bmatrix} q_0 - \mathbf{i}q_3 & -q_2 - \mathbf{i}q_1 \\ q_2 - \mathbf{i}q_1 & q_0 + \mathbf{i}q_3 \end{bmatrix}. \quad (7)$$

Proof: Suppose that the real quaternion $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \in H$ is written as $q = \boldsymbol{k}\varphi_1 + \boldsymbol{i}\varphi_2$. In this case, if we consider Hamilton operator in equation (1) then, we obtain

$$h^+(\boldsymbol{k}) = \boldsymbol{k}\varphi_1\boldsymbol{k} + \boldsymbol{i}\varphi_2\boldsymbol{k} = \boldsymbol{k}\left(\boldsymbol{k}\varphi_1\right) + \boldsymbol{i}\left(\boldsymbol{k}\varphi_2\right)$$

(8)

and

$$h^{+}(\boldsymbol{i}) = \boldsymbol{k}\varphi_1\boldsymbol{i} + \boldsymbol{i}\varphi_2\boldsymbol{i} = \boldsymbol{k}\left(\boldsymbol{k}\overline{\varphi_2}\right) + \boldsymbol{i}\left(-\boldsymbol{k}\overline{\varphi_1}\right) \tag{9}$$

where $\varphi_1\boldsymbol{k} = \boldsymbol{k}\varphi_1$, $\varphi_2\boldsymbol{k} = \boldsymbol{k}\varphi_2$, $\varphi_1\boldsymbol{i} = \boldsymbol{i}\overline{\varphi_1}$ and $\varphi_2\boldsymbol{i} = \boldsymbol{i}\overline{\varphi_2}$. Therefore, we find that the matrix form of the equations (8) and (9) is

$$\varphi_L = \begin{bmatrix} q_0 - \mathbf{i}q_3 & -q_2 - \mathbf{i}q_1 \\ q_2 - \mathbf{i}q_1 & q_0 + \mathbf{i}q_3 \end{bmatrix}.$$

This matrix φ_L is called the left Hamilton spinor matrix corresponding the left Hamilton matrix of real quaternion $q \in H$. The proof is completed.

Moreover, since $H^{+}(q)$ is the fundamental matrix for the real quaternion q we can call the left Hamilton spinor matrix φ_L the fundamental spinor matrix.

Corollary 3.4: For the fundamental spinor matrix φ_L the statement

$$q \times p \to H^{+}(q)P \to \varphi_L \rho$$

is provided where ρ is the spinor corresponds to real quaternion p with the aid of the transformation f, [11].

Now, for the right Hamilton spinor matrix, we need to define a new transformation similar to the equation (5). For this, we can write that the real quaternion q with regard to the basis $\{\boldsymbol{k},\boldsymbol{i}\}$ as

$$q = (q_3 + \mathbf{i}q_0)\boldsymbol{k} + (q_1 - \mathbf{i}q_2)\boldsymbol{i} \tag{10}$$

with right multiplication where $\mathbf{i} = \boldsymbol{ji} = -\boldsymbol{k}$. Therefore, the quaternion q can be expressed in terms of $\varphi_1 = q_3 + \mathbf{i}q_0$, $\overline{\varphi_2} = q_1 - \mathbf{i}q_2 \in C$. In this case, the set of real quaternions can be defined by

$$H = \left\{ q = \varphi_1\boldsymbol{k} + \overline{\varphi_2}\boldsymbol{i} \;\middle|\; \varphi_1 = q_3 + \mathbf{i}q_0,\ \overline{\varphi_2} = q_1 - \mathbf{i}q_2 \in C \right\}$$

where '$\bar{\ }$' is a complex conjugate. Moreover, similar to the transformation f, with the aid of equation (10) the transformation f_* can be given as

$f_* : H \to S$

$$q \to f_*(q) = f(q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3) \cong \varphi_* = \begin{bmatrix} q_3 + \mathbf{i}q_0 \\ q_1 - \mathbf{i}q_2 \end{bmatrix}. \tag{11}$$

Theorem 3.5: Let the real quaternion $q \in H$ be $q = \boldsymbol{k}\varphi_1 + \boldsymbol{i}\varphi_2 = \varphi_1\boldsymbol{k} + \overline{\varphi_2}\boldsymbol{i}$ where $\varphi_1 = q_3 + \mathbf{i}q_0$ and $\overline{\varphi_2} = q_1 - \mathbf{i}q_2 \in C$. The right Hamilton spinor matrix of this quaternion is

$$\varphi_R = \begin{bmatrix} q_0 - \mathbf{i}q_3 & q_2 - \mathbf{i}q_1 \\ -q_2 - \mathbf{i}q_1 & q_0 + \mathbf{i}q_3 \end{bmatrix}. \tag{12}$$

Proof: Assume that the real quaternion $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \in H$ is $q = \varphi_1\boldsymbol{k} + \overline{\varphi_2}\boldsymbol{i}$. Then, with the aid of the Hamilton operator in equation (2) we can write

$$h^{-}(\boldsymbol{k}) = \boldsymbol{k}\varphi_1\boldsymbol{k} + \boldsymbol{k}\overline{\varphi_2}\boldsymbol{i} = (\boldsymbol{k}\varphi_1)\boldsymbol{k} + (\boldsymbol{k}\overline{\varphi_2})\boldsymbol{i}$$

and, similarly

$$h^{-}(\boldsymbol{i}) = \boldsymbol{i}\varphi_1\boldsymbol{k} + \boldsymbol{i}\overline{\varphi_2}\boldsymbol{i} = (\boldsymbol{k}\varphi_2)\boldsymbol{k} + (-\boldsymbol{k}\overline{\varphi_1})\boldsymbol{i}$$

where $\varphi_1\boldsymbol{k} = \boldsymbol{k}\varphi_1$, $\varphi_2\boldsymbol{k} = \boldsymbol{k}\varphi_2$, $\varphi_1\boldsymbol{i} = \boldsymbol{i}\overline{\varphi_1}$ and $\varphi_2\boldsymbol{i} = \boldsymbol{i}\overline{\varphi_2}$. Consequently, we get

$$\varphi_R = \begin{bmatrix} q_0 - \mathbf{i}q_3 & q_2 - \mathbf{i}q_1 \\ -q_2 - \mathbf{i}q_1 & q_0 + \mathbf{i}q_3 \end{bmatrix}.$$

Therefore, the matrix φ_R is called the right Hamilton spinor matrix corresponding to the left Hamilton matrix of the real quaternion q.

Corollary 3.6: For the right Hamilton spinor matrix we can write

$$p \times q \to H^{-}(q)P \to \varphi_R \rho_*$$

where ρ_* is the spinor corresponding to the real quaternion p with the aid of the function f_*.

Now, we give the relationship between the Hamilton spinor matrices φ_R, φ_L and the Pauli matrices.

Theorem 3.7: Let $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \in H$ be any real quaternion and φ_R and φ_L be the left and right Hamilton spinor matrices corresponding to the real quaternion q. Therefore, these spinor matrices can be written as

$$\varphi_L = q_0 I_2 - q_1\mathbf{i}P_1 - q_2\mathbf{i}P_2 - q_3\mathbf{i}P_3$$

and

$$\varphi_R = q_0 I_2 - q_1\mathbf{i}P_1 + q_2\mathbf{i}P_2 - q_3\mathbf{i}P_3$$

where $I_2 \in C_2^2$ is unit matrix with 2×2 and $P_1, P_2, P_3 \in C_2^2$ are Pauli matrices.

Proof: Let the left Hamilton spinor matrix be φ_L. Then, from the equation (7) we get

$$for\ \ q = 1, \qquad \varphi_L^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$for\ \ q = i, \qquad \varphi_L^i = -\mathbf{i}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\mathbf{i}P_1$$

$$for\ \ q = j, \qquad \varphi_L^j = -\mathbf{i}\begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 1 \end{bmatrix} = -\mathbf{i}P_2$$

$$for\ \ q = k, \qquad \varphi_L^k = -\mathbf{i}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{i}P_3.$$

Hence, we have $\varphi_L = q_0 I_2 - q_1\mathbf{i}P_1 - q_2\mathbf{i}P_2 - q_3\mathbf{i}P_3$. Similarly, for the right Hamilton spinor matrix in equation (12) we obtain for $q=1$ $\varphi_R^1 = I_2$, for $q=\boldsymbol{i}$ $\varphi_R^i = -\mathbf{i}P_1$, for $q=\boldsymbol{j}$ $\varphi_R^j = -\mathbf{i}P_2$ and for $q=\boldsymbol{k}$ $\varphi_R^k = -\mathbf{i}P_3$. Consequently, the equation $\varphi_R = q_0 I_2 - q_1\mathbf{i}P_1 + q_2\mathbf{i}P_2 - q_3\mathbf{i}P_3$ is found.

Theorem 3.8: Let any two real quaternions be $p,q \in H$ and the left and right Hamilton spinor matrices corresponding to these quaternions be ρ_L, ρ_R and φ_L, φ_R, respectively. Then, the following statement is provided;

i) $(\varphi + \rho)_L = \varphi_L + \rho_L$, $(\varphi_* + \rho_*)_R = \varphi_R + \rho_R$,

ii) $(\mu\varphi)_L = \mu\varphi_L$, $(\mu\varphi)_R = \mu\varphi_R$,

iii) $(\lambda\varphi)_L = \varphi_L\Lambda$, $(\lambda\varphi_*)_R = \varphi_R\Lambda$,

iv) $1_L = I_2$, $1_R = I_2$,

v) $(\hat{\varphi})_L = \mathrm{i}\varphi_L P_1$, $\left(\overline{\varphi_*}\right)_R = \mathrm{i}\varphi_R P_1$,

vi) $(\varphi_L\rho)_L = \varphi_L\rho_L$, $(\varphi_R\rho_*)_R = \varphi_R\rho_R$.

where $\mu \in R$, $\lambda \in C1$ and $\Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \in C_2^2$.

Proof: Assume that ρ_L, ρ_R and φ_L, φ_R, are the left and right Hamilton spinor matrices corresponding to any two real quaternions p and q, respectively. Therefore, we can give the proof.

i) Let $\rho,\varphi \in S$ be two spinors corresponding to the real quaternions p and q, respectively. Then, with the aid of the transformation f in the equation (5) we write

$$\rho + \varphi = \begin{bmatrix} p_3 + q_3 + \mathbf{i}(p_0 + q_0) \\ p_1 + q_1 + \mathbf{i}(p_2 + q_2) \end{bmatrix}$$

and the left Hamilton spinor matrix of the spinor $\rho + \varphi$ is obtained as

$$\begin{aligned}(\rho+\varphi)_L &= \begin{bmatrix} (p_0+q_0) - \mathbf{i}(p_3+q_3) & -(p_2+q_2) - \mathbf{i}(p_1+q_1) \\ (p_2+q_2) - \mathbf{i}(p_1+q_1) & (p_0+q_0) + \mathbf{i}(p_3+q_3) \end{bmatrix} \\ &= \begin{bmatrix} p_0 - \mathbf{i}p_3 & -p_2 - \mathbf{i}p_1 \\ p_2 - \mathbf{i}p_1 & p_0 + \mathbf{i}p_3 \end{bmatrix} + \begin{bmatrix} q_0 - \mathbf{i}q_3 & -q_2 - \mathbf{i}q_1 \\ q_2 - \mathbf{i}q_1 & q_0 + \mathbf{i}q_3 \end{bmatrix} \\ &= \rho_L + \varphi_L.\end{aligned}$$

Similarly, for the right Hamilton spinor matrix, the proof is completed easily.

ii) We assume that $\mu \in R$ and the spinor $\mu\varphi = \begin{bmatrix} \mu q_3 + \mathbf{i}\mu q_0 \\ \mu q_1 + \mathbf{i}\mu q_2 \end{bmatrix}$. Then, we get the left Hamilton spinor matrix of the spinor $\mu\varphi$

$$\begin{aligned}(\mu\varphi)_L &= \begin{bmatrix} \mu q_0 - \mathbf{i}\mu q_3 & -\mu q_2 - \mathbf{i}\mu q_1 \\ \mu q_2 - \mathbf{i}\mu q_1 & \mu q_0 + \mathbf{i}\mu q_3 \end{bmatrix} \\ &= \mu\begin{bmatrix} q_0 - \mathbf{i}q_3 & -q_2 - \mathbf{i}q_1 \\ q_2 - \mathbf{i}q_1 & q_0 + \mathbf{i}q_3 \end{bmatrix} = \mu\varphi_L.\end{aligned}$$

On the other hand, we suppose that the spinor φ_* corresponds to the real quaternion q with the aid of the transformation f_* in the equation (11). In this case, the right Hamilton spinor matrix of the spinor $\mu\varphi_*$ is

$$\begin{aligned}(\mu\varphi_*)_R &= \begin{bmatrix} \mu q_0 - \mathbf{i}\mu q_3 & \mu q_2 - \mathbf{i}\mu q_1 \\ -\mu q_2 - \mathbf{i}\mu q_1 & \mu q_0 + \mathbf{i}\mu q_3 \end{bmatrix} \\ &= \mu\begin{bmatrix} q_0 - \mathbf{i}q_3 & q_2 - \mathbf{i}q_1 \\ -q_2 - \mathbf{i}q_1 & q_0 + \mathbf{i}q_3 \end{bmatrix} = \mu\varphi_R.\end{aligned}$$

iii) Suppose that the spinor φ corresponds to the real quaternion $q \in H$ with the aid of the transformation f and $\lambda = a + \mathbf{i}b \in C$. Then, we get

$$\lambda\varphi = \begin{bmatrix} (aq_3 - bq_0) + \mathbf{i}(aq_0 + bq_3) \\ (aq_1 - bq_2) + \mathbf{i}(aq_2 + bq_1) \end{bmatrix}$$

and have the left Hamilton spinor matrix of the spinor $\lambda\varphi$

$$\begin{aligned}(\lambda\varphi)_L &= \begin{bmatrix} aq_0 + bq_3 - \mathbf{i}(aq_3 - bq_0) & -aq_2 - bq_1 - \mathbf{i}(aq_1 - bq_2) \\ aq_2 + bq_1 - \mathbf{i}(aq_1 - bq_2) & aq_0 + bq_3 - \mathbf{i}(-aq_3 + bq_0) \end{bmatrix} \\ &= \begin{bmatrix} (q_0 - \mathbf{i}q_3) & (-q_2 - \mathbf{i}q_1) \\ (q_2 - \mathbf{i}q_1) & (q_0 + \mathbf{i}q_3) \end{bmatrix}\begin{bmatrix} (a + \mathbf{i}b) & 0 \\ 0 & (a - \mathbf{i}b) \end{bmatrix}.\end{aligned}$$

Consequently, we obtain $(\lambda\varphi)_L = \varphi_L\Lambda$ where $\Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}$. Similarly, the proof for the right Hamilton spinor matrix can be obtained.

iv) We consider that the real quaternion q is $q=1$. In this case, the spinor corresponding to $q=1$ is $\varphi=\begin{bmatrix}1\\0\end{bmatrix}$. Consequently, we find that the left Hamilton spinor matrix related to the spinor φ is $\varphi_L=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I_2$. Similarly, we get the right Hamilton spinor matrix for the real quaternion $q=1$ as $\varphi_R=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I_2$.

v) Let the spinor corresponding to the real quaternion q be φ and the spinor conjugate of the spinor φ be $\hat{\varphi}=\begin{bmatrix}q_2+\mathbf{i}q_1\\-q_0-\mathbf{i}q_3\end{bmatrix}$ in the equation (6). In this case, the left Hamilton spinor matrix of the spinor $\hat{\varphi}$ can be obtained

$$\begin{aligned}(\hat{\varphi})_L&=\begin{bmatrix}q_1-\mathbf{i}q_2 & q_3+\mathbf{i}q_0\\-q_3+\mathbf{i}q_0 & q_1+\mathbf{i}q_2\end{bmatrix}\\&=\mathbf{i}\begin{bmatrix}q_0-\mathbf{i}q_3 & -q_2-\mathbf{i}q_1\\q_2-\mathbf{i}q_1 & q_0+\mathbf{i}q_3\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\mathbf{i}\varphi_L P_1.\end{aligned}$$

Similarly, we obtain that the right Hamilton spinor matrix of the spinor φ_* is

$$\begin{aligned}(\varphi_*)_R&=\begin{bmatrix}q_1+\mathbf{i}q_2 & q_3+\mathbf{i}q_0\\-q_3+\mathbf{i}q_0 & q_1-\mathbf{i}q_2\end{bmatrix}\\&=\mathbf{i}\begin{bmatrix}q_0-\mathbf{i}q_3 & q_2-\mathbf{i}q_1\\-q_2-\mathbf{i}q_1 & q_0+\mathbf{i}q_3\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\mathbf{i}\varphi_R P_1.\end{aligned}$$

vi) For the spinors ρ and φ corresponding to the real quaternions $p,q\in H$ we can write

$$\begin{aligned}\varphi_L\rho&=\begin{bmatrix}q_0-\mathbf{i}q_3 & -q_2-\mathbf{i}q_1\\q_2-\mathbf{i}q_1 & q_0+\mathbf{i}q_3\end{bmatrix}\begin{bmatrix}p_3+\mathbf{i}p_0\\p_1+\mathbf{i}p_2\end{bmatrix}\\&=\begin{bmatrix}q_0p_3+q_3p_0-q_2p_1+q_1p_2+\mathbf{i}(q_0p_0-q_1p_1-q_2p_2-q_3p_3)\\q_2p_3+q_1p_0+q_0p_1-q_3p_2+\mathbf{i}(q_2p_0-q_1p_3+q_0p_2+q_3p_1)\end{bmatrix}.\end{aligned}$$

Therefore, we get the left Hamilton spinor matrix corresponding to the spinor $\varphi_L\rho\in S$ as

$$(\varphi_L\rho)_L=\begin{bmatrix}q_0-\mathbf{i}q_3 & -q_2-\mathbf{i}q_1\\q_2-\mathbf{i}q_1 & q_0+\mathbf{i}q_3\end{bmatrix}\begin{bmatrix}p_0-\mathbf{i}p_3 & -p_2-\mathbf{i}p_1\\p_2-\mathbf{i}p_1 & p_0+\mathbf{i}p_3\end{bmatrix}=\varphi_L\rho_L.$$

Similarly, we obtain

$$\varphi_R\rho_*=\begin{bmatrix}q_0p_3+q_3p_0+q_2p_1-q_1p_2+\mathbf{i}(q_0p_0-q_1p_1-q_2p_2-q_3p_3)\\-q_2p_3+q_1p_0+q_0p_1+q_3p_2-\mathbf{i}(q_2p_0+q_1p_3+q_0p_2-q_3p_1)\end{bmatrix}$$

and consequently $(\varphi_R\rho_*)_R=\varphi_R\rho_R$.

Proposition 3.9: The left and right Hamilton spinor matrices φ_L and φ_R related with the real quaternion q are normal.

Proof: Suppose that φ is the left Hamilton spinor matrix related to the real quaternion q. Then, we obtain

$$\begin{aligned}\overline{\varphi_L^t}\varphi_L&=\begin{bmatrix}q_0+\mathbf{i}q_3 & q_2+\mathbf{i}q_1\\-q_2+\mathbf{i}q_1 & q_0-\mathbf{i}q_3\end{bmatrix}\begin{bmatrix}q_0-\mathbf{i}q_3 & -q_2-\mathbf{i}q_1\\q_2-\mathbf{i}q_1 & q_0+\mathbf{i}q_3\end{bmatrix}\\&=\begin{bmatrix}q_0^2+q_1^2+q_2^2+q_3^2 & 0\\0 & q_0^2+q_1^2+q_2^2+q_3^2\end{bmatrix}\\&=N^2(q)I_2.\end{aligned}$$

Similarly, if we calculate the equation $\varphi_L\overline{\varphi_L^t}$ then, we have $\varphi_L\overline{\varphi_L^t}=N^2(q)I_2$. Consequently, we get $\overline{\varphi_L^t}\varphi_L=\varphi_L\overline{\varphi_L^t}$ and we say that the left Hamilton spinor matrix φ_L is normal. Similar to the left Hamilton spinor matrix we get $\overline{\varphi_R^t}\varphi_R=N^2(q)I_2=\varphi_R\overline{\varphi_R^t}$ and we see easily that the right Hamilton spinor matrix φ_R is normal. Consequently, the proof is completed.

Now, we calculate the determinant of the left and right Hamilton spinor matrices for the real quaternion $q\neq 0$. Hence, we get

$$\det(\varphi_L)=\det(\varphi_R)=N^2(q).$$

In this case, we can give the following corollaries.

Corollary 3.10: The left and right Hamilton spinor matrices φ_L and φ_R related with the real quaternion $q\neq 0$ are regular.

Corollary 3.11: The inverses of the left and right Hamilton spinor matrices where $q\neq 0\in H$ are

$$\varphi_L^{-1}=\frac{1}{N^2(q)}\begin{bmatrix}q_0+\mathbf{i}q_3 & q_2+\mathbf{i}q_1\\-q_2+\mathbf{i}q_1 & q_0-\mathbf{i}q_3\end{bmatrix},$$

$$\varphi_R^{-1}=\frac{1}{N^2(q)}\begin{bmatrix}q_0+\mathbf{i}q_3 & -q_2+\mathbf{i}q_1\\q_2+\mathbf{i}q_1 & q_0-\mathbf{i}q_3\end{bmatrix}.$$

Corollary 3.12: The inverses of the left and right Hamilton spinor matrices can be written as

$$\varphi_L^{-1}=\frac{1}{N^2(q)}(\varphi^*)_L,\quad \varphi_R^{-1}=\frac{1}{N^2(q)}(\varphi_*^*)_R$$

respectively, where $\left(\varphi^*\right)_L$ and $\left(\varphi^*_*\right)_R$ are the left and right Hamilton spinor matrices related with the quaternion conjugate q^*.

Especially, if we consider that the real quaternion q is unit then, we get $\varphi_L^{-1}=\left(\varphi^*\right)_L$, $\varphi_R^{-1}=\left(\varphi^*_*\right)_R$

Corollary 3.13: The left and right Hamilton spinor matrices are unitary matrices. It should be emphasized that $\varphi_L,\varphi_R\in SU(2)$

3.3 The Eigenvalues and Eigenvectors of the Fundamental Spinor Matrix

In this section, we obtain the eigenvalue and eigenvector of the fundamental spinor matrix (i.e. left Hamilton spinor matrix) φ_L. Similar equations can be obtained for the right Hamilton spinor matrix φ_R.

We consider that the real quaternion q is pure real quaternion $q=\boldsymbol{q}$. Therefore, the fundamental spinor matrix $\boldsymbol{\varphi}_L^p$ corresponding to the pure quaternion $\boldsymbol{q}$ is obtained as

$$\boldsymbol{\varphi_L}=\begin{bmatrix} -\mathbf{i}q_3 & -q_2-\mathbf{i}q_1 \\ q_2-\mathbf{i}q_1 & \mathbf{i}q_3 \end{bmatrix}.$$

Lemma 3.14: Let the fundamental spinor matrices be $\varphi_L,\boldsymbol{\varphi_L}$ corresponding to the real quaternion q and the pure real quaternion $\boldsymbol{q}$. Hence, the relationship between these fundamental spinor matrices is

$$\varphi_L=q_0I_2+\boldsymbol{\varphi_L}.$$

Proof: Suppose that the fundamental spinor matrices φ_L and $\boldsymbol{\varphi_L}$ are related with q and the pure quaternion $\boldsymbol{q}$, respectively. Then, we obtain that

$$\begin{aligned}\varphi_L&=\begin{bmatrix} q_0-\mathbf{i}q_3 & -q_2-\mathbf{i}q_1 \\ q_2-\mathbf{i}q_1 & q_0+\mathbf{i}q_3 \end{bmatrix}\\ &=q_0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} -\mathbf{i}q_3 & -q_2-\mathbf{i}q_1 \\ q_2-\mathbf{i}q_1 & \mathbf{i}q_3 \end{bmatrix}=q_0I_2+\boldsymbol{\varphi_L}.\end{aligned}$$

Theorem 3.15: The eigenvalues of the fundamental spinor matrix φ_L are $\lambda_1=q_0+\mathbf{i}N(\boldsymbol{q})$ and $\lambda_2=q_0-\mathbf{i}N(\boldsymbol{q})$ where $q=q_0+\boldsymbol{q}$ and $\boldsymbol{q}$ is vectorial part of the real quaternion q.

Proof: We suppose that the φ_L is the fundamental spinor matrix. We know that the eigenvalues of the spinor matrix φ_L are obtained with the equation $\varphi_L\vartheta=\lambda\vartheta$ where $\lambda\in C$ are the eigenvalues. Therefore, if we calculate the equation $\det\left(\varphi_L-\lambda I_2\right)=0$ we get

$$\det\left(\varphi_L-\lambda I_2\right)=\begin{vmatrix} q_0-\lambda-\mathbf{i}q_3 & -q_2-\mathbf{i}q_1 \\ q_2-\mathbf{i}q_1 & q_0-\lambda+\mathbf{i}q_3 \end{vmatrix}=0$$

and we obtain that the characteristic polynomial is

$$\lambda^2-2\lambda q_0+q_2^2+q_1^2+q_0^2+q_3^2=0.$$

Moreover, if we solve this second-order equation then we get

$$\lambda_1=q_0+\mathbf{i}N(\boldsymbol{q}),\qquad \lambda_2=q_0-\mathbf{i}N(\boldsymbol{q})\in C. \tag{13}$$

Corollary 3.16: Let $q=q_0+\boldsymbol{i}q_1+\boldsymbol{j}q_2+\boldsymbol{k}q_3$ be any real quaternion φ_L be the fundamental spinor matrix corresponding to the real quaternion q and $\boldsymbol{\varphi_L}$ be the fundamental spinor matrix corresponding to the pure real quaternion $\boldsymbol{q}=\boldsymbol{i}q_1+\boldsymbol{j}q_2+\boldsymbol{k}q_3$. Therefore, the eigenvalues of the fundamental spinor matrix φ_L are found by adding q_0 to the eigenvalues of $\boldsymbol{\varphi_L}$.

Proof: Let φ_L and $\boldsymbol{\varphi_L}$ be the fundamental spinor matrices corresponding to the real quaternion q and the pure real quaternion $\boldsymbol{q}$ respectively. Now, we calculate the roots of the characteristic polynomial $\det\left(\boldsymbol{\varphi_L}-\lambda I_2\right)=0$. Therefore, we get

$$\det\left(\boldsymbol{\varphi_L}-\lambda I_2\right)=\lambda^2+q_3^2+q_2^2+q_1^2=0$$

and

$$\lambda_1=\mathbf{i}N(\boldsymbol{q}),\quad \lambda_2=-\mathbf{i}N(\boldsymbol{q})\in C. \tag{14}$$

Consequently, if we use equations (13) and (14) we say that the eigenvalues of the fundamental spinor matrix φ_L are found by adding q_0 to the eigenvalues of the fundamental spinor matrix $\boldsymbol{\varphi_L}$.

Theorem 3.17: Assume that $\boldsymbol{\varphi_L}$ is the fundamental spinor matrix corresponding to the pure real quaternion $\boldsymbol{q}$. Moreover, we consider that the eigenvalue of $\boldsymbol{\varphi_L}$ is μ. Therefore, the eigenvalue of the spinor matrix $\boldsymbol{\varphi}_L^2$ is μ^2.

Proof: Suppose that the fundamental spinor matrix corresponding to the real pure quaternion $\boldsymbol{q}$ is $\boldsymbol{\varphi_L}$. Now, we find the spinor matrix $\boldsymbol{\varphi}_L^2$ therefore, we obtain

$$\boldsymbol{\varphi}_L^2 = \begin{bmatrix} -q_3^2 - q_2^2 - q_1^2 & 0 \\ 0 & -q_3^2 - q_2^2 - q_1^2 \end{bmatrix} = -N^2(\boldsymbol{q}) I_2.$$

If we assume that the roots of the characteristics polynomial $\det(\boldsymbol{\varphi}_L^2 - \lambda I_2) = 0$ are μ_1 and μ_2 then, we have

$$\mu_{1,2} = -N^2(\boldsymbol{q}). \tag{15}$$

Consequently, considering the equations (14) and (15) we say easily that if the eigenvalue of the spinor matrix $\boldsymbol{\varphi}_L$ is μ then, the eigenvalue of the spinor matrix $\boldsymbol{\varphi}_L^2$ is μ^2.

Corollary 3. 18: The fundamental spinor matrix φ_L corresponding to the real quaternion q is a non-defective matrix.

Proof: We see that the number of eigenvalues of the fundamental spinor matrix φ_L is equal to its number of dimensions. In this case, the fundamental spinor matrix is a non-defective matrix.

Theorem 3.19: The eigenspaces corresponding to the $\lambda_1 = q_0 + \mathbf{i}N(\boldsymbol{q})$ and $\lambda_2 = q_0 - \mathbf{i}N(\boldsymbol{q})$ eigenvalues of the fundamental spinor φ_L are, respectively,

$$\{\varsigma_L \alpha \mid \alpha \in S\} \quad \text{and} \quad \{\delta_L \alpha \mid \alpha \in S\}$$

where the spinors are $\varsigma = \begin{bmatrix} q_3 - N(\boldsymbol{q}) \\ q_1 + \mathbf{i}q_2 \end{bmatrix}$, $\delta = \begin{bmatrix} q_3 + N(\boldsymbol{q}) \\ q_1 + \mathbf{i}q_2 \end{bmatrix}$ and ς_L, δ_L are the fundamental spinor matrices of the spinors ς, δ.

Proof: Let the fundamental spinor matrix of the spinor $\varphi \in S$ corresponding to the real quaternion $q \in H$ be φ_L. Firstly, we obtain the eigenspace for the eigenvalue $\lambda_1 = q_0 + \mathrm{i}N(\boldsymbol{q})$. Then, we consider the spinor $\varsigma = \begin{bmatrix} q_3 + \mathbf{i}(\mathbf{i}N(\boldsymbol{q})) \\ q_1 + \mathbf{i}q_2 \end{bmatrix}$. In this case, the fundamental spinor matrix of the spinor ς is

$$\varsigma_L = \begin{bmatrix} \mathbf{i}(N(\boldsymbol{q}) - q_3) & -q_2 - \mathbf{i}q_1 \\ q_2 - \mathbf{i}q_1 & \mathbf{i}(N(\boldsymbol{q}) + q_3) \end{bmatrix}.$$

Moreover, let $\alpha \in S$ be any spinor such that $\alpha = \begin{bmatrix} \alpha_3 + \mathbf{i}\alpha_0 \\ \alpha_1 + \mathbf{i}\alpha_2 \end{bmatrix}$. In this case, the spinor $\varsigma_L \alpha \in S$ is calculated as

$$\varsigma_L \alpha = \begin{bmatrix} q_3\alpha_0 - q_2\alpha_1 + q_1\alpha_2 - N(\boldsymbol{q})\alpha_0 + \mathbf{i}(-q_1\alpha_1 - q_2\alpha_2 - q_3\alpha_3 + N(\boldsymbol{q})\alpha_3) \\ q_2\alpha_3 + q_1\alpha_0 - q_3\alpha_2 - N(\boldsymbol{q})\alpha_2 + \mathbf{i}(q_2\alpha_0 - q_1\alpha_3 + q_3\alpha_1 + N(\boldsymbol{q})\alpha_1) \end{bmatrix}$$

On the other hand, for the fundamental spinor matrix φ_L we know that

$$\varphi_L - \lambda_1 I_2 = \begin{bmatrix} -\mathbf{i}(N(\boldsymbol{q}) + q_3) & -q_2 - \mathbf{i}q_1 \\ q_2 - \mathbf{i}q_1 & \mathbf{i}(-N(\boldsymbol{q}) + q_3) \end{bmatrix}.$$

If we calculate the equation $(\varphi_L - \lambda_1 I_2)\varsigma_L \alpha$ then, we obtain $(\varphi_L - \lambda_1 I_2)\varsigma_L \alpha = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0.$ Consequently, the eigenspace for the eigenvalue $\lambda_1 = q_0 + \mathbf{i}N(\boldsymbol{q})$ consists of the spinors $\varsigma_L \alpha.$

Similarly, the eigenspace for the eigenvalue $\lambda_2 = q_0 - \mathbf{i}N(\boldsymbol{q})$ of the fundamental spinor matrix φ_L can be obtained. For this, we consider the spinor $\delta = \begin{bmatrix} q_3 + \mathbf{i}(-\mathbf{i}N(\boldsymbol{q})) \\ q_1 + \mathbf{i}q_2 \end{bmatrix}$. Therefore, the fundamental spinor matrix of the spinor δ is

$$\delta_L = \begin{bmatrix} -\mathbf{i}(N(\boldsymbol{q}) + q_3) & -q_2 - \mathbf{i}q_1 \\ q_2 - \mathbf{i}q_1 & \mathbf{i}(-N(\boldsymbol{q}) + q_3) \end{bmatrix}.$$

Consequently, if we make the necessary arrangements we obtain $(\varphi_L - \lambda_1 I_2)\delta_L \alpha = 0$ and see that the eigenspace for the eigenvalue $\lambda_2 = q_0 - \mathbf{i}N(\boldsymbol{q})$ consists of the spinors $\delta_L \alpha.$

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