Exact Average Run Length Evaluation on One-Sided and Two-Sided Extended EWMA Control Chart with Correlated Data

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Abstract: - The Extended Exponentially Weighted Moving Average (Extended EWMA) control chart is one of the control charts that can rapidly detect a minor shift. Using the average run length (ARL), the control charts' effectiveness can be evaluated. This research aims to derive the explicit formulations for the ARL on one-sided and two-sided Extend EWMA control charts for the MA(1) model with exponential white noise, as they have not been previously presented. The analytical solution accuracy was determined and compared to the numerical integral equation (NIE) method. The results indicate that the ARL calculated using the explicit formula and the NIE method are nearly identical, demonstrating the validity of the explicit formula. In addition, this study is extended to compare the performance with the Exponentially Weighted Moving Average (EWMA) control chart. The results show that the Extended EWMA control chart has superior efficacy to the EWMA control chart. To demonstrate the efficacy of the proposed method, the analytical solution of ARL is finally applied to real-world data on Thailand's monthly fuel price.

Key-Words: - Average Run Length, Moving Average process, explicit formula, The Extended Exponentially Weighted Moving Average control chart, the EWMA control chart, the Extended EWMA, NIE method, ARL.

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1 Introduction

Control charts have been utilized in numerous disciplines, including finance, economics, industry, health care, and medicine. A study, introduced the concept of the control chart in 1931, [1]. The Shewhart control chart detects significant shifts in the processes more effectively. Next, the Exponentially Weighted Moving Average (EWMA) control chart, [2], demonstrates that both methods are effective at detecting shift size. Numerous researchers have enhanced the EWMA control chart, making it effective at detecting small shifts rapidly for observations with both normal distribution and exponential distribution. The study, [3], proposed the Extended EWMA control chart, which is an effective performance control chart for detecting shift size in the monitored process. The performance of these control charts is also compared in terms of the average run length (ARL), [4]. It consists of two components: The ARL_0 refers to the average run length when a process is in the control state. ARL_1 is the expected number of observations taken from an out-of-control process and should be as small as feasible. Previous research has demonstrated that the average run length (ARL) can be calculated using a variety of methods. The study, [5], proposed explicit formulas and numerical integral equations of ARL for the SARX(P,r)L model based on the CUSUM chart. The study, [6], proposed the ARL of a multivariate EWMA by means of Monte Carlo simulation. The study, [7], created the effectiveness of the CUSUM control chart for trend stationary seasonal autocorrelated data. The study, [8], derived the explicit formula for the ARL on the EWMA trend exponential AR(1) process. The study, [9], derived the ARL for the Autoregressive Moving Average (ARMA) process using both the explicit formula and the NIE approach of the EWMA control chart. The study, [10], analyzed the ARL by applying the explicit formula on the EWMA control chart. The focus of the analysis was on a seasonal moving average model with exponential white noise, specifically considering an order q. Recently, [11], presented exact run length computation on the EWMA control chart for stationary moving average process with exogenous variables. The study, [12], derived designing the performance of the EWMA control chart for the seasonal moving average process with exogenous variables.

However, the explicit formula for the ARL on one-sided and two-sided Extended EWMA control charts for the MA(1) has not previously been studied. This study's objective is to derive the explicit formula for the ARL on one-sided and two-sided Extended EWMA for the MA(1). A comparison was made between the explicit formula and the NIE method for the ARL. In addition, the capability of explicit formulas for deriving the ARL on one-sided and two-sided Extended EWMA was contrasted with the EWMA control chart for both simulated and real-world data in the monthly fuel price in Thailand.

2 Materials and Methods

The study, [2], proposed the EWMA control chart, which can be detected variation in the process. The EWMA control chart is depicted below;

$$Z_t = (1-\lambda) Z_{t-1} + \lambda X_t, t = 1,2,3,... \tag{1}$$

where X_t is a process with mean, λ is exponential smoothing parameters with $0 < \lambda \le 1$ and $Z_0 = u$ is constant representing the initial value of the EWMA control chart. The upper control limit (UCL) and lower control limit (LCL) are represented as follows:

$$UCL / LCL = \mu_0 + L\sigma \sqrt{\frac{\lambda}{2-\lambda}} \tag{2}$$

where μ_0 is the parameter the target mean of the moving average process, σ is the process standard deviation parameter and L is the control limit variable for the control limit of the moving average process.

The study, [3], proposed the Extended EWMA. The Extended EWMA statistic is.

$$E_t = (1-\lambda_1+\lambda_2) E_{t-1} + \lambda_1 X_t - \lambda_2 X_{t-1}, t = 1,2,3,... \tag{3}$$

where λ_1 and λ_2 are exponential smoothing parameters with $0 \le \lambda_2 < \lambda_1 < 1$ and $E_0 = u$ is the initial value of the Extended EWMA. The upper control limit (UCL) and lower control limit (LCL) are represented as follows

$$UCL / LCL = \mu_0 \pm L\sigma \sqrt{\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}},$$

where μ_0 is the target mean parameter of the moving average process, σ is the parameter of the process standard deviation and L is the variable suitable for the control limit of the moving average process. The moving average (MA(1)) process can be characterized as

$$X_t = \eta + \varepsilon_t - \theta_1 \varepsilon_t \tag{4}$$

where ε_t is the parameter of the error term of time and presumed to be exponential white noise, η is the parameter of a constant and θ_1 is parameter of moving average coefficient with $-1 \le \theta_1 \le 1$. A describes the probability density function of ε_t is given by $f(x) = \frac{1}{\alpha} e^{-\frac{x}{\alpha}}$.

3 The Explicit formulas of ARL on Extended EWMA Control Chart of MA(1) Process

3.1 The Exact Solution of ARL on one-sided and two-side Extended EWMA for MA(1) Process

Let *H(u)* represent the ARL for the moving average (MA(1)) process. The Extended EWMA control chart has identical statistics. From equation (3) when t time equals one.

$$E_1 = (1-\lambda_1+\lambda_2)E_0 + \lambda_1\eta + \lambda_1\varepsilon_1 - \lambda_1\theta_1 X_0 - \lambda_2 X_0 \tag{5}$$

where λ_1 and λ_2 are exponential smoothing parameters with $0 \le \lambda_2 < \lambda_1 < 1$ and $E_0 = u$, $X_0 = v$, is the initial value of the Extended EWMA.

$$E_1 = (1-\lambda_1+\lambda_2)u + \lambda_1\eta - (\lambda_1\theta_1 - \lambda_2)v + \lambda_1\varepsilon_1 \tag{6}$$

The upper limit and lower limit are LCL = a and UCL = b, respectively. For the Extended EWMA statistics E_t in an in-control;

$$a \le E_1 \le b$$

H(u) denotes the ARL for the moving average (MA(1)) process as follows:

$$ARL = H(u) = E_{\infty}(\tau) \geq T, E_0 = u,$$

where E_{∞} is the expectation value.

Consider the following Fredholm integral equation of the second kind for the function $H(u) = 1 + \int H(Z_1) f(\varepsilon_1) d\varepsilon_1$:

$$H(u) = 1 + \int H((1-\lambda_1+\lambda_2)u + \lambda_1\eta - (\lambda_1\theta_1+\lambda_2)v + \lambda_1\varepsilon_1) f(\varepsilon_1) d\varepsilon_1$$

(7)

Therefore, the function *H(u)* is obtained as follows:

$$H(u) = 1 + \frac{1}{\lambda_1}\int_a^b H(y) f\left(\frac{y-(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_1+\lambda_2)v}{\lambda_1} - \eta\right) dy$$

(8)

If $\varepsilon_t = Exp(\alpha), y_t = Exp(\alpha)$ then $y = \frac{1}{\alpha} e^{-\frac{y}{\alpha}}, y \geq 0$

$$H(u) = 1 + \frac{1}{\lambda_1\alpha}\int_a^b e^{-\frac{y}{\lambda_1\alpha}} e^{\left(\frac{y-(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_1+\lambda_2)v}{\lambda_1} - \frac{\eta}{\alpha}\right)} dy \qquad (9)$$

Assume the function *G(u)* to be

$$G(u) = e^{\frac{y-(1-\lambda_1+\lambda_2)u-(\lambda_1\theta_1+\lambda_2)v}{\lambda_1} + \frac{\eta}{\alpha}}$$

$$H(u) = 1 + \frac{1}{\lambda_1\alpha}\int_a^b H(y) e^{-\frac{y}{\lambda_1\alpha}} G(u) dy$$

$$H(u) = 1 + \frac{G(u)}{\lambda_1\alpha}\int_a^b H(y) e^{-\frac{y}{\lambda_1\alpha}} dy, 0 \leq u \leq b$$

Let $d = \int_a^b H(y) e^{-\frac{y}{\lambda_1\alpha}} dy, 0 \leq u \leq b$ then

$$H(u) = 1 + \frac{G(u)}{\lambda_1\alpha} d$$

Thus $d = \int_a^b \left(1 + \frac{G(y)}{\lambda_1\alpha} d\right) e^{-\frac{y}{\lambda_1\alpha}} dy$, solving a constant *d*,

$$d = \int_a^b e^{-\frac{y}{\lambda_1\alpha}} + \int_a^b \frac{G(u)}{\lambda_1\alpha} d\left(e^{-\frac{y}{\lambda_1\alpha}}\right) dy$$

Therefore

$$d = \frac{-\lambda_1\alpha\left(e^{-\frac{b}{\lambda_1\alpha}} - e^{-\frac{a}{\lambda_1\alpha}}\right)}{1 + \frac{1}{\lambda_1-\lambda_2} e^{\frac{-(\lambda_1\theta_1+\lambda_2)v}{\lambda_1\alpha} + \frac{\eta}{\alpha}} \left(e^{\frac{-(\lambda_1-\lambda_2)b}{\lambda_1\alpha}} - e^{\frac{-(\lambda_1-\lambda_2)a}{\lambda_1\alpha}}\right)}$$

(10)

and *d* and *G(u)* are substituted in the function *H(u)*. Therefore, substituting α is equal α_0 when the process is in control, the explicit formula of ARL_0 for the two-sided Extended EWMA control chart yields the following formula;

$$ARL_0 = 1 - \frac{(\lambda_1-\lambda_2) e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha_0} + \frac{n}{\alpha_0}} e^{\frac{-(\lambda_1\theta_1-\lambda_2)}{\lambda_1\alpha_0} + \frac{n}{\alpha_0}} \left(e^{-\frac{b}{\lambda_1\alpha_0}} - e^{-\frac{a}{\lambda_1\alpha_0}}\right)}{(\lambda_1-\lambda_2) e^{\frac{(\lambda_1\theta_1+\lambda_2)}{\lambda_1\alpha_0} - \frac{n}{\alpha_0}} + \left(e^{\frac{-(\lambda_1-\lambda_2)b}{\lambda_1\alpha_0}} - e^{\frac{-(\lambda_1-\lambda_2)a}{\lambda_1\alpha_0}}\right)}$$

(11)

Consequently, substituting α is equal α_1 in *H(u)*, the explicit formula of ARL_1 for the two-sided Extended EWMA control chart can be obtained in the following

$$ARL_1 = 1 - \frac{(\lambda_1-\lambda_2) e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha_1} + \frac{n}{\alpha_0}} e^{\frac{-(\lambda_1\theta_1-\lambda_2)}{\lambda_1\alpha_1} + \frac{n}{\alpha_1}} \left(e^{-\frac{b}{\lambda_1\alpha_1}} - e^{-\frac{a}{\lambda_1\alpha_1}}\right)}{(\lambda_1-\lambda_2) e^{\frac{(\lambda_1\theta_1+\lambda_2)}{\lambda_1\alpha_1} - \frac{n}{\alpha_1}} + \left(e^{\frac{-(\lambda_1-\lambda_2)b}{\lambda_1\alpha_1}} - e^{\frac{-(\lambda_1-\lambda_2)a}{\lambda_1\alpha_1}}\right)}$$

(12)

In addition, the explicit formula of ARL_1 for the one-sided Extended EWMA control chart can be obtained as

$$ARL_1 = 1 - \frac{(\lambda_1-\lambda_2) e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha_1} + \frac{n}{\alpha_0}} e^{\frac{-(\lambda_1\theta_1-\lambda_2)}{\lambda_1\alpha_1} + \frac{n}{\alpha_1}} \left(e^{-\frac{b}{\lambda_1\alpha_1}} - 1\right)}{(\lambda_1-\lambda_2) e^{\frac{(\lambda_1\theta_1+\lambda_2)}{\lambda_1\alpha_1} - \frac{n}{\alpha_1}} + \left(e^{\frac{-(\lambda_1-\lambda_2)b}{\lambda_1\alpha_1}} - 1\right)}$$

(13)

Theorem 1.

The solution obtained by the ARL of the explicit formulas demonstrates the existence of a unique integral equation (NIE), as proven by Banach's fixed-point theorem. In this present study, let *T* denote an operation within the set of all continuous functions that are defined by**.**

$$T(H(u)) = 1 + \frac{1}{\lambda_1}\int_a^b H(y) f\left(\frac{y-(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_1+\lambda_2)v}{\lambda_1} - \eta\right) dy \qquad (14)$$

According to Banach's fixed-point theorem, if an operator *T* meets the condition of being a contraction, the fixed-point equation *T(H(u))* = *H(u)* has a unique solution, as stated. If equation (14) exists and has a unique solution, the Banach fixed-point theorem can be applied. The Banach fixed-point theorem also termed the contraction mapping theorem, appeared in explicit form in Banach's thesis in 1922, [13]. In general, it is employed to demonstrate the existence of a solution to an integral equation. Since then, due to its simplicity and utility, it has become a widely used instrument for solving existing problems in numerous mathematical disciplines, [14]. The specifics are listed below**.**

Assume that $T: X \to X$ is a contraction mapping with contraction constant $0 \le s < 1$, such that $\|T(L_1) - T(L_2)\| \le s\|L_1 - L_2\| \quad \forall L_1, L_2 \in X.$, satisfies this condition. Then by, [15], there exists a unique $L(.) \in X$ such that $T(L(u)) = L(u)$ has a unique fixed point in X.

Proof Theorem 1: To demonstrate that *T*, as defined by the equation *T(H(u))* is a contraction mapping for $H_1, H_2 \in G[a,b]$. that $\|T(H_1) - T(H_2)\| \le s\|H_1 - H_2\|, \ \forall H_1, H_2 \in G[a,b]$. with $0 \le s < 1$ under the norm $\|H_\infty\| \le \sup_{u\in[a,b]} \|H(u)\|$ From *H(u)* and *T(H(u))*.

$$\|T(H_1) - T(H_2)\|_\infty$$

$$= \sup_{u\in[a,b]} \left| \frac{1}{\lambda_1\alpha} e^{\left(\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_1+\lambda_2)v}{\lambda_1} - \frac{\eta}{\alpha}\right)} \int_a^b (H_1(y) - H_2(y)) e^{-\frac{y}{\lambda_1\alpha}} dy \right|$$

$$\le \sup_{u\in[a,b]} \left| \frac{1}{\lambda_1\alpha} e^{\left(\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_1+\lambda_2)v}{\lambda_1} - \frac{\eta}{\alpha}\right)} \int_a^b (H_1(y) - H_2(y)) e^{-\frac{y}{\lambda_1\alpha}} dy \right|$$

$$= \|H_1 - H_2\|_\infty \sup_{u\in[a,b]} \left| e^{\left(\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_1+\lambda_2)v}{\lambda_1} - \frac{\eta}{\alpha}\right)} \right| \left| 1 - \left[e^{-\frac{b}{\lambda_1\alpha}} - e^{-\frac{a}{\lambda_1\alpha}} \right] \right|$$

$$\le s\|H_1 - H_2\|_\infty$$

where

$$s = \sup_{u\in[a,b]} \left| e^{\left(\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_1+\lambda_2)v}{\lambda_1} - \frac{\eta}{\alpha}\right)} \right| \left| 1 - \left[e^{-\frac{b}{\lambda_1\alpha}} - e^{-\frac{a}{\lambda_1\alpha}} \right] \right|,$$

$0 \le s < 1$, The uniqueness of the solution is therefore ensured by Banach's fixed-point theorem**.**

3.2 The NIE method of ARL of Extended EWMA Control Chart of MA(1) Process

For the two-sided Extended EWMA control chart for the moving average (MA(1)) process. Let $H_N(u)$ be the estimated value of the ARL with the m linear equation systems by using the composite midpoint quadrature rule by, [16].

The ARL approximating NIE method on a two-sided Extended EWMA is evaluated as follows;

$$\int_a^b L(k) f(k) dk \approx \sum_{j=1}^m w_j f(x_j)$$

The system of the *m* linear equation is shown as

$L_{m\times1} = 1_{m\times1} + R_{m\times m} L_{m\times1}$ or $L_{m\times1} = (I_m - R_{m\times m})^{-1} 1_{m\times1}$

$L_{m\times1} = [L_{NIE}(x_1), L_{NIE}(x_1), \ldots, L_{NIE}(x_m)]^T$,

$I_m = diag(1,1,\ldots,1)$ and $1_{m\times1} = [1,1,\ldots,1]^T$.

Let $R_{m\times m}$ be a matrix, the definition of the *m* to m^{th} element of the matrix *R* is given by

$$[R_{ij}] \approx \frac{1}{\lambda_1} w_j f\left(\frac{y_j - (1-\lambda_1+\lambda_2)u + (\lambda_1\theta_1+\lambda_2)v}{\lambda_1} - \eta \right)$$

So, the solution of the numerical integral equation can be explained as

$$H_N(u) = 1 + \frac{1}{\lambda_1} \sum_{j=1}^m w_j f\left(\frac{y_j - (1-\lambda_1+\lambda_2)u + (\lambda_1\theta_1+\lambda_2)v}{\lambda_1} - \eta \right) \quad (15)$$

where y_j is a set of the division point on the interval [a,b] as $y_j = \left(j - \frac{1}{2}\right) w_j + a, j = 1,2,\ldots,m.$ w_j is a weight of composite midpoint formula $w_j = \frac{b-a}{m}$.

4 Numerical Results

The relative mean index (RMI), [17], is used to test the performance of a two-sided Extended EWMA on varying bound control limits [a, b] and the comparative performance of the ARL under various λ conditions. The RMI can be computed as

$$RMI = \frac{1}{n} \sum_{i=1}^n \left[\frac{ARL_i(c) - ARL_i(s)}{ARL_i(s)} \right], \quad (16)$$

where $ARL_i(c)$ is the ARL of row *i* on the tested control chart, $ARL_i(s)$ is the lowest ARL of row *i* from all the control charts such that a control chart is more effective if the RMI value is lowest, indicating that the control chart had the best performance at change detection.

The ARL was approximated by the NIE method using the composite midpoint rule on the Extended EWMA with 1,000 nodes. The absolute percentage difference to assess the veracity of ARL. When $ARL_0 = 370$, η = 0.5, λ_1 = 0.05, 0.10, λ_2 = 0.01, v = 1, θ_1 = 0.10, -0.10, 0.20, -0.20 and then the initial parameter value was studied at $\alpha_0 = 1$. The out-of-control process $\alpha_1 = (1+\delta)\alpha_0$ is computed by determining shift size (δ) to be 0.01, 0.02, 0.03, 0.05, 0.100, 0.20, 0.30, 0.50, 1.00, 2.00 and 3.00. The upper control limit of one-sided and two-sided Extended EWMA and EWMA control charts are obtained in Table 1 (Appendix). The results of ARL for the one-sided and two-sided Extended EWMA using an explicit formula with NIE are compared in Tables 2 (Appendix) and

Table 3 (Appendix). Both ARL procedures yield equivalent results.

Besides Table 4 (Appendix) and Table 5 (Appendix), the performance comparisons between the Extended EWMA control chart and the EWMA control chart when a = 0.0001 and ARL_0 = 370 are presented. The ARL_1 value of the Extended EWMA control chart is lower than the EWMA control chart at all shift sizes. Moreover, it was found that the Extended EWMA control chart had the best performance because it gave the lowest RMI. Therefore, it also can be concluded that the Extended EWMA control chart performs better than the EWMA and control chart.

5 Application to Real-world Data

The ARL was constructed using explicit formulas on one-sided and two-sided Extended EWMA control chart with ARL_0 = 370 for λ_1 = 0.05, 0.10, and λ_2 = 0.01, and its performance was compared with the EWMA control chart using real-world data on the monthly fuel price, Thailand between January 2019 and May 2023. Based on the autocorrelation function (ACF) and partial autocorrelation function (PACF), this data represents a stationary time series. The moving average (MA(1)) process was obtained as $X_t = \varepsilon_t + 0.901 X_{t-1}$ and $\varepsilon_t \sim Exp(2.1262)$.

In Table 6 (Appendix), the upper control limits for one-sided and two-sided Extended EWMA control charts are obtained. In addition, the ARL of the Extended EWMA control chart is evaluated and compared with the EWMA control chart. The ARL comparison of the one-sided and two-sided Extended EWMA control chart for MA(1) using NIE against the EWMA control chart is presented in Table 7 (Appendix). The results found that one-sided and two-sided Extended EWMA control charts outperform the EWMA control chart with small shift sizes detection as shown in Figure 1 (Appendix) and Figure 2 (Appendix). For the various λ_1 values, the performance of control charts performs better when λ_1 decreased.

6 Conclusions

In this specific study, when ARL_0 = 370, λ_1 = 0.05, 0.10, λ_2 = 0.01. The ARL was used to evaluate the efficacy of control charts. Using the numerical integral equation (NIE) method, the explicit formula is compared. Consequently, both methods demonstrate that the ARL values are near, but the explicit formula method can be calculated in less time. The Extended EWMA control chart with various λ_1 outperformed the EWMA control chart for the moving average (MA(1)) procedure in terms of performance. When considering the comparative efficacy of the ARL under different smoothing parameters, a smoothing parameter with a value of 0.05 is recommended. Eventually, the simulation studies and the performance illustration with real-world datasets using data on the monthly fuel price yielded the same outcomes. Future research could also evaluate the optimal parameters for MA(1) processes when comparing the performance of the Extended EWMA control chart with other control charts. In addition, it is possible to develop formulas for ARL values on the Extended EWMA control chart to construct new control charts or other interesting models.

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APPENDIX

Table 1. Upper control limit of the Extended EWMA and the EWMA control charts for MA(1) when η = 0.5, v = 1, a = 0.0001, α_0 = 1 for ARL_0 = 370

λ_1	θ_1	λ_2 = 0.01			
		One-sided		Two-sided	
		Extended EWMA	EWMA	Extended EWMA	EWMA
		b	h	b	h
0.05	0.1	6.930×10^{-8}	0.937×10^{-7}	1.000695×10^{-4}	1.000375×10^{-4}
	-0.1	5.680×10^{-8}	0.767×10^{-7}	1.000569×10^{-4}	1.000307×10^{-4}
	0.2	7.660×10^{-8}	1.036×10^{-7}	1.000768×10^{-4}	1.000415×10^{-4}
	-0.2	5.140×10^{-8}	0.694×10^{-7}	1.000515×10^{-4}	1.000278×10^{-4}
0.10	0.1	2.980×10^{-3}	4.459×10^{-3}	3.080×10^{-3}	4.120×10^{-3}
	-0.1	2.430×10^{-3}	3.637×10^{-3}	2.531×10^{-3}	3.380×10^{-3}
	0.2	3.300×10^{-3}	4.940×10^{-3}	3.400×10^{-3}	4.560×10^{-3}
	-0.2	2.200×10^{-3}	3.285×10^{-3}	2.300×10^{-3}	3.068×10^{-3}

Table 2. ARL comparison of one-sided Extended EWMA control chart for MA(1) using explicit formulas against NIE method when η = 0.5, v = 1, α_0 = 1 for ARL_0 = 370

λ_1	θ_1	Shift Size	Explicit	NIE	θ_1	Shift Size	Explicit	NIE
				λ_2 = 0.01				
0.05	0.1	0.00	370.04370	370.04370	0.2	0.00	370.09883	370.09883
		0.01	302.53400	302.53400		0.01	302.87861	302.87861
		0.02	248.32853	248.32853		0.02	248.85166	248.85166
		0.03	204.63007	204.63007		0.03	205.25510	205.25510
		0.05	140.53460	140.53460		0.05	141.22196	141.22196
		0.10	58.51039	58.51039		0.10	59.04442	59.04442
		0.20	13.12485	13.12485		0.20	13.33050	13.33050
		0.30	4.22723	4.22723		0.30	4.30307	4.30307
		0.50	1.38243	1.38243		0.50	1.39545	1.39545
		1.00	1.01131	1.01131		1.00	1.01189	1.01189
		2.00	1.00030	1.00030		2.00	1.00032	1.00032
		3.00	1.00004	1.00004		3.00	1.00005	1.00005
0.05	-0.1	0.00	370.44574	370.44574	-0.2	0.00	370.48339	370.48339
		0.01	302.26616	302.26616		0.01	301.99870	301.99870
		0.02	247.62957	247.62957		0.02	247.17154	247.17154
		0.03	203.66842	203.66842		0.03	203.09957	203.09957
		0.05	139.36295	139.36295		0.05	138.71968	138.71968
		0.10	57.53587	57.53587		0.10	57.02995	57.02995
		0.20	12.74016	12.74016		0.20	12.54729	12.54729
		0.30	4.08503	4.08503		0.30	4.01496	4.01496
		0.50	1.35815	1.35815		0.50	1.34645	1.34645
		1.00	1.01024	1.01024		1.00	1.00974	1.00974
		2.00	1.00026	1.00026		2.00	1.00024	1.00024
		3.00	1.00004	1.00004		3.00	1.00004	1.00004
0.10	0.1	0.00	370.77370	370.77370	0.2	0.00	370.95318	370.95318
		0.01	334.44627	334.44627		0.01	334.95718	334.95718
		0.02	302.26970	302.26970		0.02	303.04052	303.04052
		0.03	273.70977	273.70977		0.03	274.68198	274.68198
		0.05	225.67062	225.67062		0.05	226.91094	226.91094
		0.10	143.53268	143.53268		0.10	144.97215	144.97215
		0.20	64.92709	64.92709		0.20	66.08522	66.08522
		0.30	33.23285	33.23285		0.30	34.03565	34.03565
		0.50	11.61831	11.61831		0.50	11.99830	11.99830
		1.00	2.65469	2.65469		1.00	2.74326	2.74326
		2.00	1.22936	1.22936		2.00	1.24573	1.24573
		3.00	1.07846	1.07846		3.00	1.08476	1.08476

			λ_2 = 0.01					
λ_1	θ_1	Shift Size	Explicit	NIE	θ_1	Shift Size	Explicit	NIE
		0.00	370.23698	370.23698		0.00	370.87986	370.87986
		0.01	333.26829	333.26829		0.01	333.49996	333.49996
		0.02	300.59257	300.59257		0.02	300.49560	300.49560
		0.03	271.64938	271.64938		0.03	271.29140	271.29140
		0.05	223.10914	223.10914		0.05	222.38565	222.38565
		0.10	140.63698	140.63698		0.10	139.55407	139.55407
0.10	-0.1	0.20	62.64928	62.64928	-0.2	0.20	61.69210	61.69210
		0.30	31.67539	31.67539		0.30	31.00027	31.00027
		0.50	10.89489	10.89489		0.50	10.57612	10.57612
		1.00	2.49066	2.49066		1.00	2.41850	2.41850
		2.00	1.19982	1.19982		2.00	1.18700	1.18700
		3.00	1.06723	1.06723		3.00	1.06239	1.06239

Table 3. ARL comparison of two-sided Extended EWMA control chart for MA(1) using explicit formulas against NIE method when η = 0.5, v = 1, a = 0.0001, α_0 = 1 for ARL_0 = 370

			λ_2 = 0.01					
λ_1	θ_1	Shift Size	Explicit	NIE	θ_1	Shift Size	Explicit	NIE
		0.00	370.36828	370.36828		0.00	370.32315	370.32315
		0.01	302.80600	302.80600		0.01	303.06806	303.06806
		0.02	248.55644	248.55644		0.02	249.01202	249.01202
		0.03	204.82159	204.82159		0.03	205.39114	205.39114
		0.05	140.67100	140.67100		0.05	141.32054	141.32054
		0.10	58.57160	58.57160		0.10	59.09026	59.09026
0.05	0.1	0.20	13.13959	13.13959	0.2	0.20	13.34211	13.34211
		0.30	4.23157	4.23157		0.30	4.30661	4.30661
		0.50	1.38302	1.38302		0.50	1.39595	1.39595
		1.00	1.01133	1.01133		1.00	1.01191	1.01191
		2.00	1.00030	1.00030		2.00	1.00032	1.00032
		3.00	1.00004	1.00004		3.00	1.00005	1.00005
		0.00	370.35672	370.35672		0.00	370.46257	370.46257
		0.01	302.19954	302.19954		0.01	301.98769	301.98769
		0.02	247.57981	247.57981		0.02	247.16732	247.16732
		0.03	203.63139	203.63139		0.03	203.09996	203.09996
		0.05	139.34279	139.34279		0.05	138.72503	138.72503
		0.10	57.53253	57.53253		0.10	57.03698	57.03698
0.05	-0.1	0.20	12.74124	12.74124	-0.2	0.20	12.55049	12.55049
		0.30	4.08571	4.08571		0.30	4.01618	4.01618
		0.50	1.35831	1.35831		0.50	1.34666	1.34666
		1.00	1.01025	1.01025		1.00	1.00975	1.00975
		2.00	1.00026	1.00026		2.00	1.00024	1.00024
		3.00	1.00004	1.00004		3.00	1.00004	1.00004
		0.00	370.40272	370.40272		0.00	370.58201	370.58201
		0.01	334.11506	334.11506		0.01	334.62547	334.62547
		0.02	301.97340	301.97340		0.02	302.74346	302.74346
		0.03	273.44417	273.44417		0.03	274.41544	274.41544
		0.05	225.45600	225.45600		0.05	226.69512	226.69512
		0.10	143.40273	143.40273		0.10	144.84089	144.84089
0.10	0.1	0.20	64.87368	64.87368	0.2	0.20	66.03085	66.03085
		0.30	33.20800	33.20800		0.30	34.01018	34.01018
		0.50	11.61122	11.61122		0.50	11.99096	11.99096
		1.00	2.65387	2.65387		1.00	2.74239	2.74239
		2.00	1.22929	1.22929		2.00	1.24565	1.24565
		3.00	1.07844	1.07844		3.00	1.08474	1.08474
		0.00	370.02283	370.02283		0.00	370.50878	370.50878
		0.01	333.07880	333.07880		0.01	333.16970	333.16970
		0.02	300.42461	300.42461		0.02	300.20105	300.20105
		0.03	271.50019	271.50019		0.03	271.02817	271.02817
		0.05	222.99076	222.99076		0.05	222.17418	222.17418
		0.10	140.56852	140.56852		0.10	139.42776	139.42776

			$\lambda_2 = 0.01$					
λ_1	θ_1	Shift Size	Explicit	NIE	θ_1	Shift Size	Explicit	NIE
0.10	-0.1	0.20	62.62368	62.62368	-0.2	0.20	61.64139	61.64139
		0.30	31.66460	31.66460		0.30	30.97715	30.97715
		0.50	10.89241	10.89241		0.50	10.56973	10.56973
		1.00	2.49053	2.49053		1.00	2.41779	2.41779
		2.00	1.19984	1.19984		2.00	1.18693	1.18693
		3.00	1.06724	1.06724		3.00	1.06237	1.06237

Table 4. ARL comparison of the one-sided Extended EWMA for MA(1) against EWMA control charts when $\eta = 0.5$, $v = 1$, $\alpha_0 = 1$ for $ARL_0 = 370$

			$\lambda_2 = 0.01$					
λ_1	θ_1	Shift Size	Extended EWMA	EWMA	θ_1	Shift Size	Extended EWMA	EWMA
		0.00	370.04370	370.65340		0.00	370.09883	370.81573
		0.01	302.53400	303.93145		0.01	302.87861	304.36470
		0.02	248.32853	250.19939		0.02	248.85166	250.79815
		0.03	204.63007	206.75709		0.03	205.25510	207.44788
		0.05	140.53460	142.77649		0.05	141.22196	143.51578
		0.10	58.51039	60.19825		0.10	59.04442	60.76510
0.05	0.1	0.20	13.12485	13.76760	0.2	0.20	13.33050	13.98788
		0.30	4.22723	4.46429		0.30	4.30307	4.54673
		0.50	1.38243	1.42335		0.50	1.39545	1.43789
		1.00	1.01131	1.01316		1.00	1.01189	1.01384
		2.00	1.00030	1.00036		2.00	1.00032	1.00039
		3.00	1.00004	1.00006		3.00	1.00005	1.00006
		RMI	0.0000	0.0179		**RMI**	0.0000	0.0188
		0.00	370.44574	370.58084		0.00	370.48339	370.57538
		0.01	302.26616	303.27284		0.01	301.99870	302.96924
		0.02	247.62957	249.17533		0.02	247.17154	248.68553
		0.03	203.66842	205.52183		0.03	203.09957	204.92400
		0.05	139.36295	141.40508		0.05	138.71968	140.73600
		0.10	57.53587	59.12026		0.10	57.02995	58.59340
0.05	-0.1	0.20	12.74016	13.34660	-0.2	0.20	12.54729	13.14235
		0.30	4.08503	4.30739		0.30	4.01496	4.23189
		0.50	1.35815	1.39597		0.50	1.34645	1.38298
		1.00	1.01024	1.01191		1.00	1.00974	1.01132
		2.00	1.00026	1.00032		2.00	1.00024	1.00030
		3.00	1.00004	1.00005		3.00	1.00004	1.00004
		RMI	0.0000	0.0169		**RMI**	0.0000	0.0173
		0.00	370.77370	370.92612		0.00	370.92318	370.98449
		0.01	334.44627	335.97188		0.01	334.95718	336.37964
		0.02	302.26970	304.88166		0.02	303.04052	305.56730
		0.03	273.70977	277.17251		0.03	274.68198	278.07707
		0.05	225.67062	230.28954		0.05	226.91094	231.49471
		0.10	143.53268	149.11246		0.10	144.97215	150.57828
0.10	0.1	0.20	64.92709	69.55004	0.2	0.20	66.08522	70.78597
		0.30	33.23285	36.48927		0.30	34.03565	37.37372
		0.50	11.61831	13.19227		0.50	11.99830	13.63078
		1.00	2.65469	3.03257		1.00	2.74326	3.14221
		2.00	1.22936	1.30114		2.00	1.24573	1.32280
		3.00	1.07846	1.10646		3.00	1.08476	1.11508
		RMI	0.0000	0.0512		**RMI**	0.0000	0.0568
		0.00	370.23698	370.98535		0.00	370.87986	370.94168
		0.01	333.26829	335.31919		0.01	333.49996	334.92821
		0.02	300.59257	303.66321		0.02	300.49560	302.99791
		0.03	271.64938	275.50862		0.03	271.29140	274.62886
		0.05	223.10914	228.01420		0.05	222.38565	226.84395
		0.10	140.63698	146.30356		0.10	139.55407	144.89545
0.10	-0.1	0.20	62.64928	67.18474	-0.2	0.20	61.69210	66.02437
		0.30	31.67539	34.80778		0.30	31.00027	33.99388

			λ_2 = 0.01					
λ_1	θ_1	Shift Size	Extended EWMA	EWMA	θ_1	Shift Size	Extended EWMA	EWMA
		0.50	10.89489	12.36960		0.50	10.57612	11.97882
		1.00	2.49066	2.83164		1.00	2.41850	2.73882
		2.00	1.19982	1.26237		2.00	1.18700	1.24493
		3.00	1.06723	1.09122		3.00	1.06239	1.08445
		RMI	0.0000	0.0508		**RMI**	0.0000	0.0532

Table 5. ARL comparison of the two-sided Extended EWMA for MA(1) against EWMA control charts when η = 0.5 , v = 1, a = 0.0001, α_0 = 1 for ARL_0 = 370

			λ_2 = 0.01					
λ_1	θ_1	Shift Size	Extended EWMA	EWMA	θ_1	Shift Size	Extended EWMA	EWMA
		0.00	370.36828	370.41143		0.00	370.32315	370.61097
		0.01	302.80600	303.49330		0.01	303.06806	304.20274
		0.02	248.55644	249.84379		0.02	249.01202	250.66964
		0.03	204.82159	206.46739		0.03	205.39114	207.34560
		0.05	140.67100	142.58211		0.05	141.32054	143.45045
		0.10	58.57160	60.12220		0.10	59.09026	60.74287
0.05	0.1	0.20	13.13959	13.75313	0.2	0.20	13.34211	13.98502
		0.30	4.23157	4.46081		0.30	4.30661	4.54640
		0.50	1.38302	1.42301		0.50	1.39595	1.43794
		1.00	1.01133	1.01315		1.00	1.01191	1.01385
		2.00	1.00030	1.00036		2.00	1.00032	1.00039
		3.00	1.00004	1.00006		3.00	1.00005	1.00006
		RMI	0.0000	0.0170		**RMI**	0.0000	0.0180
		0.00	370.35672	370.80256		0.00	370.46257	370.46812
		0.01	302.19954	302.87129		0.01	301.98769	302.80588
		0.02	247.57981	248.85046		0.02	247.16732	248.55634
		0.03	203.63139	205.25800		0.03	203.09996	204.82151
		0.05	139.34279	141.22914		0.05	138.72503	140.67095
	-0.1	0.10	57.53253	59.05243		0.10	57.03698	58.57158
0.05		0.20	12.74124	13.33407	-0.2	0.20	12.55049	13.13959
		0.30	4.08571	4.30445		0.30	4.01618	4.23157
		0.50	1.35831	1.39569		0.50	1.34666	1.38302
		1.00	1.01025	1.01190		1.00	1.00975	1.01133
		2.00	1.00026	1.00032		2.00	1.00024	1.00030
		3.00	1.00004	1.00005		3.00	1.00004	1.00004
		RMI	0.0000	0.0168		**RMI**	0.0000	0.0170
		0.00	370.40272	370.49889		0.00	370.58201	370.97754
		0.01	334.11506	335.41433		0.01	334.62547	336.38138
		0.02	301.97340	304.38512		0.02	302.74346	305.57609
		0.03	273.44417	276.72952		0.03	274.41544	278.09154
		0.05	225.45600	229.93514		0.05	226.69512	231.51728
		0.10	143.40273	148.90387		0.10	144.84089	150.60922
0.10	0.1	0.20	64.87368	69.47016	0.2	0.20	66.03085	70.81417
		0.30	33.20800	36.45530		0.30	34.01018	37.39474
		0.50	11.61122	13.18472		0.50	11.99096	13.64174
		1.00	2.65387	3.03245		1.00	2.74239	3.14515
		2.00	1.22929	1.30129		2.00	1.24565	1.32342
		3.00	1.07844	1.10655		3.00	1.08474	1.11533
		RMI	0.0000	0.0558		**RMI**	0.0000	0.0578
		0.00	370.02283	370.33961		0.00	370.50878	370.93295
		0.01	333.07880	334.75044		0.01	333.16970	334.93694
		0.02	300.42461	303.16136		0.02	300.20105	303.02046
		0.03	271.50019	275.06501		0.03	271.02817	274.66225
		0.05	222.99076	227.66584		0.05	222.17418	226.89214
		0.10	140.56852	146.10801		0.10	139.42776	144.95643
0.10	0.1	0.20	62.62368	67.11709	-0.2	0.20	61.64139	66.07514
		0.30	31.66460	34.78228		0.30	30.97715	34.02925
		0.50	10.89241	13.36576		0.50	10.56973	11.99548

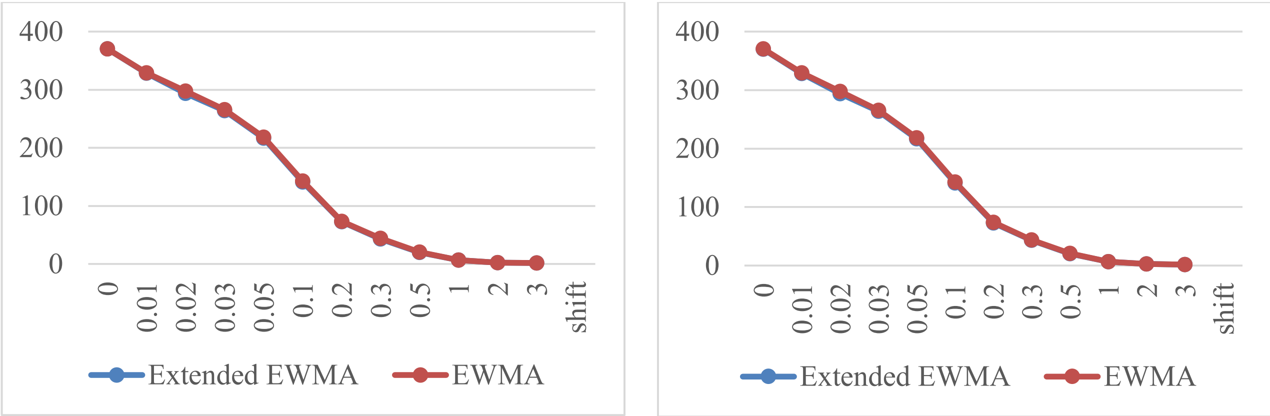
$\lambda_2 = 0.01$								
λ_1	θ_1	Shift Size	Extended EWMA	EWMA	θ_1	Shift Size	Extended EWMA	EWMA
		1.00	2.49053	2.83221		1.00	2.41779	2.74263
		2.00	1.19984	1.26261		2.00	1.18693	1.24561
		3.00	1.06724	1.09133		3.00	1.06237	1.08471
		RMI	0.0000	0.0634		**RMI**	0.0000	0.0545

Table 6. Upper control limit of the Extended EWMA and the EWMA control charts for real-world data when a = 0.0001, $\alpha_0 = 2.1262$, $\theta_1 = -0.901$ for $ARL_0 = 370$

λ_1	$\lambda_2 = 0.01$			
	One-sided		Two-sided	
	Extended EWMA	EWMA	Extended EWMA	EWMA
	b	h	b	h
0.05	0.004441	0.004361	0.004545	0.001882
0.10	0.187616	0.188629	0.187750	0.167300

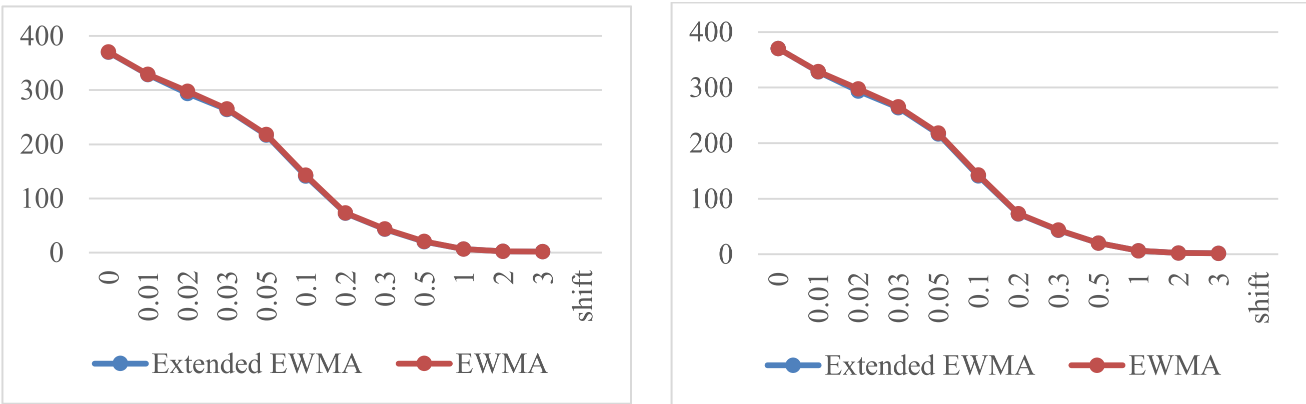
Table 7. ARL comparison of one-sided and two-sided Extended EWMA control chart for MA(1) using NIE against EWMA control chart when a = 0.0001, $\alpha_0 = 2.1262$, $\theta_1 = -0.901$ for $ARL_0 = 370$

$\lambda_2 = 0.01$					
λ_1	Shift Size	One-sided		Two-sided	
		Extended EWMA	EWMA	Extended EWMA	EWMA
0.05	0.00	370.06271	370.96000	370.05527	370.82145
	0.01	334.90280	335.64149	334.89911	335.59720
	0.02	303.79616	304.40145	303.79551	304.43282
	0.03	275.95855	276.45050	275.96037	276.54283
	0.05	229.01656	229.33199	229.02194	229.51081
	0.10	147.86130	147.92559	147.87061	148.19377
	0.20	68.54703	68.62827	68.63727	68.79191
	0.30	35.76452	35.85421	35.86091	35.94273
	0.50	12.83886	12.89523	12.89863	12.92745
	1.00	2.94598	2.96204	2.96290	2.96778
	2.00	1.28438	1.28771	1.28788	1.28859
	3.00	1.09987	1.10120	1.10127	1.10152
	RMI	0.0000	0.0023	0.0000	0.0018
0.10	0.00	370.00318	370.99628	370.23098	370.36147
	0.01	328.30959	328.37464	328.55482	329.32774
	0.02	292.81872	294.47922	293.71633	297.89702
	0.03	262.67743	263.88177	263.99881	265.43359
	0.05	214.90319	216.56967	216.64873	218.30872
	0.10	139.59408	141.35100	141.38453	142.94413
	0.20	71.39976	72.56040	72.56882	73.54454
	0.30	42.63037	43.35296	43.35576	43.94908
	0.50	19.86358	20.17653	20.17712	20.42483
	1.00	6.45577	6.52486	6.52504	6.57412
	2.00	2.51113	2.52208	2.52218	2.52733
	3.00	1.74422	1.74738	1.74745	1.74767
	RMI	0.0000	0.0088	0.0000	0.0082



(a) (b)

Fig. 1: Real-world data of the ARL values on one-sided Extended EWMA and the EWMA control charts for MA(1) with $ARL_0 = 370$; (a) $\lambda_1 = 0.05$ and (b) $\lambda_1 = 0.10$



(a) (b)

Fig. 2: Real-world data of the ARL values on two-sided Extended EWMA and the EWMA control charts for MA(1) with $ARL_0 = 370$; (a) $\lambda_1 = 0.05$ and (b) $\lambda_1 = 0.10$

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Phunsa Mongkoltawat carried out the writing-original draft preparation and simulation.
- Yupaporn Areepong has organized the conceptualization, writing, review, editing, and validation
- Saowanit Sukparungsee has implemented the methodology and software.

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Conflicts of Interest

The authors declare no conflict of interest.