Initial Coefficient Estimates of Bi-Univalent Functions Linked with Balancing Coefficients

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Abstract: - In the next study we introduce a new class $\mathfrak{B}_{\Sigma}^{\mathcal{B}}(\alpha;\ \mathcal{B}(x,z))$ of bi-univalent functions connected with Balancing numbers. For functions in this class we have derived the estimates of the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and Fekete-Szegö functional problems for functions belonging to this new subclass. The main corollaries are followed by some special manners, and the innovation of the definitions and the proofs could involve other studies for such types of similarly investigated subclasses of the bi-univalent functions.

Key-Words: - Balancing polynomial, bi-univalent, analytic function, subordination, coefficient estimates, Fekete-Szegö functional

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1 Introduction

The notion of Balancing numbers $(\mathcal{B}_n), n \geq 0$ introduced by, [1].These numbers have been studied extensively in the last twenty years. Most new studies on the topic include the articles, [2], [3], [4], [5], [6], [7], [8]. Generalizations of Balancing numbers can be obtained in various ways, [9], [10], [11], [12], [13].

Definition 1. [11]. *Assume that* $x \in \mathbb{C}$ *and* $n \geq 2$. Balancing polynomials are defined with the following recurrence relation

$$\mathcal{B}_n(x) = 6x\mathcal{B}_{n-1}(x) - \mathcal{B}_{n-2}(x), \qquad (1)$$

where

$$\mathcal{B}_0(x) = 0, \qquad (2)$$
$$\mathcal{B}_1(x) = 1.$$

By using the recurrence relation given by (1) it is easily obtained that

$$\mathcal{B}_2(x) = 6x, \qquad (3)$$
$$\mathcal{B}_3(x) = 36x^2 - 1.$$

Lemma 2. [12]. The ordinary generating function of the Balancing polynomials is defined by

$$\mathcal{B}(x,z) = \sum_{n=0}^{\infty} \mathcal{B}_n z^n = \frac{z}{1-6xz+z^2} \qquad (4)$$

Let us denote by A the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (5)$$

which are analytic in the open unit disc $D = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. Further, denote by S the class of analytic normalized and univalent functions in D.

The Koebe-one quarter theorem, [14], ensures that the image of D under every univalent function $f \in A$ contains a disc of radius 1/4. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z,\ z \in D$ and $f(f^{-1}(w)) = w,\ (|w| < r_0(f),\ r_0(f) \geq 1/4)$.

The inverse function f^{-1} is given by

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3 -(5a_2^3-5a_2a_3+a_4)w^3+\cdots. \tag{6}$$

A function $f\in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in D. Let $\sum$ denote the class of bi-univalent functions defined in D. Some examples of functions in the class $\sum$ are

$$\frac{z}{z-1},\frac{1}{2}\log\frac{1+z}{1-z},-\log(1-z).$$

Koebe function is a member of S but not in the class Σ.

The class Σ was first studied by ,[15], and showed that $|a_2|\le 1{,}51$. Later, [16], conjectured that $|a_2|\le\sqrt{2}$. After that, [17], showed that $max|a_2|=\frac{4}{3}$.

For two analytic functions, f_1 and f_2, such that $f_1(0)=f_2(0)$, we say that f_1 is subordinate to f_2 in U and write $f_1(z)\prec f_2(z),\ z\in U$, if there exists a Schwarz function $v(z)$ with $v(0)=0$ and $|v(z)|\le|z|,\ z\in U$ such that $f_1(z)=f_2(v(z)),\ z\in U$. Furthermore, if the function f_2 is univalent in U, then we have the following equivalence;

$$f_1(z)\prec f_2(z)\Leftrightarrow f_1(0)=f_2(0) \text{ and } f_1(U)\subset f_2(U).$$

In a recent study, using Balancing polynomials, [18], the authors defined the class ${}_{\mathcal{B}}C_{\Sigma}\left(I(x,z)\right)$ and examined the initial coefficients of the functions belonging to the class ${}_{\mathcal{B}}C_{\Sigma}\left(I(x,z)\right)$ as follows:

Definition 3. [18]. The function f is named to be in the class ${}_{\mathcal{B}}\mathrm{C}_{\Sigma}\left(\mathrm{I}(\mathrm{x},\mathrm{z})\right)$ if the following conditions are satisfied:

$$\frac{zf''(z)}{f'(z)}\prec\frac{\mathcal{B}(x,z)}{z}=\frac{\mathrm{z}}{1-6\mathrm{xz}+\mathrm{z}^2}$$

$$\frac{wg''(z)}{g'(z)}\prec\frac{\mathcal{B}(x,w)}{w}=\frac{\mathrm{w}}{1-6\mathrm{xw}+\mathrm{w}^2}$$

and $g(w)=f^{-1}(z)$ is defined by (2).

Theorem 4. [18]. If $f\in {}_{\mathcal{B}}C_{\Sigma}\left(I(x,z)\right)$ and $x\epsilon\square\backslash\left\{\mp\frac{1}{3\sqrt{2}}\right\}$, then

$$|a_2|\le\frac{3|x|\sqrt{6x}}{\sqrt{1-18x^2}},$$

$$|a_3|\le|x|(9|x|+1).$$

The following theorem gives the FeketeSzegö type inequality for the functions in ${}_{\mathcal{B}}C_{\Sigma}\left(I(x,z)\right)$:

Theorem 4. [18]. If $f\in {}_{\mathcal{B}}\mathrm{C}_{\Sigma}\left(\mathrm{I}(\mathrm{x},\mathrm{z})\right)$ and $\mathrm{x}\epsilon\mathbb{C}\backslash\left\{\mp\frac{\sqrt{6}}{3}\right\}$, then

$$\left|a_3-\delta a_2^2\right|$$

$$\le\begin{cases}\dfrac{|x|}{2}, & |1-\delta|\le\dfrac{|2-27x^2|}{27x^2}\\[2ex] \dfrac{27|x|^3|1-\delta|}{|4-54x^2|}, & |1-\delta|\ge\dfrac{|2-27x^2|}{27x^2}\end{cases}$$

In the last decades, to they can applicable to number theory, numerical analysis, combinatorics, and other fields, theory and applications of Fibonacci, Lucas, Chebyshev, LucasLehmer, LucasBalancing polynomials, Gregory numbers, telephone numbers in modern science has gained very importance. Nowadays, these kind of polynomials have been investigated by many authors in, [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28].

2 Coefficient Bounds of the Class $\mathcal{B}_{\Sigma}$ $(\alpha;\ \mathcal{B}(x,z))$ and the Fekete-Szegö İnequality

Consider in the next section analytic bi-univalent function class $\mathcal{B}_{\Sigma}$ $(\alpha;\ \mathcal{B}(x,z))$ deal with the Balancing polynomials to obtain the estimates of the coefficients $|a_2|$ and $|a_3|$ and Fekete Szegö functional problems, [29].

Definition 6. A function f is named to be in the class $\mathcal{B}_{\Sigma}$ $(\alpha;\ \mathcal{B}(x,z))$ if the following subordinations

$$1+\frac{z^{2-\alpha}f''(z)}{\left(zf'(z)\right)^{1-\alpha}}\prec\frac{\mathcal{B}(x,z)}{z} =\frac{\mathrm{z}}{1-6\mathrm{xz}+\mathrm{z}^2}=\mathrm{K}(\mathrm{x},\mathrm{z}),$$

$$1+\frac{w^{2-\alpha}g''(z)}{\left(wg'(w)\right)^{1-\alpha}}\prec\frac{\mathcal{B}(x,w)}{w}$$
$$=\frac{\mathrm{w}}{1-6\mathrm{xw}+\mathrm{w}^2}=\mathrm{K}(\mathrm{x},\mathrm{w}),$$

$z, w \in D$, $0\le\alpha\le 1$ and $g(w)=f^{-1}(z)$

is defined by (2).

Remark 7. If $\alpha = 0$, we say that $f \in \Sigma$ is in

$\mathcal{B}_\Sigma\left(0;\mathcal{B}(x,z)\right)=\mathcal{B}_\Sigma\left(\mathcal{B}(x,z)\right)={}_{\mathcal{B}}C_\Sigma\left(I(x,z)\right)$.

This class was introduced by, [18].

Remark 8. If $\alpha = 1$, we sat that $f \in \Sigma$ is in $\mathcal{B}_\Sigma$ $(1;\mathcal{B}(x,z))$ İf the following conditions hold true

$$1+zf''(z)\prec\mathrm{K}(\mathrm{x},\mathrm{z}),$$

$$1+wg''(w)\prec\mathrm{K}(\mathrm{x},\mathrm{w}).$$

The next lemma will be used in our study. This lemma is a generalization of Lemma 6 in, [30], which could be obtained for $l = 1$.

Lemma 9. [30]. Let $m, s \in R$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < r$ and $|z_2| < r$, then

$$|(t+s)z_1+(t-s)z_2|\le\begin{cases}2|t|r, & \text{for } |t|\ge|s|\\ 2|s|r, & \text{for } |t|\le|s|\end{cases}.$$

The next result explains the upper bounds for the first two coefficients of the functions in $\mathcal{B}_\Sigma$ $(\alpha;\mathcal{B}(x,z))$.

Theorem 10. If $f\in\mathcal{B}_\Sigma$ $(\alpha;\mathcal{B}(x,z))$ and $x\in\mathbb{C}\setminus\left\{\mp\frac{1}{3\sqrt{2|1-2\alpha|}}\right\}$, then

$$|a_2|\le\frac{3|x|\sqrt{6|x|}}{\sqrt{1-18x^2(2\alpha-1)}}, \qquad (10)$$

$$|a_3|\le|x|(9|x|+1). \qquad (11)$$

Proof. Let $f\in\mathcal{B}_\Sigma$ $(\alpha;\mathcal{B}(x,z))$Then from the Definition 3, the subordinations (8) and (9) satisfy. Thus, there exists an analytic function Λ in D with $\Lambda(0) = 0$, $|\Lambda(z)|<1$,

$$|\Lambda_i|<1 \qquad (12)$$

such that

$$1+\frac{z^{2-\alpha}f''(z)}{\left(zf'(z)\right)^{1-\alpha}}=\mathrm{K}\left(\mathrm{x},\Lambda(z)\right). \qquad (13)$$

Also, there exists an analytic function Θ in D with

$\Theta(0) = 0$, $|\Theta(z)| < 1$,

$$|\Theta_i|<1 \qquad (12)$$

such that

$$1+\frac{w^{2-\alpha}g''(z)}{\left(wg'(w)\right)^{1-\alpha}}=\mathrm{K}\left(\mathrm{x},\Theta(\mathrm{w})\right),$$

where $i \in N$ and the analytical functions Λ and Θ have the form

$$\Lambda(z) = \Lambda_1 z + \Lambda_2 z^2 + \Lambda_3 z^3 + \cdot\cdot\cdot,$$

$$\Theta(w) = \Theta_1 w + \Theta_2 w^2 + \Theta_3 w^2 + \cdot\cdot\cdot.$$

Hence, the functions $\mathrm{K}\left(\mathrm{x},\Lambda(z)\right)$ and $\mathrm{K}\left(\mathrm{x},\Theta(\mathrm{w})\right)$ are of the form

$$\begin{aligned}K\left(x,\Lambda(z)\right) &= \mathcal{B}_1(x)+\mathcal{B}_2(x)\Lambda_1 z\\ &+ [\mathcal{B}_2(x)\Lambda_2+\mathcal{B}_3(x)\Lambda_1^2]z^2\\ &+ [\mathcal{B}_2(x)\Lambda_3+2\mathcal{B}_3(x)\Lambda_1\Lambda_2\\ &+ 2\mathcal{B}_3(x){\Lambda^3}_1]z^3+\cdot\cdot\cdot.\end{aligned} \qquad (16)$$

and

$$\begin{aligned}K\left(x,\Theta(w)\right) &= \mathcal{B}_1(x)+\mathcal{B}_2(x)\Theta_1 w\\ &+ [\mathcal{B}_2(x)\Theta 2+\mathcal{B}_3(x){\Theta^2}_1]w^2\\ &+ [\mathcal{B}_2(x)\Theta 3+2\mathcal{B}_3(x)\Theta_1\Theta_2\\ &+ 2\mathcal{B}_3(x){\Theta^3}_1]w^3+\cdot\cdot\cdot.\end{aligned} \qquad (17)$$

So, comparing the corresponding coefficients in (13) by (16) and (15) by (17), we obtain that

$$2a_2=\mathcal{B}_2(x)\Lambda_1 \qquad (18)$$
$$6a_3-4(1-\alpha)a_2^2=\mathcal{B}_2(x)\Lambda_2+\mathcal{B}_2(x)\Lambda_2+\mathcal{B}_3(x)\Lambda_1^2 \qquad (19)$$

$$2a_2=-\mathcal{B}_2(x)\Theta_1 \qquad (20)$$

$$4(2+\alpha)a_2^2-6a_3=\mathcal{B}_2(x)\Theta_2+\mathcal{B}_2(x)\Theta_2+B_3(x)\Theta_1^2 \qquad (21)$$

From (18) and (20)

$$\Lambda_1 = -\Theta_1 \tag{22}$$

and

$$8a_2^2 = \mathcal{B}_2^2(x)(\Lambda_1^2 + \Theta_1^2) \tag{23}$$

Adding (18) and (20) we get

$$4(2\alpha+1)a_2^2 = \mathcal{B}_2(x)(\Lambda_2 + \Theta_2) + \mathcal{B}_3(x)(\Lambda_1^2 + \Theta_1^2) \tag{24}$$

By using (23) in (24) we have

$$a_2^2 = \frac{\mathcal{B}_2^3(x)(\Lambda_2+\Theta_2)}{4(2\alpha+1)\mathcal{B}_2^2(x) - 8\mathcal{B}_3(x)} \tag{25}$$

Considering relations (2) and (3) and using them in (25), we get

$$a_2^2 = \frac{27x^3(\Lambda_2+\Theta_2)}{1-18x^2(2\alpha-1)} \tag{26}$$

Using (12) and (14) together with the triangle's inequality in the equality (26) it follows

$$|a_2| \leq \frac{3|x|\sqrt{6x}}{\sqrt{1-18x^2(2\alpha-1)}}.$$

Also, if we subtract (21) from (19), considering (22), we have

$$12\ (\ 12a_3 - a_2^2) = \mathcal{B}_2(x)(\Lambda_2 - \Theta_2),$$

then

$$a_3 = a_2^2 + \frac{\mathcal{B}_2(x)(\Lambda_2-\Theta_2)}{12}. \tag{27}$$

This equation combined with (23) leads to

$$a_3 = \frac{\mathcal{B}_2^2(x)\left(\Lambda_1^2+\Theta_1^2\right)}{8} + \frac{\mathcal{B}_2(x)(\Lambda_2-\Theta_2)}{12} \tag{28}$$

Utilizing the triangle's inequality, (12) , (14) and (26) from (28) it follows
$|a_3| \leq |x|(9|x|+1)$.

For the special choices of the parameter α, we obtain the following :

Corollary 11. If $f \in \mathcal{B}_\Sigma\ (\mathcal{B}(x,z))$, then our result coincides with the result Thorem1 in, [8].

Corollary 12. If $f \in \mathcal{B}_\Sigma\ (1;\ \mathcal{B}(x,z))$ and $x\epsilon\square\setminus \left\{\mp\frac{\sqrt{2}}{6}\right\}$, then we obtain

$$|a_2| \leq \frac{3|x|\sqrt{6|x|}}{\sqrt{1-18x^2}},$$

and

$$|a_3| \leq |x|(9|x|+1).$$

The next theorem gives us the Fekete-Szegö İnequality :

Theorem 13. If $f \in \mathcal{B}_\Sigma\ (\alpha;\ \mathcal{B}(x,z))$ and $x\epsilon\mathbb{C}\setminus \left\{\mp\frac{1}{3\sqrt{2|1-2\alpha|}}\right\}$, then

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} |x|, & \text{for } |1-\delta| \leq \frac{|1+18x^2(2\alpha-1)|}{54x^2} \\ \frac{54|x|^3|1-\delta|}{|1+18x^2(2\alpha-1)|}, & \text{for } |1-\delta| \geq \frac{|1+18x^2(2\alpha-1)|}{54x^2} \end{cases} \tag{29}$$

Proof. If $f \in \mathcal{B}_\Sigma\ (\alpha;\ \mathcal{B}(x,z))$ has the form (5), from the equations (25) and (27), we get

$$\begin{aligned} a_3 - \delta a_2^2 &= a_2^2 + \frac{\mathcal{B}_2(x)\left(\Lambda_2 - \Theta_2\right)}{12} - \delta a_2^2 \\ &= (1-\delta)a_2^2 + \frac{\mathcal{B}_2(x)\left(\Lambda_2 - \Theta_2\right)}{12} \\ &= \mathcal{B}_2(x)\left\{\left[\text{ŧ}(\delta) + \frac{1}{12}\right]\Lambda_2 \right. \\ &\quad \left. + \left[\text{ŧ}(\delta) + \frac{1}{12}\right]\Theta_2\right\}, \end{aligned} \tag{30}$$

where $\text{ŧ}(\delta) = \frac{\mathcal{B}_2^2(x)(\Lambda_2+\Theta_2)}{4(2\alpha+1)B_2^2(x) - 8B_3(x)}$. Using the equalities (12) and (14) and applying Lemma 9, we obtain the desired result.

Corollary 14. . If $f \in \mathcal{B}_\Sigma\ \left(\mathcal{B}(x,z)\right)$, then our result coincides with the result Thorem1 in, [18].

Corollary 15. . If $f \in \mathcal{B}_\Sigma\ (1;\ \mathcal{B}(x,z))$ and $x \in \mathbb{C}\setminus\left\{\mp\frac{\sqrt{2}}{6}\right\}$, then we obtain

$$\left|a_3-\delta a_2^2\right| \leq \begin{cases} |x|, & \text{for} \quad |1-\delta| \leq \dfrac{|1+18x^2|}{54x^2} \\ \dfrac{54|x|^3|1-\delta|}{|1+18x^2|}, & \text{for} \quad |1-\delta| \geq \dfrac{|1+18x^2|}{54x^2} \end{cases}$$

4 Conclusion

In our new study, we have introduced and studied the initial coefficient problems associated with the new subclass $\mathcal{B}_\Sigma\ (\alpha;\ \mathcal{B}(x,z))$ of the well known bi-univalent class S. This bi-univalent function subclass is given by Definition 6. For the functions in our new class we have gained the estimates of the Taylor–Maclaurin coefficients $|a_2|$, $|a_3|$, and we obtained solutions for the Fekete-Szegö functional problems. New results are shown to follow upon specializing the parameters involved in our main results as given in Remark 8 and Corollary 12 for the class of $\mathcal{B}_\Sigma\left(\alpha;\ \mathcal{B}(x,z)\right)$associated with Balancing coefficients which are new and not yet studied so far.

We hope that this work encourages the researchers to obtain other characterization properties and relevant connections in other classes of univalent functions.

As an open problem, we can point out the following:

i) Other researchers may define the Hankel determinant for this de_new class
ii) Radii of starlikeness can be investigated in this class.
iii) A new class of $m-$fold symmetric analytic functions introduced by using properties of this class.

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