Application of the Bilateral Hybrid Methods to Solving Initial -Value Problems for the Volterra Integro-Differential Equations

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Abstract: - The many problems of natural sciences are reduced to solving integro-differential equations with variable boundaries. It is known that Vito Volterra, for the study of the memory of Earth, has constructed the integro-differential equations. As is known, there is a class of analytical and numerical methods for solving the Volterra integro-differential equation. Among them, the numerical methods are the most popular. For solving this equation Volterra himself used the quadrature methods. How known in solving the initial-value problem for the Volterra integro-differential equations, increases the volume of calculations, when moving from one point to another, which is the main disadvantage of the quadrature methods. Here the method is exempt from the specified shortcomings and has found the maximum value for the order of accuracy and also the necessary conditions imposed on the coefficients of the constructed methods. The results received here are the development of Dahlquist's results. Using Dahlquist’s theory in solving initial-value problem for the Volterra integro-differential equation engaged the known scientists as P.Linz, J.R.Sobka, A.Feldstein, A.A.Makroglou, V.R.Ibrahimov, M.N.Imanova, O.S.Budnikova, M.V.Bulatova, I.G.Buova and ets. The scientists taking into account the direct connection between the initial value problem for both ODEs and the Volterra integro-differential equations, the scientists tried to modify methods, that are used in solving ODEs and applied them to solve Integro-differential equations. Here, proved that some modifications of the methods, which are usually applied to solve initial-value problems for ODEs, can be adapted for solving the Volterra integro-differential equations.

Here, for this aim, it is suggested to use a multistep method with the new properties. In this case, a question arises, how one can determine the validity of calculated values. For this purpose, it is proposed here to use bilateral methods. As is known for the calculation of the validity values of the solution of investigated problems, usually have used the predictor-corrector method or to use some bounders for the step-size. And to define the value of the boundaries, one can use the stability region using numerical methods. As was noted above, for this aim proposed to use bilateral methods. For the illustration advantage of bilateral methods is the use of very simple methods, which are called Euler's explicit and implicit methods. In the construction of the bilateral methods it often becomes necessary to define the sign for some coefficients. By taking this into account, here have defined the sign for some coefficients.

Key-Words: - The Volterra integro-differential equation, multistep and hybrid methods, symmetric and bilateral methods, degree and stable, advanced methods.

Received: June 12, 2023. Revised: September 6, 2023. Accepted: October 5, 2023. Published: October 20, 2023.

1 Introduction

As was noted above, the investigation of the many problems from the different areas of the natural sciences is reduced to solving the initial value problem for the Volterra integrodifferential equations. Note that this problem can be formulated in different forms. One of the popular presentations of this problem can be presented as follows [1], [2], [3], [4], [5], [6], [7], [8]:

$$y' = f(x,y) + \lambda\int_{x_0}^{x} K(x,s,y(s))ds,\ y(x_0) = y_0, \tag{1}$$
$$x \in [x_0, X],\ \lambda = const.$$

If the continuous to the totality of arguments function $f(x,y)$ and $K(x,s,y(s))$ are given, then the problem (1) can be taken as the given. Spouse that these functions are defined in some closed area and have the partial derivatives up to p. And also suppose that the problem (1) has a unique solution, which is defined on the segment $[x_0, X]$.

It is known that there are some classes of numerical methods for solving problems (1). It is also known that the estimation obtained for the errors of these methods holds for the sufficiently small step size h. Therefore one of the main results in these areas is the construction of numerical methods, results finding by which can be accepted as the reliable information for the selected results. To solve such problems, Chapligin proposed his own two-sided or bilateral method, which now is called the bilateral analytical approximate method. The bilateral method was not developed because its advantage has not been proven. By using that, here to suggest a way for the construction of the numerical bilateral methods. And also to give some comparisons between bilateral methods and bilateral formulas, [9], [10], [11], [12], [13], [14], [15]. For the illustration of these let us consider the construction of bilateral methods.

§1.The bilateral methods and their application to solve some specific task.

As is known, the multistep method with constant coefficients is one of the popular methods for solving problem (1), which can presented as follows:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i},\quad n = 0,1,...,N-k, \tag{2}$$

here the mesh-point x_m is defined as: $x_{i+1} = x_i + h \quad (i = 0,1,...,N-1)$ and $0 < h$ is the step-size.

If method (2) is applied to solving the problem (1), then receive:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h\sum_{i=0}^{k}\beta_i \upsilon_{n+i} \tag{3}$$

here $\vartheta(x_{n+i}) = \int_{x_0}^{x_{n+i}} K(x_{n+i}, s, y(s))ds,$

but $\vartheta(x) = \int_{x_0}^{x} K(x,s,y(s))ds$ and the values $f_m = f(x_m, y_m)\ (m = 0,1,2,...,N-k).$

$\vartheta_{n+i}\ (i = 0,1,...,k)$ can be found as the values of the solution of the following problem:

$$\vartheta'(x) = K(x,x,y(x)),\ \vartheta(x_0) = 0. \tag{4}$$

In this way, the solving of the problem (1), has reduced to solving the initial-value problem for ODEs of the first order. Vito Volterra himself suggested the quadrature methods for solving problems (1). If here, has used the method for solving the problem (4), then by using that in the method (3), receive the following, [16], [17], [18], [19], [20], [21], [22]:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h\sum_{i=0}^{k}\sum_{j=i}^{k}\beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}) \tag{5}$$
$$(n = 0,1,2,...,N-k).$$

Noted that for the application of method (2), often arises the necessity to define the order of accuracy for using multistep methods which can be defined as the following form:

Definition1. Integer variable p - is called the degree for the method (2) if the following holds:

$$\sum_{i=0}^{k}(\alpha_i y(x+ih) - h\beta_i y'(x+ih)) = O(h^{p+1}),\ h \to 0. \tag{6}$$

From the asymptotic equality (6), receive that the degree for the method (2) equal to p. Dahlquist proves that if the method (2) is stable, $\alpha_k \neq 0$, then $p \le 2[k/2]$ for each value of k, there exist stable methods with the degree $p_{\max} = 2[k/2] + 2.$

Definition 2. Method (2) is called stable if the roots of the polynomial $\rho(\lambda) \equiv \alpha_k\lambda^k + \alpha_{k-1}\lambda^{k-1} + ... + \alpha_0$ lie in the unit circle on the boundary of which there are no multiple roots. For the construction stable methods with the degree $p > 2[k/2] + 2$, Prof. V.Ibrahimov has investigated advanced methods, which can be received from the (2) in the case $\beta_k \neq 0,$ and $\alpha_k = \alpha_{k-1} = ...\alpha_{k-s+1} = 0,\ \alpha_{k-s} \neq 0.$

He has constructed the concept method with the degree $p = k+2$ for the $k = 3$ and $s = 1$. In the, [23], has constructed a stable method with the degree $p = 5$. Thus receive that advanced methods are more promising. As is known one of the popular methods of type (2) is the Simpson method, which can be received from method (2) in the case $k=2$. Dahlquist result shows that in this case, the stable method with the degree $p=4$ is the Simpson method. In this case $k=2$ constructed method with degree $p=4$ is unique.

And now let us consider the case $k=1$. The popular methods in this case are the explicit and implicit Euler methods and trapezoidal rules.

Explicit Euler method for solving problem (1) can be presented as:

$$y_{n+1}=y_n+hf_n+h(K(x_n,x_n,y_n)+K(x_{n+1},x_n,y_n))/2. \quad (7)$$

But the implicit Euler method for solving problem (1) can be presented as follows:

$$y_{n+1}=y_n+hf_{n+1}+hK(x_{n+1},x_{n+1},y_{n+1}). \quad (8)$$

It is known that methods (7) and (8) correspond to the following Euler's methods:

$$y^e{}_{n+1}=y_n+hy'_n;\; y^i{}_{n+1}=y_n+hy'_{n+1}. \quad (9)$$

As is known the local traction error for these methods can be written as following, respectively:

$$h^2y''_n/2+O(h^3),\; -h^2y''_n/2+O(h^3).$$

Hence it follows that methods (7) and (8) the bilateral, so as

$y^e_{n+1}\le y(x_{n+1})\le y^i_{n+1}$ if $y''(x)\ge 0.$ And if $y''(x)<0,$ then $y^i_{n+1}\le y(x_{n+1})\le y^e_{n+1}$.

$y(x_{n+1})$ is the exact value of the solution of problem (1) at the point x_{n+1}.

It is not difficult to prove that the value

$y_{n+1}=(y^e_{n+1}+y^i_{n+i})/2$ will be exact than the y^e_{n+1} and y^i_{n+1}. y_{n+1} will be same with the value finding by the trapezoidal rule.

In our case, the simple, step-by-step algorithm for the application of Euler's methods to solving any problems can be presented as follows:

Step 1 Input (initial values);

Step 2 For $j=1$ step 1 $n-1$ do steps 3-5;

Step 3 Calculation y^e_{j+1} by the method (7);

Step 4 Calculation y^i_{j+1} by the method (8);

Step 5 Calculation $y_{j+1}=(y^i_{j+1}+y^e_{j+1})/2$; Print $(y^i_{j+1};y^e_{j+1};y_{j+1})$ end.

Step 6 STOP.

It is not difficult to believe that the method constructed in the above-mentioned way is bilateral. Let us consider the following couple of methods:

$$y^{(1)}_{n+2}=y_n+2hy'_{n+1},\; R_n=h^3y'''_n/3+h^4y^{IV}_n/3+O(h^5) \quad (10)$$

$$y^{(2)}_{n+2}=y_n+2h(y'_{n+2}+y'_{n+1}+y'_n)/3, \quad (11)$$
$$R_n=-h^3y'''_n/3-h^4y^{IV}_n/3+O(h^5).$$

Here the R_n denote the local truncation error of these methods. It is obvious that the value $\bar{y}_{n+2}=(y^{(1)}_{n+2}+y^{(2)}_{n+2})/2$ will be more accurate than the values $y^{(1)}_{n+2}$ and $y^{(2)}_{n+2}$. The receiving method is the same as the Simpson method. Note that the methods can be used to determine the limit for the value $\bar{y}_{n+2}$.

And now let us consider the application of the methods (10) and (11) to solving equation (1). In this case, receive:

$$y^{(1)}_{n+2}=y_n+2hf_{n+1}+h(K(x_{n+1},x_{n+1},y_{n+1})+K(x_{n+2},x_{n+1},y_{n+1}));$$
$$y^{(2)}_{n+2}=y_n+2h(f_{n+2}+f_{n+1}+f_n)/3+h(2K(x_{n+2},x_{n+2},y_{n+2})+K(x_{n+2},x_{n+1},y_{n+1})+K(x_{n+1},x_{n+1},y_{n+1})+K(x_{n+2},x_n,y_n)+K(x_n,x_n,y_n))/6.$$

It is easy to verify that, half sum for the values $y^{(1)}_{n+2}$ and $y^{(2)}_{n+2}$ can be presented as follows:

$$\bar{y}_{n+2}=y_n+h(f_{n+2}+4f_{n+1}+f_n)/3+h(2K(x_{n+2},x_{n+2},y_{n+2})+4K(x_{n+2},x_{n+1},y_{n+1})+4K(x_{n+1},x_{n+1},y_{n+1})+K(x_{n+2},x_n,y_n)+K(x_n,x_n,y_n))/6.$$

This is one of the modifications of Simpson's method for solving Volterra integro-differential equations. From here one can see that the presentation of multistep methods for solving initial-value problems for Volterra integro-differential equations is not unique. In the application of this method some difficulties in the calculation of the values y_{n+2}, which participates in the explicit method designed for the calculation of this value. For this aim one can use the values calculated by the midpoint method of (10), [19], [20], [21], [22], [23], [24], [25], [26], [27], [28].

However, by using the Simpson method, determining the boundary of the error is impossible. Hence, the use of bilateral methods, which is possible to find an error on any point. In the theory of numerical methods for solving integro-differential equations, often symmetrical multistep methods, therefore the following section is devoted to the study of this question.

§2. Application of symmetrical methods to solving problem (1).

The concept of symmetry is discovered in nature, for that is not a mathematical term. And then this term was more commonly used in astronomy. As is known, a planet in our galaxy is symmetrical to the plane of Earth. In the theory of multistep methods, the notion of symmetry was used by Dahlquist. However, in the theory of quadrature formulas, the concept of symmetry was used in the study of Gauss and Chebyshev methods. As is known the Gauss nodes and the coefficients are symmetrical. We believe that one of such well-known methods is the midpoint method and another is the trapezoidal rules. Note that these methods do not satisfy the Dahlquist requirement, but experts have always accepted these methods as symmetrical, which can be presented as: $y_{n+1} = y_n + hy'_{n+1/2}$; $y_{n+1} = y_n + h(y'_n + y'_{n+1})/2.$

Some authors give advantages of using symmetrical methods, which can be defined in the following form:

Definition 3. (Dahlquist), [12]. Stable method (2) is called symmetrical if the following holds:
$p = k + 2$ and $\alpha_i = -\alpha_{k-i}$; $\beta_i = \beta_{k-i}$ $(i = 0,1,...,k)$.

Taking into account these conditions, we get that k -is the even number. Hence, the amount of the mesh-points in the multistep method will be odd. In this case, by using the necessary condition of convergence, receive that $\alpha_k + \alpha_{k-1} + ... + \alpha_1 + \alpha_0 = 0$. Hence $\alpha_{k/2} = 0$. In this case, $k = 2$ the symmetrical method can be written as:

$$y_{n+2} = y_n + h(y'_{n+2} + 4y'_{n+1} + y'_n)/3, \quad (12)$$

which is unique and has the degree $p = 4$ or $p = 2k$. Note that the method receiving as the results of half sum of equalities (7) and (8) can be taken as symmetrical. Noted for the application of these methods to solve the problem (1), it is necessary to find the values of the coefficients $\alpha_i, \beta_i, \beta_i^{(j)}$ $(j, i = 0,1,...,k)$. A similar investigation was carried out by some authors, [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44]. Here, suppose that the method (2) is given. In this case, the application of method (5) to solving problem (1), must be known as the values of the coefficients $\beta_i^{(j)}$ $(j, i = 0,1,...,k)$. For this aim, one can use the following linear system of algebraic equations:

$$\sum_{j=i}^{k} \beta_i^j = \beta_i \quad (i = 0,1,...,k). \quad (13)$$

Noted that amount of the solutions in this system more than one. Therefore users have the often to choose different methods.

Note that in the generalization of the midpoint method, one can receive the hybrid methods. Hybrid methods in a general form can be presented as the following:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i+\nu_i} \quad (|\nu_i| < 1; i = 0,1,..,k). \quad (14)$$

The methods used in the previous section cannot be received from the method (13) as the partial case. As was noted above, the midpoint is stable has the degree $p = 2$, and is explicit. Let us note that all the hybrid methods of type (14) will be explicit. It does not follow from here that, the hybrid methods can be applied in direct form to solve any problems. Let us show that it's not right. For this consider the following method:

$$y_{n+1} = y_n + h(y'_{n+\nu_0} + y'_{n+\nu_1})/2, \ \nu_0 = 1/2 - \sqrt{3}/6, \quad (15)$$
$$\nu_1 = 1/2 + \sqrt{3}/6,$$

In the construction of this method have used two hybrid points, the method is stable and has the degree $p = 4$. Let us consider the application of the method (15) to solving an initial-value problem for ODEs, this can be presented as follows:

$$y_{n+1} = y_n + h(f(x_{n+\nu_0}, y_{n+\nu_0}) + f(x_{n+\nu_1}, y_{n+\nu_1}))/2,$$

, here $y'(x) = f(x, y(x))$.

It is obvious that in the application of this method are arises to calculation of the values $y_{n+(3-\sqrt{3})/6}$ and $y_{n+(3+\sqrt{3})/6}$, which are not easy. As is known the following method:

$$y_{n+1} = y_n + h(f_{n+1} + 4f_{n+1/2} + f_n)/6, \quad (16)$$

also has the degree $p = 4$ and is called the Simpson method.

Remark 1.

Let us note that method (16) has been received from Simpson's method, which resembles the Runge-Kutta method and is the one step. It is known that one-step methods have some advantages. For example, easily be applied to solving various problems. All the methods have their advantages and disadvantages. Taking into account the above mentioned, here I wanted to reduce multistep methods to the one-step methods. Runge-Kutta and (16) are not the same. Methods Runge-Kutta are explicit, but method (16) is implicit. Note that if method (16) and method Runge (constructed by Runge, which has the degree $p = 4$ then these methods will be the same. Therefore, the properties of the methods greatly depend on the problem, which must be solved.

receive any difficulty related to calculation of values $\lambda = 0$ receive any difficult related to calculation of values $y_{n+i+\nu_i}$ $(i = 0,1,2,..,k)$. As was noted above, if method (14) is applied to the calculation of definite integral, then it does not cause any difficulty. But, when applied to solve the problem (1) for the case $f(x, y) \equiv 0$, arises some difficulty, which can be solved in different ways. For example, in the application of the method (15) difficulties related to the calculation of the values $y_{n+\nu_0}$ $y_{n+\nu_1}$, which can be calculated by using the described algorithm.

It is not difficult to show that methods constructed at the intersection of methods (2) and (14) are more exact. For this let us consider the following linear method:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i} + h\sum_{i=0}^{k}\gamma_i y'_{n+i+\nu_i},\ (|\nu_i| < 1,\ i = 0,1,2,..,k), \qquad (17)$$

which was constructed on the intersection of the multistep and the hybrid methods. Note that the method of (16) is more exact than the others. For Example, if method (16) is stable, then there are stable methods of type (16) with the degree $p = 3k + 3$ and if method (14) is stable then there are in the class of methods (14), stable methods with the degree $p = 2k + 2$.

§3. On some advantages of the advanced methods and their application.

In, [36], has proved that some of the advantages of the advanced multistep methods is that if they are stable, then in this class methods exist the stable advanced methods, which are more accurate than others. At first have been constructed a lot of multistep methods and then constructed advanced as new methods for comparison with the known and multistep methods. Note that advanced methods can be presented as multistep methods. For the illustration of this let us consider to following method:

$$\sum_{i=0}^{k-m}\bar{\alpha}_i y_{n+i} = h\sum_{i=0}^{k}\bar{\beta}_i y'_{n+i} \quad (m > 0,\ n = 0,1,..,N-k+m). \qquad (18)$$

If comprise methods (2) and (18), then receive that methods (2) and (18) can be taken as the same only for the case $m = 0$. But, from the above-given condition we get that, $m > 0$. As follows from here, the class of methods (18) is the independent field of research. Formally one can say that by using the selection coefficient α_k $\alpha_k = 0$, one can receive method (18) from method (2). Let's show that it's not. For this it is enough to recall Dahlquist's condition, which can presented as the following suppose that the method (2) is convergence, then its coefficients must satisfy the following conditions, [16], [17], [18], [19], [20], [21], [22], [23], [45]:

A. Coefficients $\alpha_i, \beta_i (i = 0,1,..,k)$ are real numbers $\alpha_k \neq 0$.
B. The characteristic polynomials:
$$\rho(\lambda) \equiv \sum_{i=0}^{k}\alpha_i\lambda^i \text{ and } \delta(\lambda) \equiv \sum_{i=0}^{k}\beta_i\lambda^i$$
have no common factor different from constant.
C. The condition $\delta(1) \neq 0$ holds and $p \geq 1$ (p -is the degree).

As was noted above V.Ibrahimov constructed and investigated the method (16) and, therefore received similar conditions for the coefficients, which can be presented as:

A. The coefficients $\bar{\alpha}_i, \bar{\beta}_i (i = 0,1,...,k)$ are real numbers and $\bar{\alpha}_{k-m} \neq 0.$
B. The characteristic polynomials:
$$\bar{\rho}(\lambda) \equiv \sum_{i=0}^{k-m}\bar{\alpha}_i\lambda^i \text{ and } \bar{\delta}(\lambda) \equiv \sum_{i=0}^{k}\bar{\beta}_i\lambda^i$$
have no common factor different from constant.
C. The conditions $\bar{\delta}(1) \neq 0,\ \bar{p} \geq 1$ are hold ($\bar{p}$ -is the degree of the method (14)).

By using the condition A, receive that $\alpha_k \neq 0$. It is not difficult to understand that one of the important questions in the investigation of the define the maximum value of the degree for the investigated methods. One can prove that p≤2k (degree for the method (2) and method with the degree p=2k is unique, but $\bar{p} \leq 2k - m$ (degree for the method (13) and method with the degree $\bar{p} \leq 2k - m$-is unique, if method (2) is stable then $p \leq 2[k/2] + 2$ and there are stable methods of type (2) with the degree $p_{\max} = 2[k/2] + 2$ for each k. But if method (13) is stable and has the degree of $\bar{p}$, then $\bar{p} \leq k + 1 + m\ (k \geq 3m)$. By using these estimations one can compare methods (2) and (13) in full form. As is known all the methods have their advantages and disadvantages. The stable advanced method is more exact than the corresponding methods of type (2). But in using that there arises some difficulty with the calculation of the values of the solution investigated problems at the next points. To demonstrate this let us consider the following method:

$$y_{n+1} = y_n + h(5y'_n + 8y'_{n+1} - y'_{n+2})/12. \qquad (19)$$

One of the base properties of this method is the use of the values of the desired function on the subsequent points.

Let us apply this method to solving the problem (15). In this case receive:

$$y_{n+1} = y_n + h(5f_n + 8f_{n+1} - f_{n+2})/12 + h(2K(x_n, x_n, y_n) + \\ + 2K(x_{n+1}, x_n, y_n) + K(x_{n+2}, x_n, y_n) + 4K(x_{n+1}, x_{n+1}, y_{n+1}) + \\ + 4K(x_{n+2}, x_{n+1}, y_{n+1}) - f(x_{n+2}, x_{n+2}, y_{n+2}))/12. \quad (20)$$

Thus receive the nonlinear equation for finding the value y_{n+1}, to fined which is not easy. Therefore here recommended to use the following step-by-step algorithm (Algorithm 2):

Step 1. Input (initial values);

Step 2. For $j = 1$ step 1 to n-1 do step 3-10;

Step 3. Calculation y^e_{j+1} by the method (9);

Step 4. Calculation y^i_{j+1} by the method (9);

Step 5. Calculation $y_{j+1} = (y^i_{j+1} + y^e_{j+1})/2$;

Step 6. Calculation y^1_{j+2} by the method (9);

Step 7. Calculation y^2_{j+2} by the method (9);

Step 8. Calculation y^1_{j+2} by the method (10);

Step 9. Calculation y^2_{j+2} by the method (11);

Step 10. Calculation $y_{j+2} = (y^1_{j+2} + y^2_{j+2})/2$;

Print $y^i{}_{j+1}; y^e_{j+1}; y_{j+1})$ end.

Step 11. Stop.

Note that method (18) is stable, has the degree $p = 3$, and rises to the class of one-step and multistep methods. It is easy to understand using the method (18). It is enough to know the one value of the desired solutions at the previous point. For the sake of objectivity, note that for the application of the method (18), it is needed to have some information about the values of the solution at the current and next points. To find these values by the required exactness, one can use the above-described way. Note that if the trapezoid method to used in the application of the method (18), then some difficult liberation from which is even more difficult. Thus in the application of the method (18) to solve some problems, it is needed to use any method for calculating the values y_{n+k-m} and $y_{n+k-m} + j$ $(j = 0,1,..,m-1)$ the results of which receive the block method. The step-by-step algorithm, which is presented above is also a block method. If It is necessary one can increase the accuracy above calculated values. In our case, it is desirable to use the Simpson method, which is presented by the method (12). This method when applied to solving problem (1) can be presented as follows:

$$y_{n+2} = y_n + h(f_n + 4f_{n+1} + f_{n+2})/3 + h(k(x_{n+1}, x_n, y_n) \\ + K(x_{n+2}, x_n, y_n) + 4K(x_{n+1}, x_{n+1}, y_{n+1}) + 4K(x_{n+2}, x_{n+1}, y_{n+1}) + \\ + 2K(x_{n+2}, x_{n+2}, y_{n+2}))/6. \quad (21)$$

By using method (21) with the above-presented method (20), one can solve the problem (1) with high precision. Note that in the application of the method (21), some difficulties are related to solving the nonlinear algebraic equations. However, this problem can be solved with the predictor-corrector method. Each of these methods has its advantages and disadvantages. If you compare the methods described above, then get that the methods described above using step-by-step algorithms can be taken as better.

Remark 2. As is known in recent times scientists tried to construct simple methods for solving some problems. The above-constructed algorithms belong to the class of simple methods. Let us show that there are simple methods with a high order of accuracy that differ from the above-mentioned methods. Let us remember the midpoint roll, which in the application to solving the problem (1) can be presented as:

$$y_{n+1} = y_n + hf(x_n + h/2, y(x_n + h/2)) \\ + h(K(x_{n+1}, x_{n+1/2}, y_{n+1/2} + K(x_{n+1/2}, x_{n+1/2}, y_{n+1/2}))/2. \quad (22)$$

This method is stable, has a degree $p = 2$, and is explicit. By simple comparison, that method (21) is better than the above investigated. Note that the method (22) also has some disadvantages, for example calculating the value $y_{n+1/2}$. Some similar investigations were carried out by some authors in solving different problems, [42], [43], [44], [45], [46], [47], [48]. Note the method (22) reminds us hybrid methods that more accurate than the known. There is a lot of work dedicated to the study of hbrid methods, [49], [50], [51], [52], [53], [54], [55].

2 Numerical Results

For the demonstration receiving here result, let us consider the application of the above-presented algorithms to solve the following simple examples:

$$y' = 1 + y - x\exp(-x^2) - 2\int_0^x xt\exp(-y^2(t))dt, \\ y(0) = 0, \ 0 \le x \le 1. \quad (23)$$

The exact solution of this example: $y(x) = x$.

There are many methods for solving this problem, [54], [55], [56], [57], [58], [59], [60], [61], [62], [63], [64].

However, here we profer to useabove proposed methods.

First applied Algorithm 1 and 2 to solve problem (23), results for which are tabulated in Table 1.

Table 1. Maximum error for the above-presented algorithm 1 and algorithm 2.

Step size	Value of x	Maximal error for the Algorithm 1	Maximal error for the Algorithm 2
h=0,05	1	0.28 E -03	0.1 E -05

The receiving results are corresponded to the theoretical. Now let us consider solving the following problem:

$$y' = \lambda y - a(1 - \exp(-\lambda x) + a\lambda \int_0^x y^{-1}(s)ds,$$
$$y(0) = 1,\ 0 \le x \le 2. \tag{24}$$

The exact solution for which can presented as:

$$y(x) = \exp(\lambda x).$$

For this aim to use algorithm1 and methods (15), (18).

Note that depending on the methods applied to the calculation of the value y_{n+2}, method (18) can change its properties. For example, y'_{n+1} if y'_n to use the formula: $y_{n+2} = y_{n+1} + h(3f_{n+1} - f_n)/2$ in the method of (18), then in the resulting of which the method will be A- stable, [48].

All results are tabulated in Table 2.

Table 2. Results for step-size h=0.01.

x	Method(18) +Tr	Method(18)+ E_1	Method (18)+ E_2
0.1	4.5E-09	4.5E-09	1.8E-06
0.3	1.2E-09	1.2E-09	5.1E-07
0.5	2.1E-09	2.1E-09	8.6E-07
0.8	3.5E-09	3.5E-09	1.4E-06
1.0	1.1E-07	1.1E-07	4.4E-05

Here have used denotation Method (18)+E_2, Method(18)+E_1, Method(18)+Tr.

Method (18)+ E_2-is the predictor-corrector method. Here the predictor method has used the explicit method, but as corrector method, used method (18). Here E_1-is the implicit Euler method, E_2-is the explicit Euler method, but the Tr-Trapezodial rule.

And now let us the initial value in problem (23) to use method (15) and the Trapezoidal rule, but as the predictor method has used the explicit Euler method. Results tabulated in Table 3.

Table 3. Results for h=0.01

x	$\lambda = 1$		$\lambda = -1$	
	Method (15)	Trapezoidal rule	Method (15)	Trapezoidal rule
1.1	1.5E-7	4.6E-4	2.0E-8	5.9E-5
1.4	8.9E-7	6.2E-4	5.8E-8	4.4 E-5
1.7	2.1E-6	8.4E-4	7.5E-8	3.2 E-5
2.0	4.1E-6	1.1E-3	7.9E-8	2.4 E-5

Results received for the implicit method taken as the predictor method have been tabulated in Table 4.

Table 4. Results for h=0.01.

x	$\lambda = 1$		$\lambda = -1$	
	Method (15)	Algorithm1	Method (15)	Algorithm1
1.1	1.0E-8	6.0E-5	1.2E-9	7.2E-6
1.4	5.7E-8	8.1E-5	3.5E-9	5.3 E-6
1.7	1.3E-7	1.0E-4	4.6E-9	3.5 E-6
2.0	2.6E-7	1.4E-4	4.9E-9	2.9 E-6

But now, let us as the predictor method with the following midpoint rule.

$$y_{i+1} = y_i + hy'(x_i + h/2).$$

Table 5. Results received for the values h=0.05.

x	$\lambda = 5$		$\lambda = -5$	
	Method (15)	Trapezoidal rule	Method (15)	Trapezoidal rule
1.1	2.3E-3	5.3E-1	5.1E-8	1.2E-5
1.4	4.4E-2	2.4E-0	4.3E-8	2.9 E-6
1.7	3.4E-1	1.1E-1	1.6E-8	6.5E-7
2.0	2.2E-1	4.9E-1	5.2E-9	1.4E-7

By the results tabulated here, one can take the midpoint rule as the better, which is related to the using value, y_i calculated by the method (15). Here tabulated the results received by the method (15) with the degree $p = 4$, by the trapezoidal rule $p = 2$, and the algorithms 1 and 2. By the comparison of the results tabulated here, we see that these results correspond to the theoretical.

3 Conclusion

Here, have shown that numerical methods constructed for solving initial-value problems for ODEs can be applied to solving initial-value problems for the Volterra integro-differential equations. We apply this approach to constructing bilateral numerical methods to solving initial-value problems for the Volterra integro-differential equations. For this aim have used simple numerical methods and have proven that the results received by the bilateral methods are better. This idea has been applied to the construction of two numerical bilateral methods and has shown how one can construct a similar method. As is known, bilateral (or two-sided) methods for solving initial-value problems for the Volterra integro-differential equations, can be said to be almost uninvestigated. Therefore, in this area, the new results obtained here are of interest to many specialists from different areas of modern sciences. In the work, [45], a two-sided method is constructed for solving the ODEs, but the work, [53], has constructed the bilateral methods to solve the Volterra integral equations by using the Runge-Kutta methods. The receiving theoretical result has been demonstrated in simple examples. We hope that this method will find its followers. Therefore, they can be considered promising. It is known that one of the promising methods is the advanced method. By taking into account the results tabulated in Table 3, Table 4, and Table 5, the methods with the fractional step size give good results. These methods are reminded of the hybrid methods. Therefore, these methods are promising.

Acknowledgment:

The authors wish to express their thanks to academicians Telman Aliyev and Ali Abbasov for their suggestions to investigate the computational aspects of our problem and for their frequent valuable suggestions. This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan- Grant AEF-MCG-2022-1(42)-12/4/1-M-4

Thanks to the anonymous reviewers for their valuable comments, which improved the estimation of receiving here results.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Corresponding author V.R. Ibrahimov was responsible for used methods, research,contribution of the concept; G.Y.Mehdiyeva -data interpretation and analysis,M.N.Imanova-illustration of received results; D.A.Jurayev-data interpretation and analysis.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan- Grant AEF-MCG-2022-1(42)-12/4/1-M-4

Conflict of Interest

The authors have no conflict of interest to declare.