DELTD:
An R Package for Kernel Density Estimation using Lifetime Distributions

JAVARIA AHMAD KHAN[1], ATIF AKBAR[1], B. M. GOLAM KIBRIA[2]
[1]Department of Statistics,
Bahuddin Zakariya University, Multan,
PAKISTAN

[2]Department of Mathematics and Statistics,
Florida International University, Miami, FL 33199,
USA

Abstract: - The R package DELTD is for estimating densities by asymmetrical kernels and calculating MSE. This package is to estimate densities that are free of boundary bias. The major concern of the package is to enhance its usefulness in performing inference regarding stated kernels. For this purpose, some lifetime distributions, i.e. Beta, Birnbaum-Saunders, Erlang, Gamma, and Lognormal are considered here due to their usefulness in life data analysis, where their estimated values for density estimation can also be observed. Tuna data is also presented in this package. By using these kernels, densities will be free of boundary problems. This package is a collection of asymmetrical kernels which belong to the lifetime distribution.

Key-Words: - Asymmetrical kernel, Beta kernel, Birnbaum-Saunders kernel, Erlang kernel, Gamma kernel, Lognormal kernel, Tuna data

Received: May 28, 2023. Revised: August 29, 2023. Accepted: October 1, 2023. Published: October 20, 2023.

1 Introduction

Density estimation is a process of constructing the probability density function using underlying data. Applications of density estimation can be found in many fields of daily life, which may be:

- In Statistics: [1], stated that density estimates can be applied in the construction of smooth distribution function estimates via integration, which then can be used to generate bootstrap samples from a smooth estimate of the cumulative distribution function rather than from the empirical distribution. Other statistical applications include identifying the nonparametric part in semiparametric models, finding optimal scores for nonparametric tests, and empirical Bayes methods.
- In Engineering: The detection of abnormal or unexpected conditions from measured response data is an important issue, especially where a clear and early warning of an abnormal condition is required. For this purpose, [2], proposed a method that is based upon the probability density function (PDF) estimated using a kernel method. Other examples can be found in, [3], [4], and references therein.
- In Hydrology: The estimated density function of rainfall, river discharge data, modeling of precipitation, and other hydroclimatic variables analyzed with a probability distribution, are used to gain insight into their behavior and frequency of occurrence, [5], [6].
- In Medicine: [7], sought the study to investigate whether individuals who live near destinations (service facilities, etc.) are more intensely distributed rather than dispersed. They used the kernel density estimation technique to examine how much they are active and engage in more frequent walking for transport and recreation. For other applications in medicine see, [8], [9], [10].
- In Physics: [11], kernel density estimate in the Lamb wave-based damage detection. They showed that the distribution of data is based on the intensity of the noise. In the case of weak noise, the pdf of measured data could be

considered as the normal distribution. However, in the case of strong noise, the pdf was complex and did not belong to any type of common distribution function.

- In Bioinformatics and Genetics: in the last few decades, the importance of the development of computational systems for automated analysis of large amounts of data (high-throughput) has risen. The study, [12], discussed such problems and their solution based on LOWESS and running median. Additionally, they measured a rodent's distance from the arena's wall. They examined the density of distances from the boundary when the algorithm to estimate the boundary is being used and when it is not. Further application can be observed in, [13], [14], [15].
- In Finance and Economics: [16], suggested using the sequential method for the estimation of the size distribution of U.S. family income. Similarly, [17], [18], provide a healthy literature to enhance the importance of density estimation in this field.

  One could think of other several applications in archaeology, [19], climatology, [20], physiology, [21], astronomy, [22], [23], geoscience, [24], [25], and continues to be relevant in new areas of mathematics and information science, [26], [27], [28].

There are a variety of approaches to estimating the density; Kernel density estimation, histogram, data clustering, semi-parametric methods, etc. Kernel density estimation is one of the very famous techniques. The kernel estimator proposed by, [29], [30], is given as;

$$\hat{f}_h(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - x_i) = \frac{1}{nh}\sum_{i=1}^{n} K_h\left(\frac{x - x_i}{h}\right) \tag{1}$$

where k is the kernel function and h represents the bandwidth or smoothing parameter. Due to this, a serious problem of boundary bias or boundary effect arises. If reflected in the results variance and bias showed a sharp increase when estimating them at points near the boundary region. In other words, it affects the performance of the estimator at the boundary points due to boundary effects, then from the interior points. Such a problem occurs when variables represent some sort of physical measure such as time or length. These variables thus, have a naturally lower boundary, e.g. time of birth or zero point on a scale. So, when smoothing is carried out near the boundary and fixed the symmetric kernel is used, those kernels allocate weights outside the density support, [31]. To remove those boundary effects in kernel density estimation, a variety of methods have been developed in the literature. Some well-known methods are summarized below:

1.1 Reflection Method

The reflection method was first introduced by, [32], and then studied by, [33]. The main idea is to reflect the data points. This not only yields a twice as large sample size but most importantly yields a sample drawn from a density with unbounded support. Then kernel estimator is applied to data of size $2n$ and then the new estimate is symmetrical around the origin.

1.2 Transformation Method

To control boundary bias, [34], suggested transforming the data at both sides to a density that has its first derivative equal to 0. They suggested different transformations from a parametric family, in general, and compared with Rice's adjusted kernel method. They claimed that their proposed method produced non-negative estimates and outperformed Rice's adjustment.

1.3 Pseudo-data Method

The study, [35], presented this method, which is based on pseudo data, which is beyond the limits of density support. They claimed that their method is more adaptive because the pseudo method is more appropriate for kernels of order 2 and more. The estimators produced by this method may gain optimal orders of bias, variance, and lower mean squared error at $x = 0$. They suggested using the plug-in and least square cross-validation method for bandwidth selection.

1.4 Local Linear Method

The study, [36], introduced estimators that utilize density derivative estimators obtained from local polynomial fitting. He compared his proposed estimator and its asymptotically optimal bandwidth with Sheather and Jones's bandwidth. However, he showed that former bandwidth overcomes the boundary problems and later does not. A similar technique was further used by, [37], in which they the local polynomial smoothing technique as a possible alternative method for the problem. It was observed that such an estimator possesses desirable properties such as automatic adaptability for boundary effects near endpoints. They also obtained an optimal kernel to estimate the density at endpoints as a solution to a variation problem.

1.5 Rice's Boundary Modification

The study, [38], adapted Rice's method to the context of density estimation. The study, [39], proposed a method based on boundary modification of kernel regression estimators. In this method, a linear combination of two kernel regression estimators is used with two different bandwidths in the boundary area, as the same is used in the interior. The idea is similar to the bias reduction technique discussed in, [40].

To handle this problem, [41], suggested the solution of this problem by replacing the symmetric kernels with the asymmetric Beta kernel which never assigns weight outside the support. Many others used Chen's idea and proposed other kernels, i.e. lognormal, Weibull, Inverse Gaussian, etc. All of the methods perform well with different bandwidths. However, no package directly estimates the densities according to these asymmetrical kernels and also calculates its mean squared error. Due to the reasons stated above, we have developed a package DELTD, in R language, [42]. In which, we have used some asymmetrical kernels for which parent distribution belongs to the family of lifetime distributions to estimate the density and to calculate their mean squared error. To the best of our knowledge, no package is available that plots the density using a variety of asymmetrical kernels and calculates their MSE. There are lots of packages that are frequently used for density estimation, but almost all of them use symmetrical kernels. The problem of boundary bias occurs using the symmetrical kernel as mentioned above. The package is available from the Comprehensive R Archive Network (CRAN), [43].

This paper aims to describe this package, and also, to summarize and conveniently present the functions. This may help interested readers to apply this kind of technique to real situations. The structure of this paper is as follows. Section 2 introduces the lifetime distributions and their relevant kernel. Section 3 introduces the utility of Mean Squared Error (MSE). Section 4 presents the implementation of functions in package DELTD, with examples and argument details and lastly, Section 5 is devoted to conclusions with some suggestions for future work.

2 Life Time Distributions

Distributions that tend to better represent life data are known as lifetime distributions, [44]. Like lognormal distribution is found in environmental studies, milk production of cows, amount of rainfall, the volume of gas in a petroleum reserve, etc., [45], [46], Birnbaum-Saunders distribution describes the fatigue life studies, [47], and Beta distribution is used for percentages, proportion, rates, and fractions, [48]. Similarly, applications of gamma distribution are found in neuroscience, in bacterial gene expression, [49], [50]. The gamma distribution is widely used as a conjugate prior in Bayesian statistics, etc. Erlang distribution is a specified case of Gamma distribution, [51], and is used in queuing theory, in the mathematical study of waiting in lines. It is also used in stochastic processes mathematical biology, etc. In this paper, we are interested in only those distributions that belong to the lifetime distribution family and their asymmetrical kernels are available in the literature, e.g. Beta, Birnbaum-Saunders, Erlang, Gamma, and Lognormal.

Let $X_1, \ldots, X_n$ be a random sample from a distribution with an unknown probability density function f which has bounded support on $[0, \infty)$, with $y > 0$ and h representing the bandwidth. In the following subsections kernels are presented for the above-stated distributions that we are going to use in the package and were developed to handle the problem of boundary bias.

2.1 Beta Kernel

The study, [41], proposed a Beta kernel for estimating curves with compact support by using the Beta distribution of the first kind. The beta kernel smoother is free of boundary bias, achieving the optimal convergence rate of $n^{-\frac{4}{5}}$ for mean integrated squared error and always allocate non-negative weights. Further, they compared the beta smoothers and the local linear smoothers. Beta Kernel is

$$K_{Beta\left(\frac{x}{h}+1,\frac{1-x}{h}+1\right)}(y) = \frac{y^{\frac{x}{h}}(1-y)^{\frac{1-x}{b}}}{B\left\{\frac{x}{h}+1,\frac{(1-x)}{h}+1\right\}} \tag{2}$$

2.2 Birnbaum-Saunders Kernel

The study, [52], extended the class of non-negative, asymmetric kernel density estimators and proposed

Birnbaum-Saunders (BS) kernel density function. The density function has bounded support on $[0, \infty)$. They applied a BS kernel density estimator to high-frequency intraday time duration data. The comparisons are made on several nonparametric kernel density estimators. BS kernel performs better near the boundary in terms of bias reduction.

$$K_{BS\left(h^{\frac{1}{2}},x\right)}(y) = \frac{1}{2\sqrt{2}\pi h}\left(\sqrt{\frac{1}{xy}} + \sqrt{\frac{x}{y^3}}\right) \exp\left(-\frac{1}{2h}\left(\frac{y}{x} - 2 + \frac{x}{y}\right)\right) \quad (3)$$

2.3 Erlang Kernel

Erlang kernel is proposed by, [53]. They suggested using it for non-parametrically estimation of the probability density function (pdf). Moreover, they investigated the asymptotic normality of the proposed estimator.

$$K_{E\left(x,\frac{1}{h}\right)}(y) = \frac{1}{\Gamma\left(1+\frac{1}{h}\right)}\left[\frac{1}{x}\left(1+\frac{1}{h}\right)\right]^{\frac{h+1}{h}} y^{\frac{1}{h}} \exp\left(-\frac{y}{x}\left(1+\frac{1}{h}\right)\right) \quad (4)$$

2.4 Gamma Kernel

The study, [54], developed the Gamma kernel. As it is stated above, the reason behind this development is to handle boundary bias, which arises in symmetrical kernels. He showed that kernels are nonnegative, has naturally varying shape, and achieve the optimal rate of convergence for the mean integrated squared error. The Gamma kernel which we considered in our package is as follows:

$$K_{\mathrm{Gam\,1}\left(\frac{x}{h}+1,h\right)}(y) = \frac{y^{\frac{x}{h}}\exp\left(-\frac{y}{h}\right)}{\Gamma\frac{x}{(h+1)}h^{\frac{x}{h+1}}} \quad (5)$$

2.5 Lognormal Kernel

This kernel is also proposed by, [52]. They showed that this kernel also performs equally well with the Birnbaum-Saunders (BS) kernel.

$$K_{LN(\ln(x),4\ln(1+h))} = \frac{1}{\sqrt{(8\pi\ln(1+h))}} \exp\left[-\frac{(\ln(y)-\ln(x))^2}{8\ln(1+h)}\right] \quad (6)$$

3 Mean Squared Error

The mean squared error described the squared difference between the actual and estimated values. It measures the average of the squares of the errors. Mathematically, we can express this as

$$MSE = \frac{1}{n-k}\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = E[(y_i - \hat{y}_i)^2] \quad (7)$$

n is the number of data points, y_i represents actual values and $\hat{y}_i$ represents estimated value. MSE is a risk function, corresponding to the expected value of the squared error loss. It is always non-negative, and values closer to zero are better, [55].

4 The Package Overview

The package DELTD contains functions for density estimation by using asymmetrical kernels named Beta, Birnbaum-Saunders, Erlang, Gamma, and Lognormal. For these kernels, densities are calculated and represented graphically. The mean squared error (MSE) for each kernel can also be calculated. The following section demonstrates the use of the DELTD package with simulated examples. The functions within DELTD are briefly described in Table 1.

For density estimation, five functions (Table 3) are presented that are used to estimate the density by using Beta (plot.Beta), Birnbaum-Saunders (plot.BS), Erlang (plot.Erlang), Gamma (plot.Gamma) and lognormal (plot. LogN) kernels. For all kernels, estimated values for density estimation can also be analyzed by using Beta, BS, Erlang, Gamma, and LogN, for details, see Table 2. In our examples for observing estimated values of density, we generated a sample by using different distributions with different sample sizes. Function(s) related to density estimation depends on the grid size and h. Practically, estimation may be quite slow with small grid points, but it is important to note that for large grid points, density is smoother. In nonparametric estimation, bandwidth (h) plays a very important role. So, h also affects the smoothness of density along with grid points.

Table 1. Summary of contents of the package

Functions	Description
Beta	Estimate Density Values by the Beta kernel
BS	Estimate Density Values by Birnbaum-Saunders kernel
Erlang	Estimate Density Values by Erlang kernel
Gamma	Estimate Density Values by Gamma kernel
LogN	Estimate Density Values by Lognormal Kernel
mse	Calculate Mean Squared Error (MSE) by using different Kernels
plot.Beta	Density Plot by the Beta kernel
plot.BS	Density Plot by Birnbaum-Saunders kernel
plot.Erlang	Density Plot by Erlang kernel
plot.Gamma	Density Plot by Gamma kernel
plot.LogN	Density Plot by Lognormal kernel
TUNA	Data on Tuna fish

Functions for observing estimated values provide grid points and estimated values of density. All such functions have some default arguments, if the user does not provide such parameters then the function proceeds by using those arguments. But the user must provide at least x or k. If x is missing in the function then the package generates k grid points between minimum and maximum values of vector (y). Only in case Beta is used, with missing x, then grid points will be generated by using a uniform distribution ($U(0,1)$) as restricted by the author. Similarly, if k is missing then the function proceeds by setting $k = n$, where n is the length of the vector (y). In case, if the user does not provide the h then the function uses

$$h = 0.79IQRn^{(-1/5)} \tag{8}$$

which is described by, [56], for non-normal data.

5 Estimated Values of Density: Illustrative Examples

Here we are using BS for illustration, with all missing situations. In the following example, all arguments of a function are user-defined. Here we are using a quite small k to present results. It's better to use the same length of grid points (k) for one kernel. Although, it proceeds unequal k halt the plot () or generate NA.

Table 2. Summary of arguments of Beta, BS, Erlang, Gamma and LogN

Arguments	Description
x	a scheme for generating grid points
y	a vector of positive values
k	number of grid points
h	the bandwidth

```
> alpha = 10
> theta = 15 / 60
> k <- 10
> y <- rgamma(n = 100, shape = alpha,
> scale = theta)
> xx <- seq(min(y) , max(y), length =k)
> h <- 1.1
> den <- BS(x = xx, y = y, k = k, h = h)
```

It provides;

```
> den
$x
[1] 0.8556524 1.3077240 1.7597956 2.2118671
[5] 2.6639387 3.1160103 3.5680819 4.0201535
[9] 4.4722250 4.9242966
$y
[1] 0.1167586 0.1461795 0.1572175 0.1595425
[5] 0.1573851 0.1528002 0.1468850 0.1402692
[9] 0.1333330 0.1263139
```

If the scheme for generating grid points is unknown; then the function proceeds with the above-mentioned scheme. But for Beta, it is restricted by the author that grid points and vector (y) (either real or simulated) lie between 0 and 1. Any other scheme will produce NaN for beta-estimated values.

```
> y <- rgamma(n = 1000, shape = alpha,
```

```
> scale = theta)
> h <- 3
> BS(y = y, k = 90, h = h)
```

Similarly, the following example describes the situation, if the user does not mention the number of grid points. Further, it is not necessary that h must be fixed; it can be calculated by any other source.

```
> y <- rgamma(n = 10, shape = alpha,
> scale = theta)
> xx <- seq(0.001, 1000, length = 10)
> #any bandwidth can be used
> require(KernSmooth)
# Direct Plug-In Bandwidth
> h <- dpik(y)
> BS(x = xx, y = y, h = h)
```

It results, where the length of k is adopted by default.

```
$x
[1] 0.001 111.112 222.223 333.334 444.445
[6] 555.556 666.667 777.778 888.889 1000
$y
[1] 0.000000e + 00 4.959898e −
249.302037e − 48
[4] 1.685906e − 71 3.000456e −
95 5.238665e − 119
[7] 8.996952e − 143 1.525185e −
1662.559622e − 190
[10] 4.261985e − 214
attr(, "class")
[1] "list" "BS"
```

If both the generating scheme and the number of grid points are missing then the function is halted and will not process.

```
> y <- rgamma(n = 1000,
> shape = alpha, scale = theta)
> band = 3
> BS(y = y, h = band )
```

If bandwidth is missing then density points can be calculated as;

```
> y <- rgamma(n = 1000,
> shape = alpha, scale = theta)
> xx <- seq(0.001, 100, length = 1000)
> BS(x = xx, y = y, k = 900)
```

Similarly, Beta, Erlang, Gamma, and LogN can be used for their respective kernels. For details and examples see, [43].

6 Density Plot: Illustrative Example

To plot density, any kernel plot() can be used, for details see Table 3 For continuity, the BS kernel is used in Figure 1.

Table 3. Summary of arguments of plot.Beta, plot.BS, plot.Erlang, plot.Gamma and plot.LogN

Arguments	**Description**
x	An object of class "Beta", "BS", "Erlang", "Gamma" or "LogN"
...	Not presently used in this implementation

```
## other details can also be added
> plot (den, type = "1", ylab = "Density
> Function", lty = 1, x lab = "Time")
## To add true density along with estimated > d1 <
-density(y, bw = h)
> lines(d1, type = "p", col = "red")
> legend("topright", c("Real Density", > "Density by
Birnbaum-Saunders Kernel"), col=c("red", "black"),
lty = c(1,2))
```

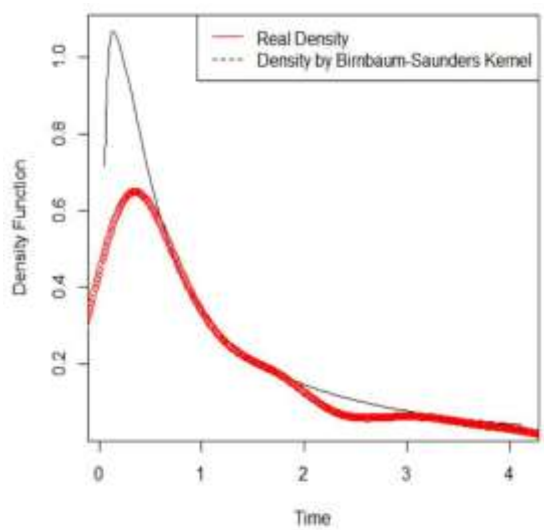

Fig. 1: Density Estimation by Using BS Kernel.

Further, the Tuna fish dataset, [57], is used to enhance the usefulness of these kernels. The data is about Tuna, which is saltwater fish. Its seasonal migration is between waters off the coast of Australia and the Indian Ocean. The data represents a line transect aerial survey of Southern Bluefin Tuna in the Great Australian Bight in summer when the tuna tends to stay on the surface. The abundance D is measured by $D = \frac{N}{A}$, where N is the total number of surface schools in the Bight and A is the survey area. To estimate D, an aircraft with two spotters on board is used to fly randomly allocated transect lines to detect tuna schools. Each school sighted from the transect is counted, and its perpendicular distance to the transect is measured.

7 Mean Squared Error (MSE): Illustrative Example

This function helps to examine the accuracy of different considered estimation methods, in terms of mean squared error (MSE). Table 4 presents argument details related to this function. These functions can be utilized only when data follows exponential, Gamma, or Weibull distribution. Similarly, Figure 2 presents the density estimation by using BS Kernel for Tuna data.

Table 4. Summary of argument of mse

Arguments	Description
kernel	type of kernel which is to be used
type	mention the distribution of vectors. If exponential distribution then use "Exp". If use gamma distribution then use "Gamma". If Weibull distribution then use "Weibull".

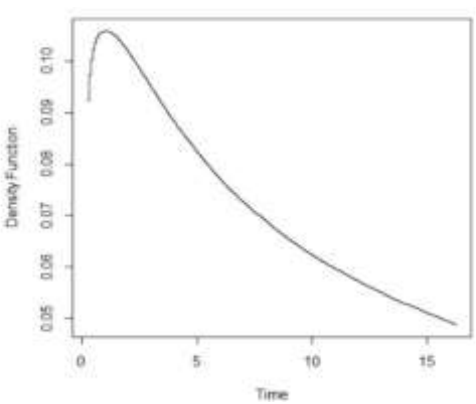

Fig. 2: Density Estimation by Using BS Kernel for Tuna data

```
> mse( kernel = den, type = "Exp")
[1] 0.002491046
```

If a distribution other than above mentioned type is used then NaN will be produced.

```
> mse (kernel = den, type ="Beta")
[1] NaN
```

8 Summary

In this paper, we have illustrated the functions of the R package DELTD. The package is about the density estimation through asymmetrical kernels when parent distribution belongs to lifetime distributions, e.g. Beta, BS, Exponential, Erlang, Gamma, Logistic, and Lognormal distribution. Additionally, their mean squared error (MSE) and plot are constructed through simulation data. Till that time, the package was the first publicly available software for the estimation of density by using asymmetrical kernel(s).

Density estimation is a powerful tool to collect information about its unknown distribution from given data. Due to this, kernel density estimation is very popular. But typically, symmetrical kernels are considered for estimation, which are sensitive to boundary bias. To overcome this problem, [41], proposed to use asymmetrical kernel which is non-negative and free of boundary bias. In this paper and the package DELTD, we combined major asymmetrical kernels that are based on lifetime distribution, i.e. Beta, BS, Erlang, Gamma, and Lognormal. The MSE criteria may be used to examine the accuracy of estimated kernels with real data.

Extensions towards the software package with more lifetime distribution kernels can be added in the package and other distributions can be introduced which can help to calculate MSE. This package can also be combined with Artificial Intelligence. In which, the function automatically identifies the suitable kernel; which has minimum MSE with estimated density. This package can be of interest to all those practitioners of different scientific fields who use any lifetime distribution(s) and those who estimate the densities for different purposes.

References:

[1] Silverman, B. & Young, G., The bootstrap: to smooth or not to smooth?, *Biometrika*, Vol.74, 1987, pp. 469-479.

[2] Desforges, M., Jacob, P. & Cooper, J., Applications of probability density estimation to the detection of abnormal conditions in engineering, *Proceedings of The Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, Vol.212, 1998, pp. 687-703.

[3] Girolami, M. & He, C., Probability density estimation from optimally condensed data samples, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, Vol.25, 2003, pp. 1253-1264.

[4] Hollands, K. & Suehrcke, H., A three-state model for the probability distribution of instantaneous solar radiation, with applications, *SolarEnergy*, Vol.96, 2013 pp. 103-112.

[5] Rajagopalan, B., Lall, U. & Tarboton, D., Evaluation of kernel density estimation methods for daily precipitation resampling, *Stochastic Hydrology and Hydraulics*, Vol.11, 1997, pp. 523-547.

[6] Kim, K. & Heo, J., Comparative study of flood quantiles estimation by nonparametric models, *Journal of Hydrology*, Vol.260, 2002, pp. 176-193.

[7] King, T., Thornton, L., Bentley, R. & Kavanagh, A., The use of kernel density estimation to examine associations between neighborhood destination intensity and walking and physical activity, *PLoS One*, Vol.10, No.e0137402, 2015.

[8] Shankar, P., The use of the compound probability density function in ultrasonic tissue characterization, *Physics in Medicine & Biology*, Vol.49, 2004, pp. 1007.

[9] Rosado-Mendez, I., Drehfal, L., Zagzebski, J. & Hall, T., Analysis of coherent and diffuse scattering using a reference phantom, *IEEE Transactions on Ultrasonics, Ferroelectrics, And Frequency Control*, Vol.63, 2016, pp. 1306-1320.

[10] Kang, E., Lee, E., Jang, M., Kim, S., Kim, Y., Chun, M., Tai, J., Han, W., Kim, S. & Kim, J., Reliability of computer-assisted breast density estimation: comparison of interactive thresholding, semiautomated, and fully automated methods, *American Journal of Roentgenology*,Vol.207, 2016, pp. 126-134.

[11] Yu, L. & Su, Z., Application of kernel density estimation in lamb wave-based damage detection, *Mathematical Problems in Engineering*, Vol.2012, 2012.

[12] Sakov, A., Golani, I., Lipkind, D., Benjamini, Y. & Others., High-throughput data analysis in behavior genetics, *The Annals of Applied Statistics*, Vol.4, 2010, pp. 743-763.

[13] Ewens, W. & Grant, G., *Statistical methods in bioinformatics: an introduction*, Springer Science & Business Media, 2006.

[14] Knapp, B., Frantal, S., Cibena, M., Schreiner, W. & Bauer, P., Is an intuitive convergence definition of molecular dynamics simulations solely based on the root mean square deviation possible?, *Journal of Computational Biology*, Vol.18, 2011, pp. 997-1005.

[15] Sawle, L. & Ghosh, K., Convergence of molecular dynamics simulation of protein native states: Feasibility vs self-consistency dilemma, *Journal of Chemical Theory and Computation*, Vol.12, 2016, pp. 861-869.

[16] Wu, X., Calculation of maximum entropy densities with application to income distribution, *Journal of Econometrics*, Vol.115, 2003, pp.347-354.

[17] Tortosa-Ausina, E., Cost efficiency and product mix clusters across the Spanish banking industry, *Review of Industrial Organization*, Vol.20,2002, pp. 163-181.

[18] Alemany, R., Bolancé, C. & Guillén, M., A nonparametric approach to calculating value-at-risk, *Insurance: Mathematics and Economics*, Vol.52, 2013, pp. 255-262.

[19] Baxter, M., Beardah, C. & Westwood, S., Sample size and related issues in the analysis of lead isotope data, *Journal of Archaeological Science*, Vol.27, 2000, pp. 973-980.

[20] Hannachi, A., Quantifying changes and their uncertainties in probability distribution of climatevariables using robust statistics, *Climate Dynamics*, Vol.27, 2006, pp. 301-317.

[21] Paulsen, O. & Heggelund, P., The quantal size at retinogeniculate synapses determined from spontaneous and evoked EPSCs in guineapig thalamic slices, *The Journal of Physiology*, Vol.480, 1994, pp. 505-511.

[22] Rau, M., Seitz, S., Brimioulle, F., Frank, E., Friedrich, O., Gruen, D. & Hoyle, B., Accurate

photometric redshift probability density estimation–method comparison and application, *Monthly Notices of The Royal Astronomical Society*, Vol.452, 2015, pp. 3710-3725.

[23] Cavuoti, S., Amaro, V., Brescia, M., Vellucci, C., Tortora, C. & Longo, G., METAPHOR: a machine-learning-based method for the probability density estimation of photometric redshifts, *Monthly Notices of The Royal Astronomical Society*, Vol.465, 2016, pp. 1959-1973.

[24] Li, X. & Gong, F., A method for fitting probability distributions to engineering properties of rock masses using Legendre orthogonal polynomials, *Structural Safety*, Vol.31, 2009, pp.335- 343.

[25] Woodbury, A., A FORTRAN program to produce minimum relative entropy distributions, *Computers & Geosciences*, Vol.30, 2004, pp. 131-138.

[26] Lu, N., Wang, L., Jiang, B., Lu, J. & Chen, X., Fault prognosis for process industry based on information synchronization, *IFAC Proceedings Volumes*, Vol.44, 2011, pp. 4296-301.

[27] Hajihosseini, P., Salahshoor, K. & Moshiri, B., Process fault isolation based on transfer entropy algorithm, *ISA Transactions*, Vol.53, 2014, pp. 230-240.

[28] Xu, S., Baldea, M., Edgar, T., Wojsznis, W., Blevins, T. & Nixon, M., Root cause diagnosis of plant-wide oscillations based on information transfer in the frequency domain, *Industrial & Engineering Chemistry Research*, Vol.55, 2016, pp. 1623-1629.

[29] Nadaraya, E., On estimating regression, *Theory of Probability & Its Applications*, Vol.9, 1964, pp. 141-142.

[30] Watson, G., Smooth regression analysis, *Sankhyā: The Indian Journal of Statistics, Series A*, 1964, pp. 359-372.

[31] Jou, P., Akhoond-Ali, A., Behnia, A. & Chini-pardaz, R., Parametric and nonparametric frequency analysis of monthly precipitation in Iran, *Journal of Applied Sciences*, Vol.8, 2008, pp. 3242-3248.

[32] Schuster, E., Incorporating support constraints into nonparametric estimators of densities, *Communications in Statistics-Theory and Methods*, Vol.14, 1985, pp. 1123-1136.

[33] Cline, D. & Hart, J., Kernel estimation of densities with discontinuities or discontinuous derivatives, *Statistics: A Journal of Theoretical and Applied Statistics*, Vol.22, 1991, pp. 69-84.

[34] Marron, J. & Ruppert, D., Transformations to reduce boundary bias in kernel density estimation, *Journal of The Royal Statistical Society: Series B (Methodological)*, Vol.56, 1994, pp. 653-671.

[35] Cowling, A. & Hall, P., On pseudo data methods for removing boundary effects in kernel density estimation, *Journal of The Royal Statistical Society: Series B (Methodological)*, Vol.58, 1996,pp. 551-563.

[36] Cheng, M., Boundary aware estimators of inte grated density derivative products, *Journal of The Royal Statistical Society: Series B (Statistical Methodology)*, Vol.59, 1997, pp. 191-203.

[37] Zhang, S. & Karunamuni, R., On kernel density estimation near endpoints, *Journal of Statistical Planning and Inference*, Vol.70, 1999, pp. 301-316.

[38] Cheng, M. & Others, Choice of the bandwidth ratio in Rice's boundary modification, *Journal of The Chinese Statistical Association*, Vol.44, 2006, pp. 235-251.

[39] John, R., Boundary modification for kernel regression, *Communications in Statistics-Theory and Methods*, Vol.13, 1984, pp. 893-900.

[40] Schucany, W. & Sommers, J., Improvement of kernel type density estimators, *Journal of The American Statistical Association*, Vol.72, 1977, pp. 420-423.

[41] Chen, S., Beta kernel smoothers for regression curves, *Statistica Sinica*, 2000, pp. 73-91

[42] R Core Team R: A Language and Environment for Statistical Computing, (R Foundation for Statistical Computing), 2023, https://www.R-project.org/, ISBN 3-900051-07-0

[43] DELTD: Kernel Density Estimation using Lifetime Distributions Manual (Online), https://CRAN.R-project.org/package=DELTD (Accessed Date: October 13, 2023)

[44] Bekker, P., *A lifetime distribution model of depreciable and reproducible capital assets*, Amsterdam: Vrije Universiteit, 1991.

[45] Limpert, E., Stahel, W. & Abbt, M., Log-normal distributions across the sciences: keys and clues: on the charms of statistics, and how mechanical

models resembling gambling machines offer a link to a handy way to characterize log-normal distributions, which can provide deeper insight into variability and probability—normal or log-normal: that is the question, *BioScience*, Vol.51, 2001, pp. 341-352.

[46] Foster, J., Bevis, M. & Raymond, W., Precipitable water and the lognormal distribution, *Journal of Geophysical Research: Atmospheres*, Vol.111, 2006.

[47] Birnbaum, Z. & Saunders, S., A new family of life distributions, *Journal of Applied Probability*, Vol.6, 1969, pp. 319-327.

[48] Ferrari, S. & Cribari-Neto, F., Beta regression for modelling rates and proportions, *Journal of Applied Statistics*, Vol.31, 2004, pp. 799-815.

[49] Robson, J. & Troy, J., Nature of the maintained discharge of Q, X, and Y retinal ganglion cells of the cat, *JOSA A*, Vol.4, 1987, pp. 2301-2307.

[50] Wright, M., Winter, I., Forster, J. & Bleeck, S., Response to best-frequency tone bursts in the ventral cochlear nucleus is governed by ordered inter-spike interval statistics, *Hearing Research*, Vol.317, 2014, pp. 23-32.

[51] Jambunathan, M., Some properties of beta and gamma distributions, *The Annals of Mathematical Statistics*, 1954, pp. 401-405.

[52] Jin, X., Kawczak, J. & Others., Birnbaum-Saunders and lognormal kernel estimators for modelling durations in high frequency financial data, *Annals of Economics and Finance*, Vol.4, 2003, pp. 103-124.

[53] Salha, R., El Shekh Ahmed, H. & Alhoubi, I., Hazard rate function estimation using Erlang kernel, *Hazard Rate Function Estimation Using Erlang Kernel*, Vol.3, 2014.

[54] Chen, S., Probability density function estimation using gamma kernels, *Annals of The Institute of Statistical Mathematics*, Vol.52, 2000,pp. 471-480.

[55] Mood, A., Graybill, F. & Boes, D., *Introduction to the theory of statistics*, McGraw-Hill Inc, New York, 1974.

[56] Silverman, B., *Density Estimation*, Chapman & Hall/CRC, 1986.

[57] Buckland, S., Burnham, K., Anderson, D. & Laake, J., *Density estimation using distance sampling*, Chapman Hall, London, 1993.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Javaria Ahmad Khan: conceived the idea, developed and maintained the package, programming, graph making, writing, performed the computations, and final approval of the version to be published.
- Atif Akbar: editing, proofreading, and final approval for publication.
- B M Golam Kibria: proofreading, formatting, and critical review.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflict of interest to declare.

Creative Commons Attribution License 4.0 (Attribution 4.0 International, CC BY 4.0)

This article is published under the terms of the Creative Commons Attribution License 4.0
https://creativecommons.org/licenses/by/4.0/deed.en_US