A Study of Generalized Fuzzy Dishkant Implications

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Abstract: - In this paper, we revisit the generalized Dishkant implications and provide analytical proof that they are a new fuzzy implications' class that contains the known class of Dishkant implications. Both classes are not always fuzzy implications. For this reason we use the term operations instead of implications in general. Nonetheless, it will be demonstrated that a necessary but not sufficient condition for a generalized Dishkant operation to be a fuzzy implication exists. Furthermore, the intersection of the sets of generalized Dishkant operations and Dishkant operations (respectively, implications) is provided. At the end, we prove a theorem for F- conjugation in GD-operations.

Key-Words: - fuzzy negation, t- norm, t- conorm, D- implication

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1 Introduction

As we said in [1], many fuzzy logic concepts are derived from generalizations of classical tautologies. Many classes of fuzzy implications and many of their features are also such generalizations, as shown in [2], [3], [4], [5], [6], [7], [8], [9], [10].

Fuzzy implications play an important part in many applications, [11], [12], and are used in a wide range of scientific areas. Fuzzy mathematical morphology, approximate reasoning, image processing, control theories, expert systems, and others are examples.

In this research, we review and investigate GD-implications, [1], a generalization from an existing class of fuzzy implications known as Dishkant implications (abbreviated D- implications). The following questions drove the inspiration for this study:

1. *What happens if we are not restricted to use only one fuzzy negation in a formula of a fuzzy implication that contains a fuzzy negation, more times than one time?*

2. *What are the results if we use different fuzzy negations?*

Indeed, using different fuzzy negations in such formulas is not forbidden, [1], [6], [8], [9], [10], [13]. As a result, [1], introduces a new class of fuzzy implications known as generalized Dishkant implications (shortly GD- implications). In this paper, we will prove that this is a new class of fuzzy implications as well as a hyper class of the known as D- implications' class. Furthermore, this hyper class broadens the required range of fuzzy implications.

The following is how the paper is structured: Section 2 introduces the key principles for comprehending the article. Section 3 contains the analytical proofs for the results we have presented in [1], some examples that establish these results and the intersection of the sets of D- and GD- operations (respectively, implications). We shall observe that GD- operations are not necessarily fuzzy implications, and we will provide a necessary but not sufficient condition for a GD- operation to be a fuzzy implication. Furthermore, we will exclude some quadruples $(\perp, \top, \neg_1, \neg_2)$ that do not produce GD- implications. Finally, a theorem for F- conjugation in GD- operations will be demonstrated. Section 4 contains the conclusions.

2 Preliminaries

Definition 1. *[2], [14], [15], [16]. A decreasing function $\neg : [0,1] \to [0,1]$ is called fuzzy negation, if $\neg(0) = 1$ and $\neg(1) = 0$. Moreover, a fuzzy negation $\neg$ is called strong, if it is an involution, i.e.,*

$$\neg(\neg(\varepsilon)) = \varepsilon, \textit{ for all } \varepsilon \in [0,1].$$

Remark 1. *(i) The so called, least and greatest fuzzy negations(see Example 1.4.4 in [2]) are respectively*

$$\neg_0(\varepsilon) = \begin{cases} 0, & \textit{if } \varepsilon > 0 \\ 1, & \textit{if } \varepsilon = 0 \end{cases} \tag{1}$$

and

$$\neg^1(\varepsilon) = \begin{cases} 0, & \textit{if } \varepsilon = 1 \\ 1, & \textit{if } \varepsilon < 1 \end{cases}. \tag{2}$$

(ii) We call $\neg_C(\varepsilon) = 1-\varepsilon$ the classical fuzzy negation, which is a strong negation. Moreover, in this paper we will use another type of fuzzy negations, which is $\neg_K(\varepsilon) = 1 - \varepsilon^2$ (see Example 1.4.4 and Table 1.6 in [2]).

Definition 2. *[2], [15], [16]. A function*

$$\top : [0,1] \times [0,1] \to [0,1]$$

is called a t- norm, if it satisfies, $\forall \varepsilon, \zeta, \delta \in [0,1]$*:*

$$\top(\varepsilon,\zeta) = \top(\zeta,\varepsilon), \quad (3)$$

$$\top(\varepsilon,\top(\zeta,\delta)) = \top(\top(\varepsilon,\zeta),\delta), \quad (4)$$

$$\zeta \leq \delta \Rightarrow \top(\varepsilon,\zeta) \leq \top(\varepsilon,\delta), \quad (5)$$

$$\top(\varepsilon,1) = \varepsilon. \quad (6)$$

Dually, a t- conorm is a function

$$\bot : [0,1] \times [0,1] \to [0,1]$$

if it satisfies, for all $\forall \varepsilon, \zeta, \delta \in [0,1]$*, the above conditions (3), (4), (5) and additionally*

$$\bot(\varepsilon,0) = \varepsilon. \quad (7)$$

Remark 2. *A t- norm we will use in this paper is* $\top_P(\varepsilon,\zeta) = \varepsilon\cdot\zeta$ *(see Table 2.1 in [2]) and a t- conorm is* $\bot_M(\varepsilon,\zeta) = max\{\varepsilon,\zeta\}$ *(see Table 2.2 in [2]).*

Definition 3. *(See Definition 2.2.2 in [2]). We call a t- conorm* $\bot$
(i) idempotent, if

$$\bot(\varepsilon,\varepsilon) = \varepsilon, \forall \varepsilon \in [0,1], \quad (8)$$

(ii) positive, if

$$\bot(\varepsilon,\zeta) = 1 \Rightarrow \varepsilon = 1 \text{ or } \zeta = 1. \quad (9)$$

Definition 4. *[2], [16]. A t- conorm* $\bot$ *is strictly monotone, if* $\bot(\varepsilon,\zeta) < \bot(\varepsilon,\delta)$*, whenever* $\varepsilon < 1$ *and* $\zeta < \delta$*.*

Proposition 1. *(See Proposition 9 in [4]).* $\forall \varepsilon, \zeta \in [0,1]$*:*

$$\top(\varepsilon,\zeta) \leq \varepsilon \leq \bot(\varepsilon,\zeta) \text{ and } \top(\varepsilon,\zeta) \leq \zeta \leq \bot(\varepsilon,\zeta). \quad (10)$$

Remark 3. *By Proposition 1, it follows that*

$$\bot(1,\varepsilon) = \bot(\varepsilon,1) = 1, \varepsilon \in [0,1] \quad (11)$$

and

$$\top(0,\varepsilon) = \top(\varepsilon,0) = 0, \varepsilon \in [0,1]. \quad (12)$$

Definition 5. *(See Definition 2.3.8 in [2]). Let* $\neg$ *be a fuzzy negation and* $\bot$ *a t- conorm. We say that the pair* $(\bot,\neg)$ *satisfies the law of excluded middle if*

$$\bot(\neg(\varepsilon),\varepsilon) = 1, \varepsilon \in [0,1]. \quad (13)$$

Definition 6. *[2], [17]. By* F *we denote the family of all increasing bijections from* $[0,1]$ *to* $[0,1]$*. We say that functions* $\lambda, \nu : [0,1]^n \to [0,1]$ *are* F*- conjugate, if there exists a* $f \in F$ *such that* $\nu = \lambda_f$*, where for any* $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n \in [0,1]$*:*

$$\lambda_f(\varepsilon_1,\varepsilon_2,...,\varepsilon_n) = f^{-1}(\lambda(f(\varepsilon_1),f(\varepsilon_2),...,f(\varepsilon_n))). \quad (14)$$

Remark 4. *(See Proposition 1.4.8, Remarks 2.1.4(vii) and 2.2.5(vii) in [2]). It is easy to prove that if* $f \in F$ *and* $\top$ *is a t- norm,* $\bot$ *is a t- conorm and* $\neg$ *is a fuzzy negation (respectively strong), then* $\top_f$ *is a t- norm,* $\bot_f$ *is a t- conorm and* $\neg_f$ *is a fuzzy negation (respectively strong).*

Definition 7. *[2], [14]. A function*

$$\Sigma : [0,1] \times [0,1] \to [0,1]$$

is called a fuzzy implication if

$$\Sigma(\cdot,\zeta) \text{ is decreasing}, \quad (15)$$

$$\Sigma(\varepsilon,\cdot) \text{ is increasing}, \quad (16)$$

$$\Sigma(0,0) = 1, \quad (17)$$

$$\Sigma(1,1) = 1, \quad (18)$$

$$\Sigma(1,0) = 0. \quad (19)$$

Remark 5. *By axioms (16) and (17) we deduce the normality condition*

$$\Sigma(0,1) = 1. \quad (20)$$

Moreover, by Definition 7 it is easy to prove the left and right boundary conditions, [2]

$$\Sigma(0,\zeta) = 1, \zeta \in [0,1], \quad (21)$$

$$\Sigma(\varepsilon,1) = 1, \varepsilon \in [0,1]. \quad (22)$$

Definition 8. *(See Definition 1.3.1 in [2]). A fuzzy implication* Σ *is said to satisfy the left neutrality property, if*

$$\Sigma(1,\zeta) = \zeta, \zeta \in [0,1], \quad (23)$$

Remark 6. *(i) Property (23) is not limited to fuzzy implications, but in any function*

$$\Sigma : [0,1] \times [0,1] \to [0,1],$$

(ii) It is proved that, if $f \in F$ *and*

$$\Sigma : [0,1] \times [0,1] \to [0,1],$$

satisfies (15) (respectively (16), (17), (18), (19)), then Σ_f is also satisfies (15) (respectively (16), (17), (18), (19)). Moreover, if Σ is a fuzzy implication, then Σ_f is also a fuzzy implication (see Proposition 1.1.8 in [2]).

Lemma 1. *(See Lemma 1.4.14 in [2]). If a function*

$$\Sigma : [0,1] \times [0,1] \to [0,1],$$

satisfies (15), (17) and (19), then the function $\neg_\Sigma : [0,1] \to [0,1]$ is a fuzzy negation, where

$$\neg_\Sigma(\varepsilon) = \Sigma(\varepsilon, 0), \varepsilon \in [0,1]. \quad (24)$$

Definition 9. *(See Definition 1.4.15 in [2]). Let*

$$\Sigma : [0,1] \times [0,1] \to [0,1],$$

be a fuzzy implication. The function $\neg_\Sigma$ defined by Lemma 1 is called the natural negation of Σ.

Definition 10. *[2], [18], [19]. A function*

$$\Sigma : [0,1] \times [0,1] \to [0,1],$$

is called a D- operation if there exist a t-norm $\top$, a t-conorm $\bot$ and a fuzzy negation $\neg$ such that

$$\Sigma(\varepsilon, \zeta) = \bot(\top(\neg(\varepsilon), \neg(\zeta)), \zeta), \varepsilon, \zeta \in [0,1]. \quad (25)$$

If Σ is a D-operation generated from the triple $(\top, \bot, \neg)$, then we will often denote it by $\Sigma^{\top,\bot,\neg}$.

Remark 7. *[2], [18], [19]. D- operations are not fuzzy implications in general since (16) could not hold. Only if the D- operation is a fuzzy implication, we will use the term D- implication.*

3 Generalized Dishkant Implications

In this Section all the statements of [1], will be proved in detail and supplemented with some more results and a figure.

Definition 11. *[1]. A function*

$$\Sigma : [0,1] \times [0,1] \to [0,1],$$

is called a GD- operation, if there exist a t- conorm $\bot$, a t- norm $\top$ and two fuzzy negations $\neg_1$, $\neg_2$, such that

$$\Sigma(\varepsilon, \zeta) = \bot(\top(\neg_1(\varepsilon), \neg_2(\zeta)), \zeta), \varepsilon, \zeta \in [0,1]. \quad (26)$$

If Σ is a GD- operation generated by the quadruple $(\bot, \top, \neg_1, \neg_2)$, then we denote it by $\Sigma_{\bot,\top,\neg_1,\neg_2}$.

Theorem 1. *[1]. $\Sigma_{\bot,\top,\neg_1,\neg_2}$ satisfies (15), (17), (18), (19), (20) and (22). Furthermore $\neg_{\Sigma_{\bot,\top,\neg_1,\neg_2}} = \neg_1$, where $\neg_{\Sigma_{\bot,\top,\neg_1,\neg_2}}(\varepsilon) = \Sigma_{\bot,\top,\neg_1,\neg_2}(\varepsilon, 0), \varepsilon \in [0,1]$.*

Proof. Let $\Sigma_{\bot,\top,\neg_1,\neg_2}$ be a GD- operation, then for $\varepsilon, \zeta, \delta \in [0,1]$, if

$$\begin{aligned}
&\varepsilon \le \zeta \Rightarrow \neg_1(\varepsilon) \ge \neg_1(\zeta) \\
&\overset{(5)}{\Rightarrow} \top(\neg_2(\delta), \neg_1(\varepsilon)) \ge \top(\neg_2(\delta), \neg_1(\zeta)) \\
&\overset{(3)}{\Rightarrow} \top(\neg_1(\varepsilon), \neg_2(\delta)) \ge \top(\neg_1(\zeta), \neg_2(\delta)) \\
&\overset{(5)}{\Rightarrow} \bot(\delta, \top(\neg_1(\varepsilon), \neg_2(\delta))) \ge \bot(\delta, \top(\neg_1(\zeta), \neg_2(\delta))) \\
&\overset{(3)}{\Rightarrow} \bot(\top(\neg_1(\varepsilon), \neg_2(\delta)), \delta) \ge \bot(\top(\neg_1(\zeta), \neg_2(\delta)), \delta) \\
&\Rightarrow \Sigma_{\bot,\top,\neg_1,\neg_2}(\varepsilon, \delta) \ge \Sigma_{\bot,\top,\neg_1,\neg_2}(\zeta, \delta),
\end{aligned}$$

which means that $\Sigma_{\bot,\top,\neg_1,\neg_2}$ satisfies (15).

$\Sigma_{\bot,\top,\neg_1,\neg_2}$ satisfies (17), since

$$\begin{aligned}
\Sigma_{\bot,\top,\neg_1,\neg_2}(0,0) &= \bot(\top(\neg_1(0), \neg_2(0)), 0) \\
&\overset{(7)}{=} \top(\neg_1(0), \neg_2(0)) \\
&= \top(1,1) \\
&\overset{(6)}{=} 1.
\end{aligned}$$

$\Sigma_{\bot,\top,\neg_1,\neg_2}$ satisfies (18), since

$$\begin{aligned}
\Sigma_{\bot,\top,\neg_1,\neg_2}(1,1) &= \bot(\top(\neg_1(1), \neg_2(1)), 1) \\
&= \bot(\top(0,0), 1) \\
&\overset{(12)}{=} \bot(0,1) \\
&\overset{(11)}{=} 1.
\end{aligned}$$

$\Sigma_{\bot,\top,\neg_1,\neg_2}$ satisfies (19), since

$$\begin{aligned}
\Sigma_{\bot,\top,\neg_1,\neg_2}(1,0) &= \bot(\top(\neg_1(1), \neg_2(0)), 0) \\
&\overset{(7)}{=} \top(\neg_1(1), \neg_2(0)) \\
&= \top(0,1) \\
&\overset{(6)}{=} 0.
\end{aligned}$$

$\Sigma_{\bot,\top,\neg_1,\neg_2}$ satisfies (20), since

$$\begin{aligned}
\Sigma_{\bot,\top,\neg_1,\neg_2}(0,1) &= \bot(\top(\neg_1(0), \neg_2(1)), 1) \\
&\overset{(11)}{=} 1.
\end{aligned}$$

$\Sigma_{\bot,\top,\neg_1,\neg_2}$ satisfies (22), since $\forall \varepsilon \in [0,1]$:

$$\begin{aligned}
\Sigma_{\bot,\top,\neg_1,\neg_2}(\varepsilon,1) &= \bot(\top(\neg_1(\varepsilon), \neg_2(1)), 1) \\
&\overset{(11)}{=} 1.
\end{aligned}$$

Lastly, $\forall \varepsilon \in [0,1]$ we have

$$\begin{aligned}
\neg_{\Sigma_{\bot,\top,\neg_1,\neg_2}}(\varepsilon) &= \Sigma_{\bot,\top,\neg_1,\neg_2}(\varepsilon, 0) \\
&= \bot(\top(\neg_1(\varepsilon), \neg_2(0)), 0) \\
&\overset{(7)}{=} \top(\neg_1(\varepsilon), \neg_2(0)) \\
&= \top(\neg_1(\varepsilon), 1) \\
&\overset{(6)}{=} \neg_1(\varepsilon).
\end{aligned}$$

□

Proposition 2. *[1].* $\Sigma_{\perp,\top,\neg_1,\neg_2}$ *satisfies (23).*

Proof. $\forall\zeta \in [0,1]$ it is

$$\begin{aligned}\Sigma_{\perp,\top,\neg_1,\neg_2}(1,\zeta) &= \perp(\top(\neg_1(1),\neg_2(\zeta)),\zeta)) \\ &= \perp(\top(0,\neg_2(\zeta)),\zeta)) \\ &\overset{(12)}{=} \perp(0,\zeta) \\ &\overset{(3)}{=} \perp(\zeta,0) \\ &\overset{(7)}{=} \zeta.\end{aligned}$$

Thus, $\Sigma_{\perp,\top,\neg_1,\neg_2}$ satisfies (23). ☐

Remark 8. *[1]. If* $\neg_1 = \neg_2$ *the corresponding GD- operation is a D- operation. Thus, GD- operations sometimes do not satisfy (16). The same happens even if we use different negations according to the following Example 1. For these reasons, we use the term GD- operations, instead of GD- implications.*

Example 1. *Consider the quadruple* $(\perp_M, \top_P, \neg_C, \neg_K)$. *The corresponding GD- operation is*

$$\begin{aligned}\Sigma_{\perp_M,\top_P,\neg_C,\neg_K}(\varepsilon,\zeta) &= \perp_M(\top_P(\neg_C(\varepsilon),\neg_K(\zeta)),\zeta) \\ &= \perp_M(\neg_C(\varepsilon)\cdot\neg_K(\zeta),\zeta) \\ &= \perp_M((1-\varepsilon)\cdot(1-\zeta^2),\zeta) \\ &= max\{(1-\varepsilon)\cdot(1-\zeta^2),\zeta\}\end{aligned}$$

which is not a fuzzy implication, since

$$0.1 \leq 0.2 \Rightarrow \Sigma_{\perp_M,\top_P,\neg_C,\neg_K}(0.1,0.1) = 0.891 > 0.864 = \Sigma_{\perp_M,\top_P,\neg_C,\neg_K}(0.1,0.2).$$

Thus, $\Sigma_{\perp_M,\top_P,\neg_C,\neg_K}$ *does not satisfy (16).*

Proposition 3. *[1]. If* $\Sigma_{\perp,\top,\neg_1,\neg_2}$ *satisfies (16), then we call it GD- implication.*

Proof. The proof is obvious. ☐

Proposition 4. *[1].* $\Sigma_{\perp,\top,\neg_1,\neg_2}$ *satisfies (21) if and only if the pair* $(\perp,\neg_2)$ *satisfies (13).*

Proof. If $\Sigma_{\perp,\top,\neg_1,\neg_2}$ satisfies (21), then $\forall\zeta \in [0,1]$:

$$\begin{aligned}\Sigma_{\perp,\top,\neg_1,\neg_2}(0,\zeta) = 1 &\Rightarrow \perp(\top(\neg_1(0),\neg_2(\zeta)),\zeta) = 1 \\ &\Rightarrow \perp(\top(1,\neg_2(\zeta)),\zeta) = 1 \\ &\overset{(3)}{\Rightarrow} \perp(\top(\neg_2(\zeta),1),\zeta) = 1 \\ &\overset{(6)}{\Rightarrow} \perp(\neg_2(\zeta),\zeta) = 1.\end{aligned}$$

Thus, the pair $(\perp,\neg_2)$ satisfies (13).
Conversely, if the pair $(\perp,\neg_2)$ satisfies (13), then $\forall\zeta \in [0,1]$ it is

$$\begin{aligned}\Sigma_{\perp,\top,\neg_1,\neg_2}(0,\zeta) &= \perp(\top(\neg_1(0),\neg_2(\zeta)),\zeta) \\ &= \perp(\top(1,\neg_2(\zeta)),\zeta) \\ &\overset{(3)}{=} \perp(\top(\neg_2(\zeta),1),\zeta) \\ &\overset{(6)}{=} \perp(\neg_2(\zeta),\zeta) \\ &\overset{(13)}{=} 1.\end{aligned}$$

Therefore, $\Sigma_{\perp,\top,\neg_1,\neg_2}$ satisfies (21). ☐

Corollary 1. *[1]. If* $\Sigma_{\perp,\top,\neg_1,\neg_2}$ *is a GD- implication, then the pair* $(\perp,\neg_2)$ *satisfies (13).*

Proof. If $\Sigma_{\perp,\top,\neg_1,\neg_2}$ is a GD- implication, then it satisfies (21). So, by Proposition 4 we deduce that the pair $(\perp,\neg_2)$ satisfies (13). ☐

Remark 9. *[1]. (i) Corollary 1 gives a necessary, but not sufficient condition for the generation of a GD- operation* $\Sigma_{\perp,\top,\neg_1,\neg_2}$. *Note that every D- operation (respectively implication)* $\Sigma^{\top,\perp,\neg}$ *is also a GD- operation (respectively implication)* $\Sigma_{\perp,\top,\neg,\neg}$. *Mas et al. in Proposition 3 in [18] mention that if* $\Sigma^{\top,\perp,\neg}$ *is a D- implication, then the pair* $(\perp,\neg)$ *satisfies (13), where* $\neg$ *is a strong fuzzy negation. They also mention after Proposition 3 that this condition (they mean (13)) given in the previous proposition (i.e. Proposition 3 in [18]) is necessary but not sufficient. Moreover, we must note that this proposition is proved for strong fuzzy negations only, but the proof is similar and holds for any fuzzy negation* $\neg$.
(ii) By Corollary 1 it is obvious that, if the pair $(\perp,\neg_2)$ *does not satisfy (13), i.e.* $\perp(\neg(\varepsilon),\varepsilon) \neq 1$, *for some* $\varepsilon \in (0,1)$, *then the obtained* $\Sigma_{\perp,\top,\neg_1,\neg_2}$ *GD- operation is not a fuzzy implication.*

Example 2. *Consider the quadruple /..l; where* $\perp$ *and* $\top$ *are any t- conorm and t-norm, respectively. The corresponding GD- operation, which is a GD- implication (the proof is simple) is*

$$\begin{aligned}I_{\perp,\top,\neg_0,\neg^1}(\varepsilon,\zeta) &= \perp(\top(\neg_0(\varepsilon),\neg^1(\zeta)),\zeta) \\ &= \begin{cases}\perp(\top(\neg_0(\epsilon),0),1), & \textit{if } \zeta = 1 \\ \perp(\top(\neg_0(\epsilon),1),\zeta), & \textit{if } \zeta < 1\end{cases} \\ &\underset{(6)}{\overset{(12)}{=}} \begin{cases}\perp(0,1), & \textit{if } \zeta = 1 \\ \perp(\neg_0(\varepsilon),\zeta), & \textit{if } \zeta < 1\end{cases} \\ &\overset{(11)}{=} \begin{cases}1, & \textit{if } \zeta = 1 \\ \perp(0,\zeta), & \textit{if } \varepsilon > 0 \textit{ and } \zeta < 1 \\ \perp(1,\zeta), & \textit{if } \varepsilon = 0 \textit{ and } \zeta < 1\end{cases} \\ &\underset{(11)}{\overset{(3)}{=}} \begin{cases}1, & \textit{if } \zeta = 1 \\ \perp(\zeta,0), & \textit{if } \varepsilon > 0 \textit{ and } \zeta < 1 \\ 1, & \textit{if } \varepsilon = 0 \textit{ and } \zeta < 1\end{cases}\end{aligned}$$

$$\overset{(7)}{=} \begin{cases} 1, & \textit{if}\ \zeta = 1 \\ \zeta, & \textit{if}\ \varepsilon > 0\ \textit{and}\ \zeta < 1 \\ 1, & \textit{if}\ \varepsilon = 0\ \textit{and}\ \zeta < 1 \end{cases}$$
$$= \begin{cases} \zeta, & \textit{if}\ \varepsilon > 0 \\ 1, & \textit{otherwise} \end{cases}$$
$$= \begin{cases} 1, & \textit{if}\ \varepsilon = 0 \\ \zeta, & \textit{otherwise} \end{cases}$$
$$= I_{12}(\varepsilon, \zeta).$$

See Figure 5 in page 509 in [3], for the formula of I_{12}.

Remark 10. *By Remark 9 and Example 2 we deduce that there are GD- implications, that are not D- implications. Firstly,* $\neg_{\Sigma^{\top,\bot,\neg}} = \neg_{\Sigma_{\bot,\top,\neg,\neg}} = \neg$. *Moreover, there does not exist any t-conorm* $\bot$ *such that the pair* $(\bot, \neg_0)$ *satisfies (13), since*

$$\bot(\neg_0(0.3), 0.3) = \bot(0, 0.3) \overset{(3)}{=} \bot(0.3, 0) \overset{(7)}{=} 0.3 \neq 1.$$

Thus, there does not exist any D- implication that has $\neg_0$ *as its natural negation. On the other hand* $\Sigma_{\bot,\top,\neg_0,\neg^1} = I_{12}$ *is a GD- implication with* $\neg_0$ *as its natural negation, that means it is not a D- implication. Therefore, the class of GD- implications is a new hyper class of that of D- implications, which contains them.*

These results lead us to the following Figure 1.

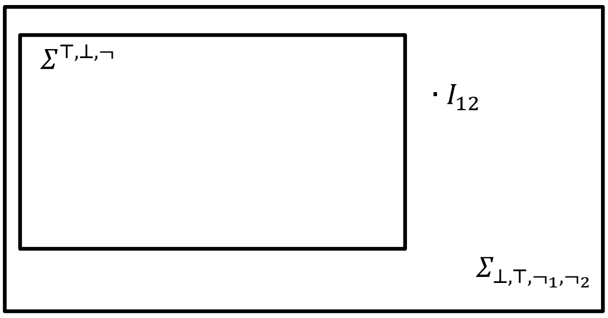

Figure 1: The intersection among the sets of D- operations (respectively, implications) and GD- operations (respectively, implications).

Theorem 2. *[1]. If* $\bot$ *is any idempotent, strict or positive t- conorm,* $\top$ *is any t- norm,* $\neg_1$ *is any fuzzy negation and* $\neg_2$ *is any continuous fuzzy negation, then* $\Sigma_{\bot,\top,\neg_1,\neg_2}$ *is not a fuzzy implication.*

Proof. Firstly, it has been proved (see Theorem 1.4.7 in [2]), there exists exactly one $\xi \in (0, 1)$, such that $\neg(\xi) = \xi$, where $\neg$ is any continuous fuzzy negation. If $\bot$ is strict t- conorm, then

$$\bot(\neg(\xi), \xi) = \bot(\xi, \xi) \neq 1,$$

because if

$$\bot(\xi, \xi) = 1 \Leftrightarrow \bot(\xi, \xi) = \bot(e, 1),$$

a contradiction, since

$$\xi < 1 \Rightarrow \bot(\xi, \xi) < \bot(\xi, 1).$$

So, $\Sigma_{\bot,\top,\neg_1,\neg_2}$ is not a fuzzy implication according to Remark 9(ii). Furthermore, since the only idempotent, which is also a positive t- conorm is $\bot_M$ (see Remark 2.2.5(ii) and Table 2.2 in [2]), we will continue the proof only for positive t-conorms $\bot$. If we assume that $\bot$ is any positive t- conorm, then

$$\bot(\neg(\xi), \xi) = S(\xi, \xi) \neq 1,$$

since $\xi < 1$. So, $\Sigma_{\bot,\top,\neg_1,\neg_2}$ is not a fuzzy implication according to Remark 9(ii). □

Theorem 3. *[1]. If* $f \in F$ *and* $\Sigma_{\bot,\top,\neg_1,\neg_2}$ *is a GD- operation (respectively implication), then* $(\Sigma_{\bot,\top,\neg_1,\neg_2})_f$ *is a GD- operation (respectively implication) and moreover*

$$(\Sigma_{\bot,\top,\neg_1,\neg_2})_f = \Sigma_{\bot_f,\top_f,(\neg_1)_f,(\neg_2)_f}.$$

Proof. According to the Remark 6(ii) if $\Sigma_{\bot,\top,\neg_1,\neg_2}$ is a GD- operation (respectively implication), then $(\Sigma_{\bot,\top,\neg_1,\neg_2})_f$ is a GD- operation (respectively implication). Moreover, $\forall \varepsilon, \zeta \in [0, 1]$:

$$\begin{aligned}
&(\Sigma_{\bot,\top,\neg_1,\neg_2})_f(\varepsilon, \zeta) = f^{-1}(\Sigma_{\bot,\top,\neg_1,\neg_2}(f(\varepsilon), f(\zeta))) \\
&= f^{-1}(\bot(\top(\neg_1(f(\varepsilon)), \neg_2(f(\zeta))), f(\zeta))) \\
&= f^{-1}(\bot(\top(f(f^{-1}(\neg_1(f(\varepsilon)))), \\
&\qquad f(f^{-1}(\neg_2(f(\zeta))))), f(\zeta))) \\
&= f^{-1}(\bot(\top(f((\neg_1)_f(\varepsilon)), f((\neg_2)_f(\zeta))), f(\zeta))) \\
&= f^{-1}(\bot(f(f^{-1}(\top(f((\neg_1)_f(\varepsilon)), f((\neg_2)_f(\zeta))))), \\
&\qquad f(\zeta))) \\
&= \bot_f(\top_f((\neg_1)_f(\varepsilon), (\neg_2)_f(\zeta)), \zeta) \\
&= \Sigma_{\bot_f,\top_f,(\neg_1)_f,(\neg_2)_f}(\varepsilon, \zeta).
\end{aligned}$$

□

4 Conclusion

There exist fuzzy implications, which have a fuzzy negation function more than once in their formula. Is the use of only one fuzzy negation in these formulations binding? It is self-evident that the response is negative. In this study, we revisit a hyper class of the well-known class of fuzzy implications known as Dishkant implications. This hyperclass is known as generalized Dishkant implications (or GD- implications for short). The downside of GD- implications is that they do not always satisfy (16). As a result, we refer to GD- operations rather than implications in general. The characterization of quadruples $(\perp, \top, \neg_1, \neg_2)$, such that $\Sigma_{\perp,\top,\neg_1,\neg_2}$ satisfies (16) remains unsolved. The same problem holds for the characterization of triples $(\top, \perp, \neg)$, such that $\Sigma^{\top,\perp,\neg}$ satisfies (16) (see page 108 in [2]).

On the other hand, it has been demonstrated that the set of D- operations is a subset of the set of GD-operations, and the findings are depicted in Figure 1. It has been demonstrated that a necessary but not sufficient condition for a GD- operation to be a fuzzy implication exists (see Corollary 1 and Remark 9). Theorem 2 excludes quadruples $(\perp, \top, \neg_1, \neg_2)$ that do not generate GD- implications, and Theorem 3 investigates the relationship of $F-$ conjugation in GD-operations.

References:

[1] D.S. Grammatikopoulos, B. Papadopoulos, Generalized Fuzzy Dishkant Implications, *ICCMSE 2022 In: Prof. T. E. Simos (Chairman), 18th International Conference of Computational Methods in Sciences and Engineering (ICCMSE 2022)*, 2022, Waiting for AIP Conference Proceedings publication.

[2] M. Baczyński, B. Jayaram, *Fuzzy Implications*, Springer Berlin, Heidelberg, 2008.

[3] J. Drewniak, Invariant fuzzy implications, *Soft Computing*, Vol.10, 2006, pp. 506-513.

[4] D.S. Grammatikopoulos, B.K. Papadopoulos, A Method of Generating Fuzzy Implications with Specific Properties, *Symmetry*, Vol.12, No.1, 2020, pp. 155-170.

[5] D.S. Grammatikopoulos, B.K. Papadopoulos, An Application of Classical Logic's Laws in Formulas of Fuzzy Implications, *Journal of Mathematics*, Vol.2020, 2020, Article ID 8282304, 18 pages.

[6] D.S. Grammatikopoulos, B.K. Papadopoulos, A study of $(T, N)-$ and $(N', T, N)-$ Implications, *Fuzzy Information and Engineering*, Vol.13, No.3, 2021, pp. 277-295.

[7] G.P. Dimuro, B. Bedregal, H. Bustince, A. Jurio, M. Baczyński, K. Mis, QL-operations and QL-implication functions constructed from triples (O,G,N) and the generation of fuzzy subsethood and entropy measures, *International Journal of Approximate Reasoning*, Vol.82, 2017, pp. 170-192.

[8] J. Pinheiro, B. Bedregal, R.H.N. Santiago, H. Santos, (N′, T, N)-Implications, *Fuzzy Systems (FUZZ-IEEE) in: 2018 IEEE International Conference*, 2018, pp. 1-6.

[9] D.S. Grammatikopoulos, B.K. Papadopoulos, A Study of GD′- Implications, a New Hyper Class of Fuzzy Implications, *Mathematics*, Vol.9, No.16, 2021, 1925, 16 pages.

[10] D.S. Grammatikopoulos, B. Papadopoulos, A Study of Generalized QL′- Implications. *Mathematics*, Vol.10, No.20, 2022, 3742, 17 pages.

[11] M. Baczyński, On the applications of fuzzy implication functions, *In: Balas, V.E., Fodor, J., Várkonyiczy, A.R., Dombi, J., Jain, L.C. (eds.) Soft Computing Applications. AISC,* **2013**, Vol.195, 2013, pp. 9—10.

[12] M. Baczyński, G. Beliakov, H. Bustince, A. Pradera, *Advances in Fuzzy Implication Functions*, Springer Berlin, Heidelberg, 2013.

[13] D.S. Grammatikopoulos, B. Papadopoulos, Generalized R′-Implications: A Hyper Class of R- and R′-Implications, *Journal of Mathematics*, Vol.2023, 2023, Article ID 7111888, 13 pages.

[14] J.C. Fodor; M. Roubens, *Fuzzy preference modelling and multicriteria decision support*, Kluwer Academic Publishers, 1994.

[15] S. Gottwald, *A treatise on many-valued logics*, Research Studies Press, 2001.

[16] E.P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publishers, 2000.

[17] M. Kuczma, *Functional equations in a single variable*, PWN–Polish Scientific Publishers, 1968.

[18] M. Mas, M. Monserrat, J. Torrens, QL-implications versus D-implications, *Kybernetika*, Vol.42, 2006, pp. 351-366.

[19] S. Massanet, J. Torrens, Intersection of Yager's implications with QL and D-implications, *International Journal of Approximate Reasoning*, Vol.53, 2012, pp. 467-479.

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