Investigation of Affine Factorable Surfaces in Pseudo-Galilean Space

MOHAMED SAAD[1*], HOSSAM ABDEL-AZIZ[2], HAYTHAM ALI[2]

[1]Mathematics Department, Faculty of Science, Islamic University of Madinah, Al-Madinah, SAUDI ARABIA

[2]Mathematics Department, Faculty of Science, Sohag University, EGYPT

Abstract: In this paper, we investigate affine factorable surfaces of the second kind in the three-dimensional pseudo-Galilean space G1 3. We use the invariant theory and theory of diffeerential equations to study the geometric properties of these surfaces, namely, the first and second fundamental forms, Gaussian and mean curvatures. Also, we present some special cases by changing the partial diffeerential equation into the ordinary diffeerential equation to simplify our special cases. Furthermore, we give some theorems according to zero and non-zero Gaussian and mean curvatures of the meant surfaces. Finally, we give some examples to confifirm and demonstrate our results.

KeyWords: Affine factorable surfaces, minimal surfaces, Gaussian and mean curvatures, pseudo-Galilean space.

Received: April 29, 2023. Revised: August 2, 2023. Accepted: August 23, 2023. Published: September 26, 2023.

1 Introduction

In classical differential geometry, the problem of obtaining Gaussian and mean curvatures of a surface in Euclidean space and other spaces is one of the most important problems, so we are interested here to study such a problem for a surface known as affine factorable surface in the three-dimensional pseudo-Galilean space G_3^1.

The geometry of Galilean Relativity acts like a “bridge” from Euclidean geometry to special Relativity. The Galilean space which can be defined in three-dimensional projective space $P_3(R)$ is the space of Galilean Relativity, [1]. The geometries of Galilean and pseudo-Galilean spaces have similarities, but, of course, are different. In the Galilean and pseudo Galilean spaces, some special surfaces such as surfaces of revolution, ruled surfaces, translation surfaces and tubular surfaces have been studied in [2], [3], [4], [5], [6], [7], [8], [9], [10]. For further study of surfaces in the pseudo-Galilean space, we refer the reader to [9]. Recall that the graph surfaces are also known as Monge surfaces, [11]. In this work, we are interested here in studying a special type of Monge surface, namely the factorable surface of the second kind that is a graph of the function $y(x,z) = f(x)g(z)$. Such surfaces with non-zero constant Gaussian and mean curvatures in various ambient spaces have been classified (see, [12], [13], [14], [15], [16]). Our purpose is to analyze the factorable surfaces in the pseudo-Galilean space G_3^1 that is one of real Cayley-Klein spaces (for more details see, [17], [18], [19]). There exist three different kinds of factorable surfaces, explicitly, a Monge surface in G_3^1 is said to be factorable (so-called a homothetic) if it is given in one of the following forms: $\Phi_1 : z(x,y) = f(x)g(y)$ is the first kind, $\Phi_2 : y(x,z) = f(x)g(z)$ the second kind, and $\Phi_3 : x(y,z) = f(y)g(z)$ the third kind where f, g are smooth functions, [14]. These surfaces have different geometric structures in different spaces such as metric, curvatures, etc. We hope that this work will be useful for the specialists in this field.

2 Basic concepts

The pseudo-Galilean space G_3^1 is one of the Cayley-Klein spaces with absolute figure that consists of the ordered triple $\{\omega, f, I\}$, where ω is the absolute plane given by $x_o = 0$, in the three-dimensional real projective space $P_3(R)$, f the absolute line in ω given by $x_o = x_1 = 0$ and I the fixed hyperbolic involution of points of f and represented by $(0 : 0 : x_2 : x_3) \to (0 : 0 : x_3 : x_2)$, which is equivalent to the requirement that the conic $x_2^2 - x_3^2 = 0$ is the absolute conic. The metric connections in G_3^1 are introduced with respect to the absolute figure. In terms of the affine coordinates given by $(x_o : x_1 : x_2 : x_3) = (1 : x : y : z)$, the distance between the points $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ is defined by (see for instance, [9], [18])

$$d(p,q) = \begin{cases} |q_1 - p_1|, & if\ p_1 \neq q_1, \\ \sqrt{|(q_2-p_2)^2 - (q_3-p_3)^2|}, & if\ p_1 = q_1. \end{cases}$$

The pseudo-Galilean scalar product of the vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ is given by

$$\langle X, Y \rangle_{G_3^1} = \begin{cases} x_1 y_1, & if\ x_1 \neq 0\ or\ y_1 \neq 0, \\ x_2 y_2 - x_3 y_3, & if\ x_1 = 0\ and\ y_1 = 0. \end{cases}$$

In this sense, the pseudo-Galilean norm of a vector X is $\|X\| = \sqrt{|X.X|}$. A vector $X = (x_1, x_2, x_3)$ is called isotropic (non-isotropic) if $x_1 = 0$ ($x_1 \neq 0$). All unit non-isotropic vectors are of the form $(1, x_2, x_3)$. The isotropic vector $X = (0, x_2, x_3)$ is called spacelike, timelike and lightlike if $x_2^2 - x_3^2 > 0$, $x_2^2 - x_3^2 < 0$ and $x_2 = \pm x_3$, respectively. The pseudo-Galilean cross product of X and Y on G_3^1 is given as follows

$$X \wedge_{G_3^1} Y = \begin{vmatrix} 0 & -e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where e_2 and e_3 are canonical basis.

Let M be a connected, oriented 2-dimensional manifold and $\phi : M \to G_3^1$ be a surface in G_3^1 with parameters (u, v). The surface parametrization ϕ is expressed as

$$\phi(u,v) = (x(u,v), y(u,v), z(u,v)).$$

On the other hand, we denote by E, F, G and L, M, N the coefficients of the first and second fundamental forms of ϕ, respectively. The Gaussian curvature K and mean curvature H are expressed as

$$K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{EN + GL - 2FM}{2|EG - F^2|}, \tag{1}$$

where

$$\begin{aligned} E &= \phi'_u.\phi'_u, \quad F = \phi'_u.\phi'_v, \quad G = \phi'_v.\phi'_v, \\ L &= \phi''_{uu}.n, \quad M = \phi''_{uv}.n, \quad N = \phi''_{vv}.n, \end{aligned}$$

where the normal surface is given by

$$n = \frac{\phi'_u \wedge \phi'_v}{|\phi'_u \wedge \phi'_v|}.$$

3 Factorable surfaces in pseudo-Galilean space G_3^1

In what follows, we consider the factorable surface of second kind in G_3^1 which can be locally written as

$$\phi(x,z) = (x, f(x)g(z), z). \tag{2}$$

Definition 1 *An affine factorable surface in pseudo-Galilean space G_3^1 is defined as a parameter surface $\phi(u,v)$ and can be written as*

$$\begin{aligned} \phi(u,v) &= (x(u,v), y(u,v), z(u,v)) \\ &= (u, f(u)g(v+au), v) \\ &= (x, f(x)g(z+ax), z), \end{aligned} \tag{3}$$

for non zero constant a, and functions $f(x)$ and $g(z+ax)$, [19].

Now, from Eq. (3) by a straightforward calculation, the first fundamental form with its coefficients of ϕ is given by

$$\begin{aligned} I &= Edx^2 + 2Fdxdy + Gdy^2, \\ E &= 1, \quad F = 0, \quad G = (fg')^2 - 1, \end{aligned}$$

$$g' = \frac{dg(z+ax)}{d(z+ax)}.$$

Also, the second fundamental form of ϕ is

$$II = Ldx^2 + 2Mdxdy + Ndy^2,$$

$$\begin{aligned} L &= \frac{\left(f''g + 2af'g' + a^2 fg''\right)}{D}, \\ M &= \frac{(f'g' + afg'')}{D}, \qquad N = \frac{fg''}{D}, \end{aligned}$$

where

$$D(x,z) = \sqrt{1 - (fg')^2}.$$

In addition, the Gaussian and mean curvature of ϕ can be obtained

$$K = \frac{f'^2 g'^2 - f'' f g'' g}{\left(1 - (fg')^2\right)^2}, \tag{4}$$

$$H = \frac{\Omega(x,z)}{2\left(1 - (fg')^2\right)^{\frac{3}{2}}}, \tag{5}$$

such that

$$\begin{aligned} \Omega(x,z) = &(1-a^2)fg'' - f''g - 2af'g' \\ &+ f^2 f'' g'^2 g + 2af' f^2 g'^3 + a^2 f^3 g'^2 g''. \end{aligned}$$

A surface in G_3^1 is said to be *an isotropic minimal* (resp. *flat*) if H (resp. K) vanishes identically. Further, it is said to have constant an isotropic mean (resp. Gaussian) curvature if H (resp. K) is a constant function on a whole surface.

4 Affine factorable surfaces with zero curvatures

In this section, if the Gaussian and mean curvatures of Eq. (3) are vanished, then we get the following result.

Theorem 2 *Let $\phi : I \subset R \to G_3^1$ be an affine factorable surface of second kind given in the form*

$$\phi(x, z) = (x, f(x)g(z + ax), z),$$

if its Gaussian curvature is zero, then the surface is one of the following forms:

(1) $y(x, z) = f_o g(z + ax)$,

(2) $y(x, z) = g_o f(x)$,

(3) $y(x, z) = ce^{c_5 x + c_4 z}$,

(4) $y(x, z) = [(1 - k)(c_6 x + c_7)]^{\frac{1}{1-k}}$

Theorem 3 $.\left[\left(\frac{k-1}{k}\right)(c_8(z + ax) + c_9)\right]^{\frac{k}{k-1}}$.

Proof. If the Gaussian curvature of ϕ is zero, then from Eq. (4), we have

$$f'^2 g'^2 - f'' f g'' g = 0. \tag{6}$$

To solve this equation we have the following cases:
Case 1. if $f' = 0$, then $f'' = 0$, $f = f_o = const.$, then $y(x, z) = f_o g(z + ax)$.
Case 2. if $g' = 0$, then $g'' = 0$, $g = g_o = const.$, then $y(x, z) = g_o f(x)$.
Case 3. if $f' \neq 0$ and $g' \neq 0$, and let

$$\begin{cases} u = x, \\ v = z + ax, \end{cases}$$

where $\partial(u, v)/\partial(x, z) \neq 0$. Then Eq. (7) can be written as

$$f_u^2 g_v^2 - f f_{uu} g g_{vv} = 0,$$

or

$$\left(\frac{df}{du}\right)^2 \left(\frac{dg}{dv}\right)^2 = f \frac{df_u}{df} \frac{df}{du} g \frac{dg_v}{dg} \frac{dg}{du}. \tag{7}$$

From Eq. (8), we find

$$\frac{df}{du}\frac{dg}{dv} = f \frac{df_u}{df} g \frac{dg_v}{dg}.$$

Since, $\frac{df}{du}\frac{dg}{dv} \neq 0$ and $g\frac{dg_v}{dg} \neq 0$, then

$$\left(\frac{f\frac{df_u}{df}}{f_u}\right) = \left(\frac{g_v}{g\frac{dg_v}{dg}}\right), \tag{8}$$

let's rewrite the last equation as follows:

$$\left(\frac{f\frac{df_u}{df}}{f_u}\right) = \left(\frac{g_v}{g\frac{dg_v}{dg}}\right) = k; \quad k = const. \tag{9}$$

(a) If $k = 1$, then from Eq. (10), we have

$$\frac{df_u}{f_u} = \frac{df}{f}, \quad \frac{dg_v}{g_v} = \frac{dg}{g}, \tag{10}$$

it leads to

$$f = c_1 e^{c_2 u}, \quad g = c_3 e^{c_4 v},$$

where c_1, c_2, c_3, c_4 are constants. And then

$$\begin{aligned} y(x, z) &= f(x)g(z + ax) = c_1 e^{c_2 x} c_3 e^{c_4(z+ax)} \\ &= c_5 e^{c_6 x + c_4 z}, \end{aligned}$$

where $c_5 = c_1 c_3$ and $c_6 = c_2 + ac_4$ are constants.
(b) When $k \neq 1$, then from Eq. (10), we get

$$f\frac{df_u}{df} = kf_u, \quad kg\frac{dg_v}{dg} = g_v,$$

which has the solution

$$\begin{aligned} f(x) &= [(1 - k)(c_7 x + c_8)]^{\frac{1}{1-k}}, \\ g(z + ax) &= \left[\left(\frac{k-1}{k}\right)(c_9(z + ax) + c_{10})\right]^{\frac{k}{k-1}}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} y(x, z) &= [(1 - k)(c_7 x + c_8)]^{\frac{1}{1-k}} \\ &\quad .\left[\left(\frac{k-1}{k}\right)(c_9(z + ax) + c_{10})\right]^{\frac{k}{k-1}}, \end{aligned}$$

where c_7, c_8, c_9 and c_{10} are constants. ■

Theorem 4 *For given affine factorable surface of second kind in a three-dimensional pseudo-Galilean space in the form*

$$\phi(x, z) = (x, f(x)g(z + ax), z).$$

Let its mean curvature be zero, then this surface will be one of the following forms:

(1) $y(x, z) = f_o(b_1(z + ax) + b_2)$, *or* $y(x, z) = f_o\left(\sqrt{\frac{a^2-1}{a^2 f_o^2}}(z + ax) + b_3\right)$,

(2) $y(x, z) = g_o(b_4 x + b_5)$,

(3) $y(x, z) = b_8(b_6 x + b_7)$, *or* $y(x, z) = (b_6 x + b_7)(b_9(z + ax) + b_{10})$,

(4) $y(x, z) = (b_{12} x + b_{13})(b_{11}(z + ax) + b_{12})$, *or* $y(x, z) = \frac{1}{b_{11}}(b_{11}(z + ax) + b_{12})$.

Proof. If $H = 0$, then from Eq. (6), we find

$$(1-a^2)fg'' - f''g - 2af'g' + f^2f''g'^2g + 2af'f^2g'^3 + a^2f^3g'^2g'' = 0. \quad (11)$$

This equation can be solved with the aid of the following:
(1) If $f' = f'' = 0$, then $f = f_o = const.$, and (4.6) becomes

$$(1-a^2)fg'' + a^2f^3g'^2g'' = 0,$$

it can be written in a simple form as

$$g'' = 0 \quad \text{or} \quad g' = \sqrt{\frac{a^2-1}{a^2f_o^2}},$$

which has the solution

$$g = b_1(z+ax)+b_2 \quad \text{or} \quad g = \sqrt{\frac{a^2-1}{a^2f_o^2}}(z+ax)+b_3,$$

it leads to

$$y(x,z) = f_o(b_1(z+ax)+b_2),$$

and then, we get

$$y(x,z) = f_o\left(\sqrt{\frac{a^2-1}{a^2f_o^2}}(z+ax)+b_3\right),$$

where b_1, b_2, and b_3 are constants.
(2) When $g' = g'' = 0$, then $g = g_o = const.$, and Eq. (12) becomes

$$f''g = 0,$$

it has the solution

$$f = b_4x + b_5.$$

Using what we got from solutions, we can write

$$y(x,z) = g_o(b_4x+b_5),$$

where b_4, b_5 are constants.
(3) When $f'' = 0$, this leads to $f' = b_6$ which gives $f = b_6x + b_7$. From Eq. (12), we have

$$(1-a^2)fg'' - 2af'g' + 2af'f^2g'^3 + a^2f^3g'^2g'' = 0,$$

which can be written as

$$(1-a^2)fg_{vv} - 2af_ug_v + 2af_uf^2g_v^3 + a^2f^3g_v^2g_{vv} = 0,$$

therefore, by differentiating this equation three times with respect to u, we obtain

$$g_v^2g_{vv} = 0,$$

which gives

$$g_v = 0 \ \rightarrow \ g = b_8,$$

and so

$$g_{vv} = 0 \ \rightarrow \ g = b_9(z+ax)+b_{10},$$

in light of this, we get

$$y(x,z) = b_8(b_6x+b_7),$$

and then, we have

$$y(x,z) = (b_6x+b_7)(b_9(z+ax)+b_{10}),$$

where b_6, b_7, b_8, b_9 and b_{10} are constants.
(4) If $g'' = 0$, it means that $g' = b_{11} \rightarrow g = b_{11}(z+ax)+b_{12}$ and then from Eq. (12), we obtain

$$f''g + 2af'g' - f^2f''g'^2g - 2af'f^2g'^3 = 0,$$

which can be written as

$$f_{uu}g + 2af_ug_v - f^2f_{uu}g_v^2g - 2af_uf^2g_v^3 = 0.$$

Differentiate this equation with respect to v, we find

$$b_{11}f_{uu} - b_{11}^3f^2f_{uu} = 0,$$

$$f_{uu} = 0 \ \rightarrow \ f = b_{12}x + b_{13},$$

it leads to

$$f = \frac{1}{b_{11}},$$

Therefore, we get

$$y(x,z) = (b_{12}x+b_{13})(b_{11}(z+ax)+b_{12}),$$

it follows that

$$y(x,z) = \frac{1}{b_{11}}(b_{11}(z+ax)+b_{12}).$$

Taking into consideration that b_{11}, b_{12} and b_{13} are constants. Thus, this completes the proof. ■

5 Affine factorable surfaces with non-zero curvatures

In this section, we describe the affine factorable surfaces of the second kind in G_3^1 with non-zero constant Gaussian and mean curvatures.

Theorem 5 *Let $\phi : I \subset R \to G_3^1$ be an affine factorable surface of the second kind in G_3^1, and it has a non-zero constant Gaussian curvature, then this surface takes the form:*

$$y(x,z) = (g_o(z+ax)+\lambda_2)$$
$$.\left(\pm\frac{1}{g_o}\tanh\left[\sqrt{K_o}x \mp g_o\lambda_1\right]\right), \quad \lambda_1,\lambda_2 \in R.$$

Proof. Let K_o be a non-zero constant Gaussian curvature. Hence, we get

$$K_o = \frac{f'^2g'^2 - f''fg''g}{\left(1-(fg')^2\right)^2}, \tag{12}$$

Since, K_o vanishes identically when f or g is a constant function. Then f and g must be non-constant functions. So, we can distinguish two cases for Eq. (13), as follows:
Case 1. $f' = f_o$, $f_o \in R - \{0\}$, then from Eq. (13), we get a polynomial equation in (g'):

$$K_o - (2K_of^2 + f_o^2)g'^2 + K_of^4g'^4 = 0,$$

which it yields a contradiction.
Case 2. If $g' = g_o$; $g_o \in R - \{0\}$. Then, Eq. (13) leads to

$$f' = \frac{\pm\sqrt{K_o - 2K_og_o^2f^2 + K_og_o^4f^4}}{g_o},$$

therefore, it has the solution:

$$f(x) = \pm\frac{1}{g_o}\tanh\left[g_o\sqrt{K_o}x \mp g_o\lambda_1\right], \quad \lambda_1 \in R.$$

Case 3. If $f'' \neq 0$; $g'' \neq 0$. Then, Eq. (13) leads to

$$K_o = \frac{f'^2g'^2 - f''fg''g}{\left(1-(fg')^2\right)^2},$$

So, using $u = x$, $v = z + ax$ and $\partial(u,v)/\partial(x,y) \neq 0$, we can obtain

$$K_o = \frac{f_u^2g_v^2 - f_{uu}fg_{vv}g}{\left(1-(fg_v)^2\right)^2}, \tag{13}$$

it leads to

$$\frac{f'}{f^2f''} + \frac{3f'f^2}{f''}g'^4 = 0, \tag{14}$$

which means that all coefficients must vanish, therefore the contradiction $f' = 0$ is obtained. Thus the proof is completed. ■

Theorem 6 *For given affine factorable surface of the second kind in G_3^1 which has a non-zero constant mean curvature H_o. Then*

$$\begin{aligned} y(x,z) &= f_o\left(\frac{\sqrt{9H_o^2 - a^4f_o^2\lambda_3^2}}{3f_oH_o}(z+ax)+\lambda_4\right), \\ &= \left(-\frac{2H_o}{g_o}x^2 + cx + c\right)g_o. \end{aligned}$$

Proof. From Eq. (6), we have

$$H_o = \frac{\begin{pmatrix} (1-a^2)fg'' - f''g - 2af'g' \\ +f^2f''g'^2g + 2af'f^2g'^3 + a^2f^3g'^2g'' \end{pmatrix}}{2\left(1-(fg')^2\right)^{3/2}},$$

Solving this equation leads to the following two cases:

Case 1. If $f = f_o$, $g'' = \lambda_3 = const.$, we obtain

$$2H_o\left(1-(fg')^2\right)^{3/2} = (1-a^2)fg'' + a^2f^3g'^2g'',$$

and using $u = x$, $v = z + ax$ and $\partial(u,v)/\partial(x,y) \neq 0$, we have

$$2H_o\left(1-(fg_v)^2\right)^{3/2} = (1-a^2)fg_{vv} + a^2f^3g_v^2g_{vv}, \tag{15}$$

it leads to

$$g_v = \frac{\sqrt{9H_o^2 - a^4f_o^2\lambda_3^2}}{3f_oH_o},$$

it has the solution:

$$g = \pm\frac{\sqrt{9H_o^2 - a^4f_o^2\lambda_3^2}}{3f_oH_o}(z+ax) + \lambda_4; \ \lambda_4 \in R,$$

and then we get

$$y(x,z) = f_o\left(\frac{\sqrt{9H_o^2 - a^4f_o^2\lambda_3^2}}{3f_oH_o}(z+ax)+\lambda_4\right).$$

Case 2. If $g = g_o$, we have

$$2H_o = -f''g,$$

it leads to

$$f = -\frac{H_o}{g_o}x^2 + \lambda_5x + \lambda_6,$$

where $\lambda_5, \lambda_6 \in R$. Hence, the result is clear. ■

Proposition 7 *Let $\phi : I \subset R \to G_3^1$ be an affine factorable surface in G_3^1. Then, the relation between its Gaussian and mean curvatures is given by*

$$H = A(x, z)K, \tag{16}$$

where $A(x, z) = \frac{D^3(a^2 f g'' + 2af'g' + f''g) - fg''D}{f''fg''g - f'^2 g'^2}$;

$D = \sqrt{1-(fg')^2}$. *Further, if $D = 0$, then ϕ is an isotropic minimal affine factorable surface of the second kind.*

6 Examples

In this section, we present some examples of the affine factorable surfaces of the second kind. So, let us consider the affine factorable surfaces of the second kind in G_3^1 given as follows:

(1) $\phi : y(x, z) = 8e^{6x+z}; (x, z) \in [-1, 1] \times [0, 2\pi]$ (an isotropic flat; $K = 0$, see Fig. 1),

(2) $\phi : y(x, z) = \sqrt{\frac{3}{4}}(2x+z)+9; (x, z) \in [0, 15] \times [-1, 30]$ (an isotropic minimal; $H = 0$, see Fig. 2),

(3) $\phi : y(x, z) = (10x+z)\tanh[x]; (x, z) \in [-1, 1]$ ($K = constant$, see Fig. 3),

(4) $\phi : y(x, z) = -x^2 + 2x + 1; (x, z) \in [-1, 1]$ ($H = constant$, see Fig. 4).

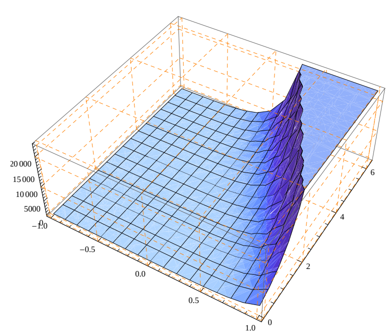

Figure 1: The isotropic flat surface of the second kind.

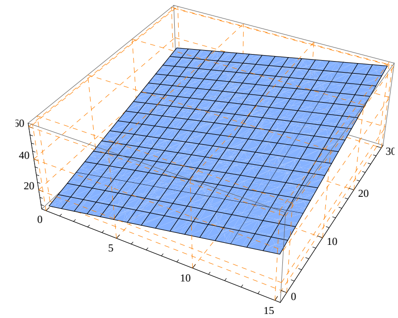

Figure 2: The isotropic minimal surface of the second kind.

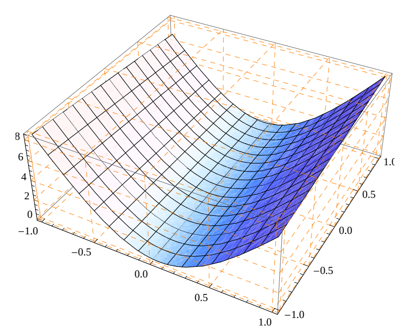

Figure 3: The affine factorable surface of the second kind with $K = constant$.

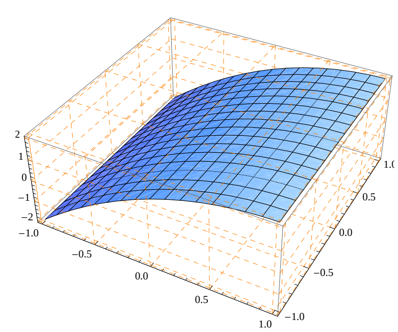

Figure 4: The affine factorable surface of the second kind with $H = constant$.

7 Concluding Remarks

In the surface theory, especially factorable surfaces, there are three kinds of these surfaces known as first, second and third kinds. In this paper, the factorable surface of the second kind which has an affine form in the three-dimensional pseudo-Galilean space G_3^1 has been studied. The classification of these surfaces with zero and non-zero Gaussian and mean curvatures has been investigated. Also, an essential relation between the curvatures of these surface has been obtained. Finally, some computational examples to support our findings are given and plotted. In future works, we plan to study the factorable surfaces in Lorentz-Minkowski space for different queries and further improve the results in this paper, combined with the techniques and results in [20], [21], [22].

Acknowledgments: We gratefully acknowledge the constructive comments from the editor and the anonymous referees. Also, the author (M. Khalifa Saad) would like to express his gratitude to the Islamic University of Madinah.

References

[1] M. P. Do Carmo, Differential Geometry of Curves and Surfaces, Prentice Hall, Englewood Cliffs, NJ, (1976).

[2] B. Divjak and Ž. M. Šipuš, Some special surfaces in the pseudo-Galilean space, Acta Math. Hungar. 118 (2008), 209–226.

[3] B. Divjak and Ž. M. Šipuš, Minding isometries of ruled surfaces in pseudo-Galilean space, J. Geom. 77 (2003), 35–47.

[4] B. Divjak and Ž. M. Šipuš, Special curves on ruled surfaces in Galilean and pseudo-Galilean spaces, Acta Math. Hungar. 98 (2003), 203–215.

[5] I. Kamenarović, Existence theorems for ruled surfaces in the Galilean space G_3, Rad Hrvatske Akad. Znan. Umjet. No. 456 (1991), 183–196.

[6] M. K. Karacan and Y. Tuncer, Tubular surfaces of Weingarten types in Galilean and pseudo-Galilean, Bull. Math. Anal. Appl. 5 (2013), 87–100.

[7] Ž. Milin Šipuš, Ruled Weingarten surfaces in the Galilean space, Period. Math. Hungar. 56 (2008), 213–225.

[8] Ž. Milin Šipuš and B. Divjak, Translation surface in the Galilean space, Glas. Mat. Ser. III 46(66) (2011), 455–469.

[9] Ž. Milin Šipuš and B. Divjak, Surfaces of constant curvature in the pseudo-Galilean space, Int. J. Math. Math. Sci. (2012), Art ID375264, 28pp.

[10] A. O. Öğrenmiş and M. Ergüt, On the Gauss map of ruled surfaces of type II in 3-dimensional pseudo-Galilean space, Bol. Soc. Parana. Mat. (3) 31 (2013), 145–152.

[11] A. Gray, Modern Differential Geometry of Curves and Surfaces with Mathematica, CRC Press LLC, (1998).

[12] M. Bekkar and B. Senoussi, Factorable surfaces in the three-dimensional Euclidean and Lorentzian spaces satisfying $\triangle r_i = \lambda_i r_i$, J. Geom. 103 (2012), 17–29.

[13] R. López and M. Moruz, Translation and homothetical surfaces in Euclidean space with constant curvature, J. Korean Math. Soc. 52(3) (2015), 523-535.

[14] H. Meng and H. Liu, Factorable surfaces in Minkowski space, Bull. Korean Math. Soc. 46(1) (2009), 155–169.

[15] Y. Yu and H. Liu, The factorable minimal surfaces, Proceedings of The Eleventh International Workshop on Diff. Geom. 11 (2007), 33-39.

[16] P. Zong, L. Xiao and H. Liu, Affine factorable surfaces in three-dimensional Euclidean space, Acta Math. Sinica Chinese Serie 58(2) (2015), 329-336.

[17] D. Klawitter, Clifford Algebras: Geometric Modelling and Chain Geometries with Application in Kinematics, Springer Spektrum, (2015).

[18] A. Onishchick and R. Sulanke, Projective and Cayley-Klein Geometries, Springer, (2006).

[19] U. Simon, A. Schwenk-Schellschmidt and H. Viesel, Introduction to the affine differential geometry of hypersurfaces, Lecture Notes of

the Science University of Tokyo, Sci. Univ. Tokyo, Tokyo, (1991).

[20] M. Khalifa Saad, R. A. Abdel-Baky, F. Alharbi and A. Aloufi, Characterizations of some special curves in Lorentz-Minkowski space, Mathematics and Statistics 8(3) (2020), 299-305.

[21] H. S. Abdel-Aziz, H. Serry, M. Khalifa Saad, Evolution equations of pseudo spherical images for timelike curves in Minkowski 3-space, Mathematics and Statistics 10(4) (2022), 884-893.

[22] Y. Li, A. Alkhaldi, A. Ali, R. A. Abdel-Baky and M. Khalifa Saad, Investigation of ruled surfaces and their singularities according to Blaschke frame in Euclidean 3-space, AIMS Math 8(6) (2023), 13875-13888.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors declare that there is no conflict of interests.