New results on contractive type in cone 2-metric space

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Abstract: - The common fixed point for self-contractive mappings in cone 2-metric spaces over Banach algebra is established in this study. The acquired results enhance and generalise the corresponding conclusions from the literature. A numerical example and a counterexample were then provided at the end.

Key-Words: - Metric spaces; contraction principle; fixed point; contractive mapping.

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1 Introduction

The principle of 2-metric space (2-MS) was established in [1] and [2], using generalizing the metric space(MS) and showed numerous fixed point theorems (FPTs) in such space. Many papers have investigated the necessary factors for the existence / uniqueness of FPT for contraction mappings in 2-MS, [3], [4], [5], [6], [7], [8]. On another hand, [9], introduced sundry FPTs in cone 2-MS. The authors in [9], [10], [11], [12], [13], established various FPTs in new MSs for an ordered Banach space (BS) in the codomain. Over Banach algebras, [14] and [15], worked on cone MS. [16] presented cone 2-MS generalizing both 2-MS and cone MS and proved some FPTs for self-mappings satisfying certain contractive conditions, [16], [17], [18]. The analysis of the existence / uniqueness of coincide /common points of diverse operators in the context of MS is also one of the most alluring research topics in FPTs, [19], [20], [21], [22]. Banach contraction principle to prove the exist a FP for a given space was introduced by Banach [23]. The method of FP development is either developing a type of used space or a type of contractive mapping. The development of space depends on decrease or changing the metric conditions. Consider that abusing or debilitating a portion of the metric conditions rise to the loss of some topological advantages, thus getting hard in proving some FPTs. Hardy-Rogers' theory (H-R) [24], is one of the most main findings that developed the Banach contraction principle by contractive type, many researchers have developed various FPTs on this important finding, [25], [26], [27], [28]. For this reason, we have seen generalize some FPTs in a cone 2-MS by using H-R' mappings, which opens the entrance to a similar study on cone n-MS.

2 Preliminaries

Definition 2.1. [29] Suppose $\mathfrak{G}$ be a Banach algebra (BG), then $\forall \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3 \in \mathfrak{G}, \alpha \in \mathcal{R}$:

(i) $(\mathfrak{u}_1\mathfrak{u}_2)\mathfrak{u}_3 = \mathfrak{u}_1(\mathfrak{u}_2\mathfrak{u}_3)$;

(ii) $\mathfrak{u}_1(\mathfrak{u}_2 + \mathfrak{u}_3) = \mathfrak{u}_1\mathfrak{u}_2 + \mathfrak{u}_1\mathfrak{u}_3$ and $(\mathfrak{u}_1 + \mathfrak{u}_2)\mathfrak{u}_3 = \mathfrak{u}_1\mathfrak{u}_3 + \mathfrak{u}_2\mathfrak{u}_3$;

(iii) $\alpha(\mathfrak{u}_1\mathfrak{u}_2) = (\alpha\mathfrak{u}_1)\mathfrak{u}_2 = \mathfrak{u}_1(\alpha\mathfrak{u}_2)$;

(iv) $\|\mathfrak{u}_1\mathfrak{u}_2\| \leq \|\mathfrak{u}_1\|\|\mathfrak{u}_2\|$.

In this work, a BG has a unit e: $e\mathfrak{u}_1 = \mathfrak{u}_1 e = \mathfrak{u}_1$ $\forall \mathfrak{u}_1 \in \mathfrak{G}$, where $\mathfrak{u}_1$ if there is an inverse element, then is said to be invertible. $\mathfrak{u}_2 \in \mathfrak{G}$, $\mathfrak{u}_1\mathfrak{u}_2 = \mathfrak{u}_2\mathfrak{u}_1 = e$. $\mathfrak{u}_1$'s inverse is represented by $\mathfrak{u}_1^{-1}$. see [31] for further information. The set $\{\mathfrak{u}_1, \mathfrak{u}_2, \cdots, \mathfrak{u}_n\} \subset \mathfrak{G}$ is commute if $\mathfrak{u}_i\mathfrak{u}_j = \mathfrak{u}_j\mathfrak{u}_i$ $\forall i, j \in \{1, 2, \cdots, n\}$.

Definition 2.2. [30] Suppose that $\mathcal{U}$ be a nonempty set and the mapping $\delta : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathfrak{G}$ satisfies

(i) $\delta(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3) \neq 0$ for every pair $\mathfrak{u}_1 \neq \mathfrak{u}_2 \in \mathcal{U}$, and $\mathfrak{u}_3 \in \mathcal{U}$,

(ii) $\delta(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3) \geq 0$ for all $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3 \in \mathcal{U}$ and $\delta(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3) = 0$ if and only if at least two of $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3$ are equal,

(iii) $\delta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3) = \delta(p(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3))$ for all $\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3 \in \mathcal{U}$ and for all permutations $p(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)$ of $(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3)$,

(iv) $\delta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3) \preccurlyeq \delta(\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_4) + \delta(\mathfrak{u}_1,\mathfrak{u}_4,\mathfrak{u}_3) + \delta(\mathfrak{u}_4,\mathfrak{u}_2,\mathfrak{u}_3)$, for all $\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3,\mathfrak{u}_4 \in \mathcal{U}$.

Then δ is called a cone 2-M on $\mathcal{U}$ and $(\mathcal{U},\delta)$ is called a cone 2-MS.

Definition 2.3. [30] Suppose $(\mathcal{U},\delta)$ be a cone 2-MS. Let $\mathfrak{u} \in \mathcal{U}$ and $\{\mathfrak{u}_n\}$ be a sequence in $\mathcal{U}$. Then

(i) $\{\mathfrak{u}_n\}$ is convergence sequence if $\mathfrak{u}_n \to \mathfrak{u}$ whenever for every $c \in \mathfrak{G}$ with $0 \ll c$, there is a natural number $\mathcal{N}$ such that $\delta(\mathfrak{u}_n,\mathfrak{u},\mathfrak{u}_3) \ll c$, for all $\mathfrak{u}_3 \in \mathcal{U}$ and $n \geq \mathcal{N}$.

(ii) $\{\mathfrak{u}_n\}$ is a Cauchy sequence if for every $c \in \mathfrak{G}$ with $0 \ll c$, there is a natural number $\mathcal{N}$ such that $\delta(\mathfrak{u}_n,\mathfrak{u}_k,\mathfrak{u}_3) \ll c$, for all $\mathfrak{u}_3 \in \mathcal{U}$ and $n,k \geq \mathcal{N}$.

(iii) $(\mathcal{U},\delta)$ is a complete cone 2-MS if every Cauchy sequence is convergent in $\mathcal{U}$.

Proposition 2.4. [31] Let $\mathfrak{G}$ be a BG with a unite e and $\mathfrak{u} \in \mathfrak{G}$. If the spectral radius $\mathfrak{r}(\mathfrak{u}) < 1$, which implies that

$$\mathfrak{r}(\mathfrak{u}) = \lim_{n\to\infty} \|\mathfrak{u}^n\|^{\frac{1}{n}} = \inf_{n\to\infty} \|\mathfrak{u}^n\|^{\frac{1}{n}} < 1.$$

Then $(e-\mathfrak{u})$ is invertible. Actually, $(e-\mathfrak{u})^{-1} = \sum_{i=0}^{+\infty} \mathfrak{u}^i$.

Remark 2.5.

(i) $\mathfrak{r}(\mathfrak{u}) \leq \|\mathfrak{u}\|$ for any $\mathfrak{u} \in \mathfrak{G}$, refer [31].

(ii) In Proposition 2.4, if $\mathfrak{r}(\mathfrak{u}) < 1$ is replaced by $\|\mathfrak{u}\| < 1$ then the conclusion remains true.

Lemma 2.6. [33] If $\mathfrak{G}$ is a real BS with a solid cone $\mathcal{P}$ and if $\|\mathfrak{u}_n\| \to 0$ *as* $n \to \infty$, then for any $0 \ll c$, there exists $n_1 \in \mathcal{N}$ such that for all $n > n_1$, we have $\mathfrak{u}_n \ll c$.

3 Main Results

In this section, we will prove the uniqueness of the common FP in con 2-MS using H-R contractive self mappings of BG.

Theorem 3.1. Let $(\mathcal{U},\delta)$ be a complete cone 2-MS on a BG $\mathfrak{G}$ and $\mathcal{P}$ the underlying cone. Assume that, T, F are self-mappings of $\mathcal{U}$ satisfying the condition

$$\begin{aligned}
&\delta(T_i^{ki}\mathfrak{u}_1, T_j^{kj}\mathfrak{u}_2,\mathfrak{u}_3)\\
&\preccurlyeq \alpha_1\delta(F_i^{ki}\mathfrak{u}_1, F_j^{kj}\mathfrak{u}_2,\mathfrak{u}_3) + \alpha_2\delta(F_i^{ki}\mathfrak{u}_1, T_i^{ki}\mathfrak{u}_1,\mathfrak{u}_3)\\
&\quad + \alpha_3\delta(F_j^{kj}\mathfrak{u}_2, T_j^{kj}\mathfrak{u}_2,\mathfrak{u}_3) + \alpha_4\delta(F_i^{ki}\mathfrak{u}_1, T_j^{kj}\mathfrak{u}_2,\mathfrak{u}_3)\\
&\quad + \alpha_5\delta(F_j^{kj}\mathfrak{u}_2, T_i^{ki}\mathfrak{u}_1,\mathfrak{u}_3), \qquad (1)
\end{aligned}$$

where $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5 \in \mathcal{P}$, for all $\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3 \in \mathcal{U}$. If $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ are commute and $\mathfrak{r}(\alpha_1)+\mathfrak{r}(\alpha_2)+\mathfrak{r}(\alpha_3)+\mathfrak{r}(\alpha_4)+\mathfrak{r}(\alpha_5) < 1$. Then $\{T_i^{ki}\}_{i=1}^{\infty}$ and $\{F_i^{ki}\}_{i=1}^{\infty}$ have a unique common FP.

Proof. Consider $T_i^{ki} = \mathfrak{h}_i$ and $F_i^{ki} = \mathfrak{f}_i$, for all $i \in \mathcal{N}$. Inequality (1) it will become

$$\begin{aligned}
&\delta(\mathfrak{h}_i\mathfrak{u}_1, \mathfrak{h}_j\mathfrak{u}_2,\mathfrak{u}_3)\\
&\preccurlyeq \alpha_1\delta(\mathfrak{f}_i\mathfrak{u}_1,\mathfrak{f}_j\mathfrak{u}_2,\mathfrak{u}_3) + \alpha_2\delta(\mathfrak{f}_i\mathfrak{u}_1,\mathfrak{h}_i\mathfrak{u}_1,\mathfrak{u}_3)\\
&\quad + \alpha_3\delta(\mathfrak{f}_j\mathfrak{u}_2,\mathfrak{h}_j\mathfrak{u}_2,\mathfrak{u}_3) + \alpha_4\delta(\mathfrak{f}_i\mathfrak{u}_1,\mathfrak{h}_j\mathfrak{u}_2,\mathfrak{u}_3)\\
&\quad + \alpha_5\delta(\mathfrak{f}_j\mathfrak{u}_2,\mathfrak{h}_i\mathfrak{u}_1,\mathfrak{u}_3). \qquad (2)
\end{aligned}$$

Let $u_0 \in \mathcal{U}$ be arbitrary and define the sequence $\mathfrak{u}_n$ as $\mathfrak{u}_n = \mathfrak{h}_n(\mathfrak{u}_{n-1}) = \mathfrak{f}_n\mathfrak{u}_n$, for all $n \in \mathcal{N}$. Now we prove that $\{\mathfrak{u}_n\}$ is a Cauchy sequence in $\mathcal{U}$. Take

$$\begin{aligned}
&\delta(\mathfrak{u}_{n+1},\mathfrak{u}_n,\mathfrak{u}_3)\\
&=\\
&\preccurlyeq\alpha_1\delta(\mathfrak{f}_n\mathfrak{u}_n,\mathfrak{f}_n\mathfrak{u}_{n-1},\mathfrak{u}_3) + \alpha_2\delta(\mathfrak{f}_n\mathfrak{u}_n,\mathfrak{h}_n\mathfrak{u}_n,\mathfrak{u}_3)\\
&\quad + \alpha_3\delta(\mathfrak{f}_n\mathfrak{u}_{n-1},\mathfrak{h}_n\mathfrak{u}_{n-1},\mathfrak{u}_3) + \alpha_4\delta(\mathfrak{f}_n\mathfrak{u}_n,\mathfrak{h}_n\mathfrak{u}_{n-1},\mathfrak{u}_3)\\
&\quad + \alpha_5\delta(\mathfrak{f}_n\mathfrak{u}_{n-1},\mathfrak{h}_n\mathfrak{u}_n,\mathfrak{u}_3)\\
&\preccurlyeq\alpha_1\delta(\mathfrak{u}_n,\mathfrak{u}_{n-1},\mathfrak{u}_3) + \alpha_2\delta(\mathfrak{u}_n,\mathfrak{u}_{n+1},\mathfrak{u}_3) + \alpha_3\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3)\\
&\quad + \alpha_4\delta(\mathfrak{u}_n,\mathfrak{u}_n,\mathfrak{u}_3) + \alpha_5\delta(\mathfrak{u}_{n-1},\mathfrak{u}_{n+1},\mathfrak{u}_3)\\
&\preccurlyeq\alpha_1\delta(\mathfrak{u}_n,\mathfrak{u}_{n-1},\mathfrak{u}_3) + \alpha_2\delta(\mathfrak{u}_n,\mathfrak{u}_{n+1},\mathfrak{u}_3) + \alpha_3\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3)\\
&\quad + \alpha_5[\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3) + \delta(\mathfrak{u}_n,\mathfrak{u}_{n+1},\mathfrak{u}_3)]\\
&\preccurlyeq\alpha_1\delta(\mathfrak{u}_n,\mathfrak{u}_{n-1},\mathfrak{u}_3) + \alpha_2\delta(\mathfrak{u}_n,\mathfrak{u}_{n+1},\mathfrak{u}_3) + \alpha_3\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3)\\
&\quad + \alpha_5\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3) + \alpha_5\delta(\mathfrak{u}_n,\mathfrak{u}_{n+1},\mathfrak{u}_3)\\
&\preccurlyeq(e-\alpha_2-\alpha_5)^{-1}(\alpha_1+\alpha_3+\alpha_5)\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3)\\
&\preccurlyeq\eta_1\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3),
\end{aligned}$$

where

$$\eta_1 = (e-\alpha_2-\alpha_5)^{-1}(\alpha_1+\alpha_3+\alpha_5) \in \mathcal{P}.$$

By symmetrical probability of cone 2-MS, we have

$$\begin{aligned}
&\delta(\mathfrak{u}_{n+1},\mathfrak{u}_n,\mathfrak{u}_3)\\
&= \delta(\mathfrak{u}_n,\mathfrak{u}_{n+1},\mathfrak{u}_3)\\
&= \delta(\mathfrak{h}_n\mathfrak{u}_{n-1},\mathfrak{h}_{n+1}\mathfrak{u}_n,\mathfrak{u}_3)\\
&\preccurlyeq\alpha_1\delta(\mathfrak{f}_n\mathfrak{u}_{n-1},\mathfrak{f}_n\mathfrak{u}_n,\mathfrak{u}_3) + \alpha_2\delta(\mathfrak{f}_n\mathfrak{u}_{n-1},\mathfrak{h}_n\mathfrak{u}_{n-1},\mathfrak{u}_3)\\
&\quad + \alpha_3\delta(\mathfrak{f}_n\mathfrak{u}_n,\mathfrak{h}_n\mathfrak{u}_n,\mathfrak{u}_3) + \alpha_4\delta(\mathfrak{f}_n\mathfrak{u}_{n-1},\mathfrak{h}_n\mathfrak{u}_n,\mathfrak{u}_3)\\
&\quad + \alpha_5\delta(\mathfrak{f}_n\mathfrak{u}_n,\mathfrak{h}_n\mathfrak{u}_{n-1},\mathfrak{u}_3)\\
&\preccurlyeq\alpha_1\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3) + \alpha_2\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3) + \alpha_3\delta(\mathfrak{u}_n,\mathfrak{u}_{n+1},\mathfrak{u}_3)\\
&\quad + \alpha_4\delta(\mathfrak{u}_{n-1},\mathfrak{u}_{n+1},\mathfrak{u}_3) + \alpha_5\delta(\mathfrak{u}_n,\mathfrak{u}_n,\mathfrak{u}_3)\\
&\preccurlyeq\alpha_1\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3) + \alpha_2\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3) + \alpha_3\delta(\mathfrak{u}_n,\mathfrak{u}_{n+1},\mathfrak{u}_3)\\
&\quad + \alpha_4[\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3) + \delta(\mathfrak{u}_n,\mathfrak{u}_{n+1},\mathfrak{u}_3)]\\
&\preccurlyeq\alpha_1\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3) + \alpha_2\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3) + \alpha_3\delta(\mathfrak{u}_n,\mathfrak{u}_{n+1},\mathfrak{u}_3)\\
&\quad + \alpha_4\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3) + \alpha_4\delta(\mathfrak{u}_n,\mathfrak{u}_{n+1},\mathfrak{u}_3)\\
&\preccurlyeq(e-\alpha_3-\alpha_4)^{-1}(\alpha_1+\alpha_2+\alpha_4)\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3)\\
&\preccurlyeq\eta_2\delta(\mathfrak{u}_{n-1},\mathfrak{u}_n,\mathfrak{u}_3),
\end{aligned}$$

where

$$\eta_2 = (e - \alpha_3 - \alpha_4)^{-1}(\alpha_1 + \alpha_2 + \alpha_4) \in \mathcal{P}.$$

We pretend that, either $\mathfrak{r}(\eta_1) < 1$ or $\mathfrak{r}(\eta_2) < 1$. If $\mathfrak{r}(\eta_1) > 1$, we obtain

$$\big(\mathfrak{r}(\alpha_1) + \mathfrak{r}(\alpha_3) + \mathfrak{r}(\alpha_5)\big)\big(1 - \mathfrak{r}(\alpha_2) - \mathfrak{r}(\alpha_5)\big)^{-1} \geq \mathfrak{r}(\eta_1) > 1.$$

Which leads to,

$$\mathfrak{r}(\alpha_1) + \mathfrak{r}(\alpha_2) + \mathfrak{r}(\alpha_3) + 2r(\alpha_5) > 1. \tag{3}$$

If $\mathfrak{r}(\eta_2) > 1$, we obtain

$$\big(\mathfrak{r}(\alpha_1) + \mathfrak{r}(\alpha_2) + \mathfrak{r}(\alpha_4)\big)\big(1 - \mathfrak{r}(\alpha_3) - \mathfrak{r}(\alpha_4)\big)^{-1} \geq \mathfrak{r}(\eta_2) > 1.$$

Which implies

$$\mathfrak{r}(\alpha_1) + \mathfrak{r}(\alpha_2) + \mathfrak{r}(\alpha_3) + 2r(\alpha_4) > 1. \tag{4}$$

By adding (3) and (4), we have $\mathfrak{r}(\alpha_1)+\mathfrak{r}(\alpha_2)+\mathfrak{r}(\alpha_3)+\mathfrak{r}(\alpha_4) + \mathfrak{r}(\alpha_5) > 1$. Which is a contradiction. Hence our pretension is correct.

$$\delta(\mathfrak{u}_{n+1}, \mathfrak{u}_n, \mathfrak{u}_3) \preccurlyeq \alpha\delta(\mathfrak{u}_n, \mathfrak{u}_{n-1}, \mathfrak{u}_3), \tag{5}$$

for all $n \geq 1$ and $\mathfrak{r}(\alpha) < 1$. Jointly with Proposition 2.4 we have

$$\begin{aligned} \delta(\mathfrak{u}_{n+1}, \mathfrak{u}_n, \mathfrak{u}_3) &\preccurlyeq \alpha\delta(\mathfrak{u}_n, \mathfrak{u}_{n-1}, \mathfrak{u}_3) \\ &\preccurlyeq \alpha^2\delta(\mathfrak{u}_{n-1}, \mathfrak{u}_{n-2}, \mathfrak{u}_3) \\ &\vdots \\ &\preccurlyeq \alpha^n\delta(\mathfrak{u}_1, \mathfrak{u}_0, \mathfrak{u}_3), \end{aligned}$$

where

$$\alpha = \begin{cases} \eta_1, when\ \mathfrak{r}(\eta_1) < 1, \\ \eta_2, when\ \mathfrak{r}(\eta_2) < 1, \\ \eta_1\ or\ \eta_2\ when\ \mathfrak{r}(\eta_1) < 1\ and\ \mathfrak{r}(\eta_2) < 1. \end{cases}$$

Then $\alpha \in \mathcal{P}$ and $\mathfrak{r}(\alpha) < 1$. For all $\tau < n$ we have

$$\begin{aligned} \delta(\mathfrak{u}_n, \mathfrak{u}_{n-1}, \mathfrak{u}_\tau) &\preccurlyeq \alpha\delta(\mathfrak{u}_{n-1}, \mathfrak{u}_{n-2}, \mathfrak{u}_\tau) \\ &\preccurlyeq \alpha^2\delta(\mathfrak{u}_{n-2}, \mathfrak{u}_{n-3}, \mathfrak{u}_\tau) \\ &\vdots \\ &\preccurlyeq \alpha^{n-\tau-1}\delta(\mathfrak{u}_{\tau+1}, \mathfrak{u}_\tau, \mathfrak{u}_\tau). \end{aligned}$$

Therefore, for all $\tau < n$, we obtain $\delta(\mathfrak{u}_n, \mathfrak{u}_{n-1}, \mathfrak{u}_\tau) = 0$. Now, for such $n > s$ we have

$$\begin{aligned} &\delta(\mathfrak{u}_n, u_s, \mathfrak{u}_3) \\ \preccurlyeq&\delta(\mathfrak{u}_n, \mathfrak{u}_s, \mathfrak{u}_{n-1}) + \delta(\mathfrak{u}_n, \mathfrak{u}_{n-1}, \mathfrak{u}_3) + \delta(\mathfrak{u}_{n-1}, \mathfrak{u}_s, \mathfrak{u}_3) \\ \preccurlyeq&\alpha^{n-1}\delta(\mathfrak{u}_1, \mathfrak{u}_0, \mathfrak{u}_3) + \delta(\mathfrak{u}_{n-1}, \mathfrak{u}_s, \mathfrak{u}_{n-2}) \\ &+ \delta(\mathfrak{u}_{n-1}, \mathfrak{u}_{n-2}, \mathfrak{u}_3) + \delta(\mathfrak{u}_{n-2}, \mathfrak{u}_s, \mathfrak{u}_3) \\ \preccurlyeq&(\alpha^{n-1} + \alpha^{n-1})\delta(\mathfrak{u}_1, \mathfrak{u}_0, \mathfrak{u}_3) + \delta(\mathfrak{u}_{n-2}, \mathfrak{u}_s, \mathfrak{u}_3) \\ \preccurlyeq&(\alpha^{n-1} + \alpha^{n-1} + \cdots + \alpha^{s+1})\delta(\mathfrak{u}_1, \mathfrak{u}_0, \mathfrak{u}_3) \\ &+ \delta(\mathfrak{u}_{s+1}, \mathfrak{u}_s, \mathfrak{u}_3) \\ \preccurlyeq&(\alpha^{n-1} + \alpha^{n-1} + \cdots + \alpha^{s+1} + \alpha^s)\delta(\mathfrak{u}_1, \mathfrak{u}_0, \mathfrak{u}_3) \\ =&(e + \alpha + \cdots + \alpha^{n-s+1})\alpha^s\delta(\mathfrak{u}_1, \mathfrak{u}_0, \mathfrak{u}_3) \\ \preccurlyeq&\Big(\sum_{i=1}^{\infty}\alpha^i\Big)\alpha^s\delta(\mathfrak{u}_1, \mathfrak{u}_0, \mathfrak{u}_3) \\ =&\alpha^s(e - \alpha)^{-1}\delta(\mathfrak{u}_1, \mathfrak{u}_0, \mathfrak{u}_3). \end{aligned}$$

From Lemma 2.6 and the actuality

$$\|\alpha^s(e - \alpha)^{-1}\delta(\mathfrak{u}_1, \mathfrak{u}_0, \mathfrak{u}_3)\| = 0,\ as\ n \to \infty.$$

We get for any $\beta \in \mathfrak{G}$ with $0 \ll \beta$, there exist $n \in \mathcal{N}$. Such that for all $n, s > \mathcal{N}$, we get

$$\delta(\mathfrak{u}_n, u_s, \mathfrak{u}_3) \preccurlyeq \alpha^s(e - \alpha)^{-1}\delta(\mathfrak{u}_1, \mathfrak{u}_0, \mathfrak{u}_3) \ll \beta.$$

Which proves that, $\{\mathfrak{u}_n\}$ is a Cauchy sequence in $\mathcal{U}$. There exists $\mathfrak{u} \in \mathcal{U}$ such that $\mathfrak{u}_n \to \mathfrak{u}$ as $n \to \infty$ since $\mathcal{U}$ is complete. We pretend that $\mathfrak{u}$ is common FP of $\{T_i^{ki}\}_{i=1}^{\infty}$ and $\{F_i^{ki}\}_{i=1}^{\infty}$ for all $i \in \mathcal{N}$. Inequality (2) become,

$$\begin{aligned} &\delta(\mathfrak{h}_n\mathfrak{u}_n, \mathfrak{h}_m\mathfrak{u}_m, \mathfrak{u}) \\ &\preccurlyeq \alpha_1\delta(\mathfrak{f}_n\mathfrak{u}_n, \mathfrak{f}_m\mathfrak{u}_m, \mathfrak{u}) + \alpha_2\delta(\mathfrak{f}_n\mathfrak{u}_n, \mathfrak{h}_n\mathfrak{u}_n, \mathfrak{u}) \\ &+ \alpha_3\delta(\mathfrak{f}_m\mathfrak{u}_m, \mathfrak{h}_m\mathfrak{u}_m, \mathfrak{u}) \\ &+ \alpha_4\delta(\mathfrak{f}_n\mathfrak{u}_n, \mathfrak{h}_m\mathfrak{u}_m, \mathfrak{u}) + \alpha_5\delta(\mathfrak{f}_m\mathfrak{u}_m, \mathfrak{h}_n\mathfrak{u}_n, \mathfrak{u}) \\ &\preccurlyeq \alpha_1\delta(\mathfrak{u}_n, \mathfrak{u}_m, \mathfrak{u}) + \alpha_2\delta(\mathfrak{u}_n, \mathfrak{u}_{n+1}, \mathfrak{u}) + \alpha_3\delta(\mathfrak{u}_m, \mathfrak{u}_{m+1}, \mathfrak{u}) \\ &+ \alpha_4\delta(\mathfrak{u}_n, \mathfrak{u}_{m+1}, \mathfrak{u}) + \alpha_5\delta(\mathfrak{u}_m, \mathfrak{u}_{n+1}, \mathfrak{u}). \end{aligned}$$

Hence, $\delta(\mathfrak{h}_n\mathfrak{u}_n, \mathfrak{h}_m\mathfrak{u}_m, \mathfrak{u}) \preccurlyeq 0$. By (ii) in Definition 2.1 we have: $\mathfrak{h}_n\mathfrak{u}_n = \mathfrak{h}_n\mathfrak{u} = \mathfrak{u}$ when $n \to \infty$ or $\mathfrak{h}_m\mathfrak{u}_n = \mathfrak{h}_m\mathfrak{u} = \mathfrak{u}$ when $m \to \infty$. Which mean $\mathfrak{u}$ is common FP of $\mathfrak{h}_n = \{T_i^{ki}\}_{i=1}^{\infty}$. Also, we can see that, $\mathfrak{f}_n\mathfrak{u}_n = \mathfrak{f}\mathfrak{u} = \mathfrak{u}$. From this we conclude that $\mathfrak{u}$ is a common FP of $\{T_i^{ki}\}_{i=1}^{\infty}$ and $\{F_i^{ki}\}_{i=1}^{\infty}$ on $\mathcal{U}$. Assume That, w is any another common FP of $\mathfrak{h}_n$ and $\mathfrak{f}_n$ on $\mathcal{U}$, such that $w \neq \mathfrak{u}$ for all $n \in \mathcal{N}$. Then by (2) we obtain,

$$\begin{aligned} &\delta(w, \mathfrak{u}, \mathfrak{u}_3) \\ &= \delta(\mathfrak{h}_n\mathfrak{u}_n, \mathfrak{h}_m\mathfrak{u}_m, \mathfrak{u}_3) \\ &\preccurlyeq \alpha_1\delta(\mathfrak{f}_n\mathfrak{u}_n, \mathfrak{f}_m\mathfrak{u}_m, \mathfrak{u}_3) + \alpha_2\delta(\mathfrak{f}_n\mathfrak{u}_n, \mathfrak{h}_n\mathfrak{u}_n, \mathfrak{u}_3) \\ &+ \alpha_3\delta(\mathfrak{f}_m\mathfrak{u}_m, \mathfrak{h}_m\mathfrak{u}_m, \mathfrak{u}_3) \\ &+ \alpha_4\delta(f_nu_n, \mathfrak{h}_m\mathfrak{u}_m, \mathfrak{u}_3) + \alpha_5\delta(\mathfrak{f}_m\mathfrak{u}_m, \mathfrak{h}_n\mathfrak{u}_n, \mathfrak{u}_3) \\ &\preccurlyeq 0. \end{aligned}$$

This implies that $w = \mathfrak{u}$ which proven the uniqueness of a common FP $\mathfrak{u}$ of $\{T_i^{ki}\}_{i=1}^{\infty}$ and $\{F_i^{ki}\}_{i=1}^{\infty}$ on $\mathcal{U}$. □

The next conclusions can be gained from our main result.

Corollary 1. Let $(\mathcal{U}, \delta)$ be a complete cone 2-MS in a BG $\mathfrak{G}$ and $\mathcal{P}$ the underlying cone. Assume that T, F are self-mappings of $\mathcal{U}$ satisfying the condition

$$\begin{aligned}&\delta(T_i^{ki}\mathfrak{u}_1, T_j^{kj}\mathfrak{u}_2, \mathfrak{u}_3)\\ &\preccurlyeq \alpha_1\delta(F_i^{ki}\mathfrak{u}_1, F_j^{kj}\mathfrak{u}_2, \mathfrak{u}_3) + \alpha_2\delta(F_i^{ki}\mathfrak{u}_1, T_i^{ki}\mathfrak{u}_1, \mathfrak{u}_3)\\ &+ \alpha_3\delta(F_j^{kj}\mathfrak{u}_2, T_j^{kj}\mathfrak{u}_2, \mathfrak{u}_3), \qquad (6)\end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{P}$, for all $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3 \in \mathcal{U}$. If $\alpha_1, \alpha_2, \alpha_3$ are commute and $\mathfrak{r}(\alpha_1) + \mathfrak{r}(\alpha_2) + \mathfrak{r}(\alpha_3) < 1$, then $\{T_i^{ki}\}_{i=1}^{\infty}$ and $\{F_i^{ki}\}_{i=1}^{\infty}$ have a unique common FP.
The above FPT is development and generalization for Wang's result in [30].

Corollary 2. Let $(\mathcal{U}, \delta)$ be a complete cone 2-MS in a BG $\mathfrak{G}$ and $\mathcal{P}$ the underlying cone. Assume that, T, F are self-mappings of $\mathcal{U}$ satisfying the condition

$$\begin{aligned}&\delta(T_i^{ki}\mathfrak{u}_1, T_j^{kj}\mathfrak{u}_2, \mathfrak{u}_3) \qquad (7)\\ &\preccurlyeq \alpha\Big[\delta(F_i^{ki}\mathfrak{u}_1, T_i^{ki}\mathfrak{u}_1, \mathfrak{u}_3) + \delta(F_j^{kj}\mathfrak{u}_2, T_j^{kj}\mathfrak{u}_2, \mathfrak{u}_3)\Big],\end{aligned}$$

where $\alpha \in \mathcal{P}$, for all $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3 \in \mathcal{U}$. If $\mathfrak{r}(\alpha) < 1/2$, then $\{T_i^{ki}\}_{i=1}^{\infty}$ and $\{F_i^{ki}\}_{i=1}^{\infty}$ have a unique common FP.

Corollary 3. Let $(\mathcal{U}, \delta)$ be a complete cone 2-MS in a BG $\mathfrak{G}$ and $\mathcal{P}$ the underlying cone. Assume that T, F are self-mappings of $\mathcal{U}$ satisfying the condition

$$\begin{aligned}&\delta(T_i^{ki}\mathfrak{u}_1, T_j^{kj}\mathfrak{u}_2, \mathfrak{u}_3) \preccurlyeq\\ &\alpha\Big[\delta(F_i^{ki}\mathfrak{u}_1, T_j^{kj}\mathfrak{u}_2, \mathfrak{u}_3) + \delta(F_j^{kj}\mathfrak{u}_2, T_i^{ki}\mathfrak{u}_1, \mathfrak{u}_3)\Big],\end{aligned}$$

where $\alpha \in \mathcal{P}$, for all $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3 \in \mathcal{U}$. If $\mathfrak{r}(\alpha) < 1/2$, then $\{T_i^{ki}\}_{i=1}^{\infty}$ and $\{F_i^{ki}\}_{i=1}^{\infty}$ have a unique common FP.
The result of Mlaiki of FP [32] can be developed and generalized as follows.

Corollary 4. Let $(\mathcal{U}, \delta)$ be a complete cone 2-MS in a BGa $\mathfrak{G}$ and $\mathcal{P}$ the underlying cone. Assume that T, F are self-mappings of $\mathcal{U}$ satisfying the condition

$$\begin{aligned}&\delta(T_i^{ki}\mathfrak{u}_1, T_j^{kj}\mathfrak{u}_2, \mathfrak{u}_3) \preccurlyeq\\ &\alpha \max\Big[\delta(F_i^{ki}\mathfrak{u}_1, T_j^{kj}\mathfrak{u}_1, \mathfrak{u}_3), \delta(F_j^{kj}\mathfrak{u}_2, T_i^{ki}\mathfrak{u}_2, \mathfrak{u}_3)\Big],\end{aligned}$$

where, $\alpha \in \mathcal{P}$, for all $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3 \in \mathcal{U}$. If $\mathfrak{r}(\alpha) \in (0, 1)$, then $\{T_i^{ki}\}_{i=1}^{\infty}$ and $\{F_i^{ki}\}_{i=1}^{\infty}$ have a unique common FP.

Example 1. Suppose that $\mathfrak{G} = \mathcal{R}^2$ and $(\mathfrak{u}_1, \mathfrak{u}_2) \in \mathfrak{G}$ such that $\|\mathfrak{u}_1, \mathfrak{u}_2\| = |\mathfrak{u}_1| + |\mathfrak{u}_2|$. Consider the multiplication as

$$\mathfrak{u}w = (\mathfrak{u}_1.\mathfrak{u}_2)(w_1, w_2) = \varkappa_1 w_1 + \varkappa_2 w_2$$

Then $\mathfrak{G}$ is a BG with unite $e = (1, 0)$. Assume that $\mathcal{P} = \{(\mathfrak{u}_1, \mathfrak{u}_2) \in \mathcal{R}^2 | \mathfrak{u}_1, \mathfrak{u}_2, \geq 0\}$. Then $\mathcal{P}$ is a cone on $\mathfrak{G}$. Let, $\mathcal{U} = \{(\alpha, 0) \in \mathfrak{R}^2 | 0 \leq \alpha < 1\} \cup \{(0, \alpha) \in \mathfrak{R}^2 | 0 \leq \alpha < 1\}$.
Define the metric as

$$\delta(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3) = \delta(\mu_1, \mu_2),$$

where, $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3 \in \mathcal{U}$ and $\mu_1, \mu_2 \in (\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)$. Such that

$$\|\mu_1 - \mu_2\| = \min\{\|\mathfrak{u}_1 - \mathfrak{u}_2\|, \|\mathfrak{u}_2 - \mathfrak{u}_3\|, \|\mathfrak{u}_3 - \mathfrak{u}_1\|\},$$

and,

$$\begin{aligned}\delta_1\big((\alpha, 0), (\mathfrak{u}, 0)\big) &= \Big(|\alpha - \mathfrak{u}|, \frac{5}{4}|\alpha - \mathfrak{u}|\Big),\\ \delta_1\big((0, \alpha), (0, \mathfrak{u})\big) &= \Big(\frac{3}{4}|\alpha - \mathfrak{u}|, |\alpha - \mathfrak{u}|\Big),\\ \delta_1\big((\alpha, 0), (0, \mathfrak{u})\big) &= \delta_1\big((0, \mathfrak{u}), (\alpha, 0)\big)\\ &= \Big(\alpha + \frac{3}{4}\mathfrak{u}, \frac{5}{4}\alpha + \mathfrak{u}\Big).\end{aligned}$$

Thus, $(\mathcal{U}, \delta)$ is a complete cone 2-MS on the BG $\mathfrak{G}$. Define $T_i, F_i : \mathcal{U} \to \mathcal{U}$ where $i \in \mathcal{N}$ as

$$T_i\big((F_i\alpha, 0)\big) = \Big(0, 3^{\left(\frac{\frac{i}{2}-1}{i-\frac{1}{2}}\right)} 2^{\left(\frac{\frac{1}{2}-\frac{i}{2}}{i-\frac{1}{2}}\right)}\alpha\Big),$$

and

$$T_i\big((0, F_i\alpha)\big) = \Big(3^{\left(\frac{\frac{i}{2}-1}{i-\frac{1}{2}}\right)} 2^{\left(\frac{\frac{1}{2}-\frac{i}{2}}{i-\frac{1}{2}}\right)}\alpha, 0\Big).$$

Therefore, $T_i^{2i-1}(F_i^{2i-1}\alpha, 0) = (0, \frac{1}{12}\alpha)$ and $T_i^{2i-1}(0, F_i^{2i-1}\alpha) = (\frac{1}{18}\alpha, 0)$. Thus, it concludes that T_i, F_i achieves the contractive condition (1), where $Ki = 2i - 1, \alpha_1 = (\frac{1}{4}, 0)$, $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = (\frac{1}{6}, 0)$. Furthermore, $\mathfrak{r}(\alpha_1) = \frac{1}{4}$, $\mathfrak{r}(\alpha_2) = \mathfrak{r}(\alpha_3) = \mathfrak{r}(\alpha_4) = \mathfrak{r}(\alpha_5) = \frac{1}{6}$. Hence, by Theorem 3.1, we get $(0, 0)$ is a unique FP for T_i, F_i on $\mathcal{U}$ for all $i \geq 1$.

We will show that the normality condition of cone 2- metric is necessary to ensure the existence of a common FP in our result.

Example 2. Suppose that $\mathfrak{G} = C^1_{\mathcal{R}}[0,1]$, and $(\mathfrak{u}_1, \mathfrak{u}_2) \in \mathfrak{G}$ with $\|(\mathfrak{u}, v)\| = \|(\mathfrak{u}_1, \mathfrak{u}_2)\|_\infty + \|(\mathfrak{u}_1, \mathfrak{u}_2)'\|_\infty$, then $\mathfrak{G}$ is a BG with unite $e = (1, 0)$. Let, $\mathcal{P} = \{\mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{G} : \mathfrak{u}(t) \geq 0, \mathfrak{u}_2(t) \geq 0,\ t \in [0,1]\}$ be a non-normal solid cone. Consider, $T\mathfrak{u}_{1n}(t) = \frac{t^n}{n}$ and $F\mathfrak{u}_{1n}(t) = \frac{1}{n}$, then $F\mathfrak{u}_{1n} \geq T\mathfrak{u}_{1n} \geq 0$ and $\lim_{n\to\infty} F\mathfrak{u}_{1n} = 0$, but $\|\mathfrak{u}_{1n}\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} ||t^n - 1| = \frac{1}{n} + 1 > 1$. Thus, $\mathfrak{u}_n$ does not converges to 0. Hence, T and F does not have common FP.

4 Conclusion

Thus, in this work, we have obtained a unique common fixed point result in cone 2-metric space on Banach algebras. Also, we have generalized some fixed point theorems in the literature. Using the idea of n-inner product spaces on n-normed metric spaces we will make an analog study concerning the unique common fixed point of Hardy-Rogers contraction type in cone n-metric spaces over a Banach Algebra.

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