Joining Iso-Structured Models with Commutative Orthogonal Block Structure

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Abstract: - In this work, we focus on a special class of mixed models, named models with commutative orthogonal block structure (COBS), whose covariance matrix is a linear combination of known pairwise orthogonal projection matrices that add to the identity matrix, and for which the orthogonal projection matrix on the space spanned by the mean vector commutes with the covariance matrix. The COBS have least squares estimators giving the best linear unbiased estimators for estimable vectors. Our approach to COBS relies on their algebraic structure, based on commutative Jordan algebras of symmetric matrices, which proves to be advantageous as it leads to important results in the estimation. Specifically, we are interested in iso-structured COBS, applying to them the operation of models joining. We show that joining iso-structured COBS gives COBS and that the estimators for the joint model may be obtained from those for the individual models.

Key-Words: - Best linear unbiased estimators, COBS, Jordan algebra, Mixed model, Models joining, Variance components.

Received: December 23, 2022. Revised: June 8, 2023. Accepted: June 27, 2023. Published: July 27, 2023.

1 Introduction

Different areas of knowledge, such as Agriculture, Medical and Biological Sciences, Social Sciences, and others, base their experimental designs on linear models.

Using the matrix notation, a linear model can be represented as

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \quad (1)$$

where $\boldsymbol{Y}$ is the observations vector, $\boldsymbol{X}$ is the design matrix, $\boldsymbol{\beta}$ is a vector of unknown parameters and $\boldsymbol{\varepsilon}$ is the errors vector.

Classifying linear models according to the nature of the constituent parameters of the vector $\boldsymbol{\beta}$, we consider fixed effects models to have constants for all the parameters of the vector $\boldsymbol{\beta}$, random effects models if all the parameters of vector $\boldsymbol{\beta}$ are random, except for the intercept, and mixed effects models when some effects are fixed and other are random.

Linear mixed effects models, or adopting an abbreviated designation, mixed models, arise from the need to appreciate the amount of variation caused by given sources in fixed effects designs, [1], proving to be appropriate for analyzing datasets involving correlated data, or resulting from repeated measures, [2], as is common to find in experimental data in agricultural and medical sciences, for example.

In this work we address classes of mixed models, focusing on models with commutative orthogonal block structure (COBS), which constitute a special class within the subclass of mixed models introduced by, [3], [4], called models with orthogonal block structure.

To lighten the writing, we name the linear models with commutative orthogonal block structure, simply, as COBS.

Our approach to COBS relies on their algebraic structure, based on commutative Jordan algebras of symmetric matrices (CJAS). This approach proves to be advantageous as it leads to important results in the estimation of variance components and the construction of models, [5].

We are interested in the possibility of performing joint analysis of models obtained separately. In, [6], [7], [8], the theory that provides this joint analysis relies on operations between models, which are based on binary operations defined on commutative Jordan algebras.

In, [6], taking, [7], as a starting point, two operations between COBS were introduced, called models crossing and models nesting, resorting to the Kronecker matrix product and the restricted Kronecker matrix product. In, [8], was introduced another operation to build up complex models from simpler ones, named models joining, based on another binary operation defined on commutative Jordan algebras, the Cartesian product.

Since COBS has least squares estimators (LSE) giving best linear unbiased estimators (BLUE) for estimable vectors, [9], the possibility of joint analysis of COBS that were obtained independently is relevant, since, as proved by, [8], model joining operation involving COBS results in a model that is also COBS.

In previous works on operations with COBS, [7], [8], no condition was assumed to aggregate the models involved in the operations into a family of models with the same fundamental structure. The present work presents a development of the operation of joining models, considering initial models that belong to a family of iso-structured models, that is, models that are independent and have identical space spanned by their mean vectors, as well as covariance matrices given by linear combinations of the same pairwise orthogonal orthogonal projection matrices (POOPM).

The paper is structured as follows.

In carrying out the estimation for COBS we use commutative Jordan algebras of symmetric matrices in expressing the algebraic structure of those models, therefore, we will start by presenting key results about Jordan algebras in section 2. In section 3 we will present the formulation of COBS and the definition of iso-structured COBS, as well as results that will be useful when we join iso-structured COBS. In section 4 we discuss the estimation in COBS. Section 5 is devoted to the operation of model joining, involving iso-structured COBS. Some concluding remarks are presented in section 6.

2 Jordan Algebras

To formalize the notion of an algebra of observables, [10], introduced the structures that were originally designated as "*r-number systems*", and which later came to be known as Jordan Algebras. For our purposes, we will follow the approach of, [3], [4], in which Jordan algebras were used in the study of models with orthogonal block structures. Specifically, we are interested in commutative Jordan Algebras of symmetric matrices (CJAS), which are vector spaces of symmetric matrices that commute and are closed under squaring, [11].

A rediscovery of Jordan Algebras to carry out linear statistical inference, [12], showed that every CJAS has one and only one basis, called the principal basis, constituted by POOPM.

Let $Q = \{\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m\} = pb(A)$ be the principal basis of the CJAS A. A matrix, $\boldsymbol{M}$, belonging to the CJAS A is a linear combination of the matrices of the $pb(A)$, [13],

$$\boldsymbol{M} = \sum_{j=1}^{m} b_j \boldsymbol{Q}_j. \tag{2}$$

It is evident that the family of matrices $\{\boldsymbol{M}_1, \ldots, \boldsymbol{M}_m\}$, where $\boldsymbol{M}_i$, $i = 1, \ldots, m$, belongs to the CJAS A is commutative, since, as shown below, given any two matrices of the principal basis of this CJAS, $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, these matrices commute:

$$\boldsymbol{M}_1\boldsymbol{M}_2 = \left(\sum_{j=1}^{m} b_{1,j}\boldsymbol{Q}_j\right)\left(\sum_{j=1}^{m} b_{2,j}\boldsymbol{Q}_j\right)$$

$$= \sum_{j=1}^{m}\left(b_{1,j}\, b_{2,j}\right)\boldsymbol{Q}_j = \sum_{j=1}^{m}\left(b_{2,j}\, b_{1,j}\right)\boldsymbol{Q}_j \quad (3)$$

$$= \left(\sum_{j=1}^{m} b_{2,j}\boldsymbol{Q}_j\right)\left(\sum_{j=1}^{m} b_{1,j}\boldsymbol{Q}_j\right)$$

$$= \boldsymbol{M}_2\boldsymbol{M}_1$$

considering that the matrices $\boldsymbol{M}_i$, $i = 1, \dots, m$, are diagonalized by the same orthogonal matrix, [14].

The orthogonal projection matrix on the image space of $\boldsymbol{M}$ will be

$$\boldsymbol{P}(\boldsymbol{M}) = \sum_{j \in C(\boldsymbol{M})} \boldsymbol{Q}_j \ , \quad (4)$$

with $C(\boldsymbol{M}) = \{j : a_j \neq 0\}$, so, considering $\boldsymbol{M} \in A$, we have $\boldsymbol{P}(\boldsymbol{M}) \in A$.

Since the Moore-Penrose inverse of $\boldsymbol{M}$, expressed by

$$\boldsymbol{M}^{+} = \sum_{j=1}^{m} b_j^{+}\boldsymbol{Q}_j \ , \quad (5)$$

with $b_j^{+} = b_j^{-1}$ $[a_j^{+} = 0]$ if $b_j \neq 0$ $[b_j = 0]$, belongs to A, then the Moore-Penrose inverses of the matrices of A also belong to A.

When $\boldsymbol{M}$ is invertible we have $\boldsymbol{M}^{+} = \boldsymbol{M}^{-1}$, so the inverses of invertible matrices of A also belong to A.

Among the operations on CJAS, we are interested in considering the cartesian product, introduced by, [15].

Given the CJAS A_h, $h = 1, \dots, u$, with principal bases $Q_h = \{\boldsymbol{Q}_{h,1}, \dots, \boldsymbol{Q}_{h,m_h}\}$, $h = 1, \dots, u$, their cartesian product,

$$\mathrm{X}_{h=1}^{u} A_h \ , \quad (6)$$

will be the CJAS whose principal basis is constituted by the block-wise diagonal matrices

$$\mathrm{D}(\boldsymbol{M}_1, \dots, \boldsymbol{M}_u), \quad (7)$$

with principal blocks $\boldsymbol{M}_h \in pb(A_h)$, $h = 1, \dots, u$, with null sub-matrices except one belonging to the principal basis with that index, always existing a non-null sub-matrix.

3 Models with Commutative Orthogonal Block Structure

Let us consider a linear mixed model,

$$\boldsymbol{Y} = \sum_{i=0}^{w} \boldsymbol{X}_i\boldsymbol{\beta}_i \quad (8)$$

where $\boldsymbol{\beta}_i$ is fixed for $i = 0$, and independent random vectors for $i = 1, \dots, w$, having null mean vectors, covariance matrices $\sigma_1^2\boldsymbol{I}_{c_1} \dots \sigma_w^2\boldsymbol{I}_{c_w}$, where $c_i = rank(\boldsymbol{X}_i)$, $i = 1, \dots, w$. The matrices $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_w$ are known and such that $R([\boldsymbol{X}_1 \ \cdots \ \boldsymbol{X}_w]) = IR^n$. When the covariance matrix is given by

$$\boldsymbol{V}(\boldsymbol{\gamma}) = \sum_{j=1}^{m^0} \gamma_j^0\boldsymbol{Q}_j^0 \ , \quad (9)$$

where the $\boldsymbol{Q}_1^0, \dots, \boldsymbol{Q}_{m^0}^0$ are POOPM whose sum is the identity matrix

$$\sum_{j=1}^{m^0} \boldsymbol{Q}_j^0 = I_n \ , \quad (10)$$

the model (8) has an orthogonal block structure (OBS), [3], [4]. Moreover, model (8) is a model with commutative orthogonal block structure (COBS), when $\mathbf{T}$, the orthogonal projection matrix on the space, Ω, spanned by the mean vector, commute with the covariance matrix $\boldsymbol{V}(\boldsymbol{\gamma})$, whatever $\boldsymbol{\gamma}$ with nonnegative components, [5]. This commutativity between $\mathbf{T}$ and $\boldsymbol{V}(\boldsymbol{\gamma})$, characteristic condition of the COBS, is a necessary and sufficient condition for the LSE, for estimable functions, to be uniformly best linear unbiased estimators (UBLUE), as proven in, [9].

As stressed by, [16], although in OBS the estimators for estimable vectors and variance components have good behavior, the inference is somewhat complex due to the combination of estimators obtained from different orthogonal projections in the range spaces of the matrices $\boldsymbol{Q}_j^0$, $j = 1, \dots, m^0$. COBS, the class of OBS resulting from the imposition of commutativity between the matrices $\boldsymbol{T}$ and $\boldsymbol{V}(\boldsymbol{\gamma})$, allows overcoming this difficulty.

The study of COBS using an approach based on their algebraic structure leads to interesting results in the estimation of variance components and the construction of models. This approach has been adopted in several works. In, [11], an alternative condition for the definition of COBS was established, resorting to U-matrices. The focus in, [17], was on structured families of COBS. In the works, [18], [19], the relationships between COBS and other models were considered. Works on inference, in COBS, were developed by, [13], [16], [20], [21], [22], [23]. In, [7], [8], operations with models were introduced.

In this work we are interested in COBS associated with experiments carried out with the same design, that is, models with the same algebraic structure and independent observations vector.

Let us now designate by A^0 the CJAS constituted by the linear combinations of the POOPM in (8), [10]. The CJAS A^0, whose principal basis is $\boldsymbol{Q}^0 = \{\boldsymbol{Q}_1^0, \dots, \boldsymbol{Q}_{m^0}^0\} = \mathrm{pb}(A^0)$, contains the products of its matrices and, also, their Moore-Penrose inverses, since the Moore-Penrose inverse of an orthogonal projection matrix is, itself, an orthogonal projection.

$$\left(\sum_{j=1}^{m^0} b_j\,\boldsymbol{Q}_j^0\right)^+ = \sum_{j=1}^{m^0} {b_j}^+\,\boldsymbol{Q}_j^0 \qquad (11)$$

with ${b_j}^+ = {b_j}^{-1}$ $[{b_j}^+ = 0]$ when $b_j \neq 0$ $[b_j = 0]$, $j = 1, \dots, m^0$.

Now, the matrices of a family of symmetric matrices commute if and only if they are diagonalized by the same orthogonal matrix $\boldsymbol{P}$, [14]. Then that family will be contained in $A(\boldsymbol{P})$, the family of matrices diagonalized by $\boldsymbol{P}$, which is itself a CJAS.

Since intercepting CJAS gives a CJAS, intercepting all the CJAS that contain a family S of symmetric matrices that commute gives the least CJAS that contains that family, [6]. This will be the CJAS $A(S)$, generated by the family S. If $\bar{S} = \{\boldsymbol{T}, \boldsymbol{Q}_1^0, \dots, \boldsymbol{Q}_{m^0}^0, \boldsymbol{I}_n, \boldsymbol{T}^C\}$, with $\boldsymbol{T}^C = \boldsymbol{I}_n - \boldsymbol{T}$, is a family of commuting symmetric matrices, there will be a generated CJAS, $A(\bar{S})$.

We point out that

$$\boldsymbol{V}(\boldsymbol{\gamma})^+ = \sum_{j=1}^{m^0} {\gamma_j^0}^+ \boldsymbol{Q}_j^0$$

so, if A is the CJAS to which the model is associated, both $\boldsymbol{V}(\boldsymbol{\gamma})$ and $\boldsymbol{V}(\boldsymbol{\gamma})^+$will belong to A. We are assuming that A^0 is the CJAS generated by $\boldsymbol{M}_i = \boldsymbol{X}_i\boldsymbol{X}_i^T$, $i = 1, \dots, w$, and $\boldsymbol{T}$, the orthogonal projection matrix on Ω.

To obtain $pb(A)$, with A the CJAS generated by $\boldsymbol{T}$ and A^0, we reorder the $\boldsymbol{Q}_1^0, \dots, \boldsymbol{Q}_{m^0}^0$ giving the first $z_1 \geq 0$ ranks to the $\boldsymbol{Q}_j^0$ with range space $\mathrm{R}(\boldsymbol{Q}_j^0) \subseteq \Omega$, the next $z_2 \geq 0$ ranks to the $\boldsymbol{Q}_j^0$ such that we have neither $\mathrm{R}(\boldsymbol{Q}_j^0) \subseteq \Omega$ nor $\mathrm{R}(\boldsymbol{Q}_j^0) \subseteq \Omega^\perp$, with $\Omega^\perp$ the orthogonal complement of Ω, and the last $z_3 \geq 0$ ranks to the $\boldsymbol{Q}_j^0$ with range space $\mathrm{R}(\boldsymbol{Q}_j^0) \subseteq \Omega^\perp$, we will have $z_1 + z_2 + z_3 = m^0$.

Now the product of orthogonal projection matrices that commute is an orthogonal projection matrix. Then $\mathrm{A}(\bar{S})$ will contain the $\boldsymbol{Q}_1^0, \dots, \boldsymbol{Q}_m^0$, with $m = m^0 + z_2$, and

$$\begin{cases} \boldsymbol{Q}_j = \boldsymbol{Q}_j^0 = \boldsymbol{Q}_j^0\boldsymbol{T}, & j = 1, \dots, z_1 \\ \boldsymbol{Q}_j = \boldsymbol{Q}_j^0\boldsymbol{T}, & j = z_1 + 1, \dots, z_1 + z_2 \\ \boldsymbol{Q}_j = \boldsymbol{Q}_{j-z_2}^0\boldsymbol{T}^C, & j = z_1 + z_2 + 1, \dots, z_1 + 2z_2 \\ \boldsymbol{Q}_j = \boldsymbol{Q}_{j-z_2}^0 = \boldsymbol{Q}_{j-z_2}^0\boldsymbol{T}^C, & j = z_1 + 2z_2 + 1, \dots, m \end{cases} \qquad (12)$$

That is, the matrices of the second set originate pairs $(\boldsymbol{Q}_j, \boldsymbol{Q}_{j'})$ of matrices $\boldsymbol{X}$. In this construction, the matrices $\boldsymbol{Q}_j^0$ with $\mathrm{R}(\boldsymbol{Q}_j^0) \subset \mathrm{R}(\mathbf{T})$ are grouped first, followed by matrices $\boldsymbol{Q}_j^0$ with $\mathrm{R}(\boldsymbol{Q}_j)$ intersecting $\mathrm{R}(\boldsymbol{T})$ and $\mathrm{R}(\boldsymbol{T}^C)$ and finally matrices $\boldsymbol{Q}_j^0$ with $\mathrm{R}(\boldsymbol{Q}_j^0) \subset \mathrm{R}(\boldsymbol{T}^C)$.

We point out that $\boldsymbol{Q}_1 \dots \boldsymbol{Q}_m$ are POOPM that add up to $\boldsymbol{I}_n$. We now establish the following result.

Proposition 3.1

The principal basis of A corresponds to $pb(A) = \{\boldsymbol{Q}_1 \dots \boldsymbol{Q}_m\}$, with A the CJAS generated by $A^0 \cup \{\boldsymbol{T}\}$.

Proof: Any CJAS containing the matrices $\boldsymbol{Q}_1^0, \dots, \boldsymbol{Q}_{m^0}^0$ and $\boldsymbol{T}$ contains the matrices $\boldsymbol{Q}_1 \dots \boldsymbol{Q}_m$, which are POOPM thus constituting $pb(A)$.

The definition of iso-structured models with commutative orthogonal block structure (iso-structured COBS) was introduced in, [24]. According to this definition, iso-structured COBS have:

- covariance matrices that are linear combinations of the same POOPM, $\boldsymbol{Q}_1^0, \dots, \boldsymbol{Q}_{m^0}^0$;

- mean vectors that span the same space, Ω.

The use of families of iso-structured models relies on coping with inference for sets of models with the same algebraic structure and independent observation vectors. Our discussion is centered on what follows for joint analysis of independent models when they have the same algebraic structure given by CJAS.

4 Estimation

We start by pointing out that the LSE

$$\widetilde{\boldsymbol{\Psi}} = G\widetilde{\boldsymbol{\beta}}$$

of

$$\boldsymbol{\Psi} = G\boldsymbol{\beta}$$

where $\widetilde{\boldsymbol{\beta}} = \left(\boldsymbol{X}_0{}^T\boldsymbol{X}_0\right)^+\boldsymbol{X}_0{}^T\boldsymbol{Y}$ is UBLUE.

To consider the estimation of variance components we assume to have the model given by

$$\boldsymbol{Y} = \boldsymbol{X}_0\boldsymbol{\beta} + \sum_{i=1}^{w}\boldsymbol{X}_i\boldsymbol{\beta}_i \,, \qquad (13)$$

with $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_w$ having null mean vectors, null cross-covariance matrices, and covariance matrices $\sigma_1^2\boldsymbol{I}_{c_1} \dots \sigma_w^2\boldsymbol{I}_{c_w}$, where $c_i = rank(\boldsymbol{X}_i)$, $i = 1, \dots, w$. The $\sigma_1^2, \dots, \sigma_w^2$ will be the usual variance components, so $\boldsymbol{Y}$ has the covariance matrix,

$$\boldsymbol{V} = \sum_{i=1}^{w}\sigma_i{}^2\boldsymbol{M}_i \,, \qquad (14)$$

with $\boldsymbol{M}_i = \boldsymbol{X}_i\boldsymbol{X}_i{}^T$, $i = 1, \dots, w$.

If $\boldsymbol{M}_1, \dots, \boldsymbol{M}_w$ commute, these matrices generate the CJAS A^0, with $pb(A^0) = \left\{\boldsymbol{Q}_1^0, \dots, \boldsymbol{Q}_{m^0}^0\right\}$. Then we will have

$$\boldsymbol{M}_i = \sum_{j=1}^{m^0} b_{i,j}^0\,\boldsymbol{Q}_j^0 = \sum_{h=1}^{m} b_{i,h}\,\boldsymbol{Q}_h \,, \qquad (15)$$

with

$$\begin{cases} b_{i,j}^0 = b_{i,j} & ,j = 1, \dots, z_1 + z_2 \\ b_{i,j-z_2}^0 = b_{i,j} & ,j = z_1 + z_2 + 1, \dots, z_1 + 2z_2 + z_3 \,, \end{cases}$$

$i = 1, \dots, w$

where $z_1 + 2z_2 + z_3 = m$, since

$$\begin{cases} \mathbf{Q}_j^0 = \mathbf{Q}_j & , \ j = 1, \dots, z_1 \\ \mathbf{Q}_j^0 = \mathbf{Q}_j + \mathbf{Q}_{j+z_2} & , \ j = z_1 + 1, \dots, z_1 + z_2 \\ \mathbf{Q}_j^0 = \mathbf{Q}_{j+z_2} & , \ j = z_1 + z_2 + 1, \dots, z_1 + z_2 + z_3 \end{cases}$$

where $z_1 + z_2 + z_3 = m^0$.

We thus get,

$$\boldsymbol{V} = \sum_{i=1}^{w}\sigma_i{}^2\sum_{j=1}^{m^0} b_{i,j}^0\,\boldsymbol{Q}_j^0 = \sum_{j=1}^{m^0}\gamma_j^0\,\boldsymbol{Q}_j^0 \qquad (16)$$

with $\gamma_h^0 = \sum_{i=1}^{w} b_{i,j}^0\,\sigma_i{}^2$, $j = 1, \dots, m^0$, as well as

$$\boldsymbol{V} = \sum_{h=1}^{m}\gamma_h\,\boldsymbol{Q}_h \,, \qquad (17)$$

where $\gamma_h = \sum_{i=1}^{w} b_{i,h}\,\sigma_i{}^2$, $h = 1, \dots, m$, are the canonical variance components.

Given the relations between the

$$b_{i,j}^0 \,, \qquad i = 1, \dots, w, j = 1, \dots, m^0$$

and the

$$b_{i,h} \,, \ i = 1, \dots, w, h = 1, \dots, m,$$

we also get

$$\begin{cases} \gamma_j^0 = \gamma_j \,, & j = 1, \dots, z_1 \\ \gamma_j^0 = \gamma_j = \gamma_{j+z_2} \,, & j = z_1 + 1, \dots, z_1 + z_2 \\ \gamma_j^0 = \gamma_{j+z_2} \,, & j = z_1 + z_2 + 1, \dots, z_1 + z_2 + z_3 = m^0 \end{cases} \qquad (18)$$

Besides this we have

$$\boldsymbol{V}(\boldsymbol{\gamma}) = \sum_{j=1}^{m^0}\gamma_j^0\,\boldsymbol{Q}_j^0 = \sum_{j=1}^{m}\sigma_{j'}^2\,Q_{j'} \qquad (18)$$

with

$$\begin{cases} \sigma_{j'}^2 = \gamma_j^0 & , \quad j' = 1, \dots, z_1 \\ \sigma_{j'}^2 = \sigma_{j'+z_1}^2 = \gamma_j^0 & , \quad j' = z_1 + 1, \dots, z_1 + z_2 \\ \sigma_{j'}^2 = \sigma_{j'+z_2}^2 & , \quad j' = z_1 + z_2 + 1, \dots, z_1 + z_2 + z_3 \end{cases}$$

and for the $\sigma_{j'}^2$, $j' = z_1 + 1, \dots, m$, we have the estimators

$$\begin{cases} \tilde{\sigma}_{j'}^2 = \tilde{\sigma}_{j'+z_2}^2 = \dfrac{\boldsymbol{Y}^T \boldsymbol{Q}_{j'+z_2} \boldsymbol{Y}}{g_{j'+z_2}} & , j' = z_1 + 1, \dots, z_1 + z_2 \\ \tilde{\sigma}_{j'}^2 = \dfrac{\boldsymbol{Y}^T \boldsymbol{Q}_{j'} \boldsymbol{Y}}{g_{j'}} & , j' = z_1 + 2z_2 + 1, \dots, m \end{cases}$$

where $g_{h'}$ is the rank of $\boldsymbol{Q}_{h'}$, $h' = z_1 + z_2 + 1, \dots, m$. Note that, in general, we cannot estimate the $\sigma_{j'}^2$, $j' = 1, \dots, z_1$, and that the estimators for the remaining variance components follow from

$$\boldsymbol{Q}_l = B_l^{\mathrm{T}} B_l, l = z_1 + z_2 + 1, \dots, m$$

implying $B_l \boldsymbol{Y}$ to have a null mean vector and covariance matrix $\sigma_l^2 \boldsymbol{I}_{g_l}$, $l = z_1 + z_2 + 1, \dots, m$, with $g_l = rank\ (\boldsymbol{Q}_l)$.

Moreover, we will have $\boldsymbol{Y}^T \boldsymbol{Q}_{j'} \boldsymbol{Y} = \|B_l \boldsymbol{Y}\|^2$, $l = z_1 + z_2 + 1, \dots, m$, which lightens the estimator of the estimable variance components.

We now point out that,

$$\begin{cases} \sum_{j=1}^{m^0} \boldsymbol{Q}_j^0 = \sum_{h=1}^{m} \boldsymbol{Q}_h = \boldsymbol{I}_n \\ \sum_{j=1}^{z_1+z_2} \boldsymbol{Q}_j = \boldsymbol{T} \end{cases} \quad (19)$$

Let us put

$\boldsymbol{\gamma}_1 = (\gamma_1, \dots, \gamma_{z_1+z_2})$,
$\boldsymbol{\gamma}_2 = (\gamma_{z_1+z_2+1}, \dots, \gamma_{m^0})$ and
$\boldsymbol{\sigma}^2 = (\sigma_1^2, \dots, \sigma_w^2)$

and consider the partition of matrix $\boldsymbol{B}$,

$$\boldsymbol{B} = [\boldsymbol{B}_1 | \boldsymbol{B}_2], \quad (20)$$

where matrix $\boldsymbol{B}_1$ has $z_1 + z_2$ columns. Thus

$$\boldsymbol{\gamma}_l = \boldsymbol{B}_l^T \boldsymbol{\sigma}^2, \ l = 1, 2, \quad (21)$$

and when $\boldsymbol{B}_2$ has linearly independent row vectors the same happens with the column vectors of $\boldsymbol{B}_2^T$ and, [7],

$$\begin{cases} \boldsymbol{\sigma}^2 = (\boldsymbol{B}_2^T)^+ \boldsymbol{\gamma}_2 \\ \boldsymbol{\gamma}_1 = \boldsymbol{B}_1^T (\boldsymbol{B}_2^T)^+ \boldsymbol{\gamma}_2 \end{cases} \quad (22)$$

So, we may estimate $\boldsymbol{\sigma}^2$ and $\boldsymbol{\gamma}_1$, through $\boldsymbol{\gamma}_2$. Then the relevant parameters $\boldsymbol{\sigma}^2$ and $\boldsymbol{\gamma}_2$, of the random effects part of the model, determine each other. This reveals that there is segregation since that part of the model segregates a sub-model.

If $z_1 = 0$, the z_2 columns of $\boldsymbol{B}_1$ and the z_2 first columns of $\boldsymbol{B}_2$ are identical, and the corresponding components of $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are also identical. This is called matching. Thus, estimating $\boldsymbol{\gamma}_2$ leads directly to estimate $\boldsymbol{\gamma} = [\boldsymbol{\gamma}_1^T \boldsymbol{\gamma}_2^T]^T$. Besides this, since the models have COBS, the LSE is UBLUE, [9], so we only considered in detail the estimation of the variance components, both canonical, $\gamma_1^0, \dots, \gamma_{m^0}^0$, $[\gamma_1, \dots, \gamma_w]$, and usually, $\sigma_1{}^2, \dots, \sigma_w{}^2$.

6 Joining Models

The matrices $\boldsymbol{M}_i = \boldsymbol{X}_i \boldsymbol{X}_i{}^T$, $i = 1, \dots, w$ express the structure of the model. Namely $\{\boldsymbol{M}_1, \dots, \boldsymbol{M}_w\}$ and $\{\boldsymbol{M}_0, \boldsymbol{M}_1, \dots, \boldsymbol{M}_w\}$ generate the relevant CJA A^0 and A. We thus say that models with the same matrices $\boldsymbol{M}_i$, $i = 1, \dots, w$, are iso-structured.

Let us now consider u independent observations vectors of iso-structured COBS, $\boldsymbol{Y}(1), \dots, \boldsymbol{Y}(u)$, and represent by $\boldsymbol{Q}_{\mathrm{j}}$, $j = 1, \dots, m$, the POOPM of $pb(A)$.

Applying the models joining operation to these models, by overlapping the observations vectors of the initial models, [8], we obtain a joined model, whose observations vector is

$$[\boldsymbol{Y}(1)^T \ \dots \ \boldsymbol{Y}(u)^T]^T, \quad (23)$$

and the CJAS $\otimes_{l=1}^{u} A_l$, given by the cartesian product of the CJAS $A_1, \dots, A_u$, with $pb(A_l) = \{\boldsymbol{Q}_{l,1}, \dots, \boldsymbol{Q}_{l,\mathrm{h}_l}\}$, $l = 1, \dots, u$, whose principal basis is constituted by the blockwise diagonal matrices with null principal blocks, but one which will belong to $pb(A_l)$, with l as its index. The null blocks will have the same dimension as the matrices of the corresponding CJAS, [8].

Given the Independence of $\boldsymbol{Y}(1), \dots, \boldsymbol{Y}(u)$, the covariance matrix of the joined model will be the blockwise diagonal matrix,

$$\mathrm{D}(V(\boldsymbol{\gamma}(1)), \dots, V(\boldsymbol{\gamma}(u))), \quad (24)$$

with principal blocks $V(\boldsymbol{\gamma}(1)), \dots, V(\boldsymbol{\gamma}(u))$, where $\boldsymbol{\gamma}(1), \dots, \boldsymbol{\gamma}(u)$ are the vectors of canonical variance components for $\boldsymbol{Y}(1), \dots, \boldsymbol{Y}(u)$. So, the estimators for variance components obtained for the individual models can be used for the joint model.

With $\boldsymbol{\beta}(1), \dots, \boldsymbol{\beta}(u)$ the vectors of coefficients and $\boldsymbol{X\beta}(1), \dots, \boldsymbol{X\beta}(u)$ the mean vectors of the u iso-structured (individual) COBS, for the joint model we will have the vector of coefficients,

$$\boldsymbol{\beta} = [\boldsymbol{\beta}(1)^T \dots \boldsymbol{\beta}(u)^T]^T \tag{25}$$

and the mean vector

$$\boldsymbol{\mu} = (\mathbf{1}_u \otimes \boldsymbol{X})\boldsymbol{\beta} \tag{26}$$

where $\otimes$ denotes the Kronecker matrix product. Thus, the orthogonal projection matrix on the space spanned by $\boldsymbol{\mu}$ will be,

$$\mathbf{T}^{[\mathrm{u}]} = \mathrm{D_u}(\mathbf{T}) = \begin{bmatrix} \mathbf{T} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{T} \end{bmatrix} = \mathrm{I_u} \otimes \mathbf{T}\,, \tag{27}$$

with $\mathbf{T}$ the orthogonal projection matrix on Ω.

As shown below, for the joined model, the covariance matrix, $\mathrm{D}\big(V(\boldsymbol{\gamma}(1)), \dots, V(\boldsymbol{\gamma}(u))\big)$, and the orthogonal projection matrix on the space spanned by the mean vector, $\mathbf{T}^{[\mathrm{u}]}$, commute,

$$\begin{aligned} &\boldsymbol{T}^{[u]} D\left(V(\boldsymbol{\gamma}(1)), \dots, V(\boldsymbol{\gamma}(u))\right) = \\ &= D\left(\boldsymbol{T}\, V(\boldsymbol{\gamma}(1)), \dots, \boldsymbol{T}\, V(\boldsymbol{\gamma}(u))\right) = \\ &= D\big(V(\boldsymbol{\gamma}(1))\, \boldsymbol{T}, \dots, V(\boldsymbol{\gamma}(u))\, \boldsymbol{T}\big) = \\ &= D\left(V(\boldsymbol{\gamma}(1)), \dots, V(\boldsymbol{\gamma}(u))\right) \boldsymbol{T}^{[u]} \end{aligned} \tag{28}$$

This means that the joined model is also COBS. As already established, this guarantees that the LSE of the estimable functions in this model will be UBLUE. Namely, we also will have

$$\begin{aligned} &\widetilde{\boldsymbol{\beta}}^{[u]} = \left(D_u(\boldsymbol{X})^T D_u(\boldsymbol{X})\right)^+ D_u(\boldsymbol{X})^T\, \boldsymbol{Y} = \\ &= D_u(\boldsymbol{X}^T\boldsymbol{X})^+\, D\big(\boldsymbol{X}^T\boldsymbol{Y}(1), \dots, \boldsymbol{X}^T\boldsymbol{Y}(u)\big) = \\ &= D_u((\boldsymbol{X}^T\boldsymbol{X})^+) D\big(\boldsymbol{X}^T\boldsymbol{Y}(1), \dots, \boldsymbol{X}^T\boldsymbol{Y}(u)\big) = \\ &= \left[\widetilde{\boldsymbol{\beta}}(1)^T \dots \widetilde{\boldsymbol{\beta}}(u)^T\right]^T \end{aligned} \tag{29}$$

with $\widetilde{\boldsymbol{\beta}}(h)$, $h = 1, \dots, u$, the LSE estimator of the coefficients vector of the h-th model.

Now the

$$\widetilde{\boldsymbol{\beta}}_j = \left(\boldsymbol{X}_0{}^{\mathrm{T}}\boldsymbol{X}_0\right)^+ \boldsymbol{X}_0{}^{\mathrm{T}}\boldsymbol{Y}_{\boldsymbol{j}}\,,\ j = 1, \dots, u$$

will be independent and BLUE for each model. Moreover if the $\boldsymbol{\beta}_j^\alpha$, $j = 1, \dots, u$, are linear unbiased statistics obtained from the initial models the

$$\boldsymbol{V}(\boldsymbol{\beta}_j^\alpha) - \boldsymbol{V}(\widetilde{\boldsymbol{\beta}}_j),\ j = 1, \dots, u$$

will be positive semi-definite, since $\widetilde{\boldsymbol{\beta}}_j$, $j = 1, \dots, u$, are BLUE. Now the $\boldsymbol{Y}(1), \dots, \boldsymbol{Y}(u)$ are independent, so the $\widetilde{\boldsymbol{\beta}}_1, \dots, \widetilde{\boldsymbol{\beta}}_u$ $[\boldsymbol{\beta}_1^\alpha, \dots, \boldsymbol{\beta}_u^\alpha]$ are independent with

$$\begin{cases} \widetilde{\boldsymbol{\beta}} = \left[\widetilde{\boldsymbol{\beta}}_1{}^T \dots \widetilde{\boldsymbol{\beta}}_u{}^T\right]^T \\ \boldsymbol{\beta}^\alpha = \left[\boldsymbol{\beta}_1^{\alpha T}, \dots, \boldsymbol{\beta}_u^{\alpha T}\right]^T \end{cases}$$

we have

$$\begin{cases} \boldsymbol{V}(\widetilde{\boldsymbol{\beta}}) = D(\widetilde{\boldsymbol{\beta}}_1 \dots \widetilde{\boldsymbol{\beta}}_u) \\ \boldsymbol{V}(\boldsymbol{\beta}^\alpha) = D(\boldsymbol{\beta}_1^\alpha, \dots, \boldsymbol{\beta}_u^\alpha) \end{cases}$$

Going over to the estimators of variance components we point out that the joined models, being iso-structured, have identical z_1 , z_2 and z_3 in $pb(A^0)$ as well as identical CJAS A and A^0. The variance components are distinct from model to model, so we can estimate them separately.

7 Conclusion

Models joining operation opens the possibility of jointly treating models obtained separately.

When dealing with models with commutative orthogonal block structure (COBS), we obtain uniformly best linear unbiased estimators for estimable functions of these joined models and estimate their variance components, showing that the estimators for the joint model may be obtained from those for the individual models.

Addressing iso-structured COBS, that is, models whose covariance matrices are linear combinations of the same POOPM, and their mean vectors span the same space, we have shown that joining iso-structured COBS gives COBS and that the estimators for the joint model may be obtained from those for the individual models. The estimators of variance components for the individual models can therefore be used for the joint model. Moreover, the BLUE for the vector $\boldsymbol{\beta}^{[u]} = [\boldsymbol{\beta}(1)^T \dots \boldsymbol{\beta}(u)^T]^T$ of coefficients for the joint model is $\widetilde{\boldsymbol{\beta}}^{[u]} = \left[\widetilde{\boldsymbol{\beta}}(1)^T \dots \widetilde{\boldsymbol{\beta}}(u)^T\right]^T$, where the sub-vectors are the LSE for the coefficients' vectors for the sub-models. Thus, the estimators obtained for sub-models can be applied to the joint model. Since joining COBS gives COBS, the optimality of LSE then extends from the individual models to the joint model.

Acknowledgment:
The authors are grateful to the reviewers for their insightful and constructive comments and valuable suggestions, which have helped improve this work.

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Contribution of Individuals to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This work is partially funded by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 (Center for Mathematics and Applications).

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.