On Four Dimensional Absolute Valued Algebras With nonzero Omnipresent Idempotent

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Abstract: In this paper, we studies the absolute valued algebras of dimension four, containing nonzero omnipresent idempotent. And we construct algebraically some news classes of algebras.

Key-Words: Absolute valued algebras, omnipresent idempotent, central idempotent.

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1 Introduction

An absolute valued algebra, is a nonzero real algebra, that is equipped with a multiplicative norm ($\|xy\| = \|x\|\|y\|$). These algebras have attracted the attention of many mathematicians, [3], [7], [8], [9], [10], [11], [12], [13], [14], [15]. In 1947 Albert, [1]. Proved that the finite dimensional unital absolute valued algebras are classified by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. And that every finite dimensional absolute valued algebra is isotopic to one of the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. And so has dimension 1, 2, 4 or 8, [1]. Note that, the norm $\|.\|$ of any finite-dimensional absolute valued algebras, comes from an inner product $(./.)$, [2]. Urbanik and Wright proved in 1960 that, all unital absolute valued algebras are classified by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, [4]. It is easily to seen that, the one-dimensional absolute valued algebras are classified by $\mathbb{R}$. And it is well-known that the two-dimensional absolute valued algebras, are isomorphic to, $\mathbb{C}$, $^*\mathbb{C}$, $\mathbb{C}^*$, or $\overset{*}{\mathbb{C}}$, [5]. The four-dimensional absolute valued algebras, have been described by M.I. Ramírez Álvarez in 1997, [6]. The problem of classifying all four (eight)-dimensional absolute valued algebras seems still to be open.

Motivated by these facts, we became interested in the study of four-dimensional absolute valued algebras, with a nonzero omnipresent idempotent. which generalizes the studies of M.L. El-Mellah, [3]. The classification of these algebras containing only one two-dimensional sub-algebra is still an open problem. We note that there are a four-dimensional absolute valued algebras, with left unit not containing a nonzero omnipresent idempotent, [6]. On the other hand the four-dimensional absolute valued algebra with a nonzero central idempotent, contains a subalgebras of dimension two. Which means that a central idempotent is an omnipresent idempotent . The reciprocal does not hold in general, and the counter-example is given (**remark 3.2**). From the comments below, it arises in a naturel way the following question: what is the classification of four-dimensional absolute valued algebras with a nonzero omnipresent idempotent and containing two different sub-algebras of dimension two?. This paper is devoted to shed some lighe on this problem.

In section 2, we introduce the basic tools for the study of four-dimensional absolute valued algebras, with a nonzero omnipresent idempotent, and containing two different sub-algebras of dimension two.

Moreover, In section 3, we introduce news classes of four-dimensional absolute valued algebras, with a nonzero omnipresent idempotent, namely $\mathbb{M}_1$, $\mathbb{M}_2$, $\mathbb{M}_3$, $\mathbb{M}_4$, $\overset{*}{\mathbb{M}}_1$, $\overset{*}{\mathbb{M}}_2$, $\overset{*}{\mathbb{M}}_3$, $\overset{*}{\mathbb{M}}_4$, $^*\mathbb{M}_1$, $^*\mathbb{M}_2$, $^*\mathbb{M}_3$, $^*\mathbb{M}_4$, $\mathbb{M}_1^*$, $\mathbb{M}_2^*$, $\mathbb{M}_3^*$ and $\mathbb{M}_4^*$.

In section 4, we classify algebraically, all four-dimensional absolute valued algebras, containing at least, two different subalgebras of dimension two.

In section 5, we summarize our study in the table.6.

2 Notations and Preliminary Results

Throughout this paper, the word algebra refers to a non-necessarily associative algebra, over the field of real numbers $\mathbb{R}$.

Definition 2.1 *Let A be an arbitrary algebra.*

i) A is called a normed algebra (resp, absolute valued algebra) if it's endowed with a space norm: $\|.\|$ such that $\|xy\| \leq \|x\|\|y\|$ (resp, $\|xy\| = \|x\|\|y\|$), for all $x, y \in A$.

ii) A is called a division algebra if, for all nonzero $a \in A$, the operators $L_a(x) = ax$ and $R_a(x) = xa$ (for all $x \in A$) of left and right multiplication by a are bijectives. Note that every finite-dimensional absolute valued algebra is a division algebra.

iii) We mean by a nonzero omnipresent idempotent, an idempotent which is contained in all two-dimensional sub-algebras of A.

iv) A(x,y) denote the sub-algebra of A generated by x, and y.

The most natural **examples** of absolute valued algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (the algebra of Hamilton quaternion), and $\mathbb{O}$ (the algebra of Cayley numbers). The algebras $^*\mathbb{C}$, $\mathbb{C}^*$, and $\overset{*}{\mathbb{C}}$ (obtained by endowing the space $\mathbb{C}$ with the products defined respectively by

$$x * y = \bar{x}y, \quad x * y = x\bar{y}, \quad and \quad x * y = \bar{x}\bar{y} \quad (P)$$

Where $x \to \bar{x}$ is the standard conjugation of $\mathbb{C}$.

We shall also denote by $^*\mathbb{H}$, $\mathbb{H}^*$, or $\overset{*}{\mathbb{H}}$ the real algebras obtained by endowing the space $\mathbb{H}$ with the products defined by (P) respectively, with $x \to \bar{x}$ is the standard conjugation of $\mathbb{H}$. The reader is referred to [11] for more informations of these classical absolute valued algebras.

We need the following results.

Theorem 2.2 *.[2]. The norm of any finite dimensional absolute valued algebra come from an inner product.*

Lemma 2.3 .[9]. *Every algebra in which $x^2 = 0$ only if $x = 0$. Contains a nonzero idempotent.*

Lemma 2.4 . *Let A be an absolute valued algebra of dimension $n \geq 2$, containing a nonzero central idempotent e, and let B a 2-dimensional sub-algebra of A. Then B contains a nonzero element orthogonal to e.*

Proof.. Let a, b be an orthonormal basis of B. Then there exists $\lambda, \beta \in \mathbb{R}$ and $u, v \in \{e\}^\perp$ such that $a = \lambda e + u$, $b = \beta e + v$. Now $\beta a - \lambda b = w \in B \setminus \{0\}$ and $w = \beta u - \lambda v \in \{e\}^\perp$

Lemma 2.5 . *Let A be a four-dimensional absolute valued algebra, containing a nonzero central idempotent f, then the following statements hold:*

i) A contains a 2-dimensional sub-algebra.

ii) $x^2 = -\|x\|^2 f$, for all $x \in \{f\}^\perp$.

iii) If $e \in A$ is another nonzero idempotent such that $e \neq f$, then the subalgebra $A(e, f)$ is isomorphic to $\overset{*}{\mathbb{C}}$.

Proof..

i) We can induce isometries from the commutative linear isometries L_f and R_f on the orthogonal space $\{e\}^\perp := E$ of dimension 3. So there exist common norm-one eigenvector $u \in E$ for both L_f and R_f associated to eigenvalues $\alpha, \beta \in \{-1, 1\}$. That is, $u^2 = -f$. Consequently $A(u, f)$ is a two-dimensional subalgebra of A.

ii) As A has an inner product space, we can assum that $\|x\| = 1$. We have

$$\|x^2 - f\| = \|x - f\|\|x + f\| = 2$$

That is $(x^2/f) = -1$, then $x^2 = -f$.

iii) As $e \neq f$. We have

$$\|e - f\| = \|e^2 - f^2\| = \|e - f\|\|e + f\|$$

That is $\|e + f\| = 1$, this imply $(e/f) = -\frac{1}{2}$. So

$$e + f + ef = 0$$

Consequently, $A(e, f)$ is isomorphic to $\overset{*}{\mathbb{C}}$.

Lemma 2.6 . *Let A be a four-dimensional absolute valued algebra containing 2-dimensional subalgebra B.*

1) If $x \in B^\perp$, then $x^2 \in B$.

2) If f is a nonzero central idempotent of A, then f is an omnipresent idempotent.

Proof.. By Rodríguez theorem, [5]. B is isomorphic to $\mathbb{C}$, $^*\mathbb{C}$, $\mathbb{C}^*$, or $\overset{*}{\mathbb{C}}$, and by lemma 2.3, B contains a nonzero idempotent e. We can set $B = A(e, i)$, where $ei = \pm i$, $ie = \pm i$, and $i^2 = \pm e$.

1) We have $\{e, i\}$ is an orthonormal basis of B, which can be extended to an orthonormal basis $F = \{e, i, j, k\}$ of A. Since L_j is bijective, there exist j_1, and j_2, such that

$$jj_1 = i \;\; and \;\; jj_2 = e$$

Let $x = aj + bk \in B^\perp$, we have

$$(j_1/e) = (jj_1/je) = (i/je) = \pm(ie/je) = \pm(i/j) = 0$$

And

$$(j_1/i) = (jj_1/ji) = (i/ji) = \pm(ei/ji) = \pm(e/j) = 0$$

Hence, $j_1 = \alpha j + \beta k$, likewise $j_2 = \alpha' j + \beta' k$. So we have

$$i = jj_1 = \alpha j^2 + \beta jk$$

and

$$e = jj_2 = \alpha' j^2 + \beta' jk$$

As $\alpha\beta' - \beta\alpha' = \pm 1$, then $j^2 \in B$, and $jk \in B$. Similarly we show that $k^2 \in B$ and $kj \in B$, so

$$x^2 = a^2 j^2 + b^2 k^2 + ab(jk + kj) \in B$$

2) Let x be a nonzero element of A orthogonal to f, so $x^2 = -\|x\|^2 f$ then $f = -\|x\|^{-2} x^2 \in B$ which mean that f is an omnipresent idempotent.

Remark 2.7 . *For any orthogonal two elements $x, y \in e^{\perp}$, we have $(xy/yx) = -(x^2/y^2)$.*

Proof. A simple linearisation of the identity $\|x^2\| = \|x\|^2$ give this result.

3 New class of four-dimensional absolute valued algebras with a nonzero omnipresent idempotent

In this paragraph we construct some news classes of four-dimensional absolute valued algebras with a nonzero omnipresent idempotent.

3.1 Construction of $\mathbb{M}_1$, $\mathbb{M}_2$, $\mathbb{M}_3$, and $\mathbb{M}_4$

Let $\{e, i, j, k\}$ be the orthonormal basis of the algebra $\mathbb{H}$ of quaternions with the usual multiplication table:

$Table.1.\quad \mathbb{H}$

	e	i	j	k
e	e	i	j	k
i	i	-e	k	-j
j	j	-k	-e	i
k	k	j	-i	-e

Let ϕ, ψ, Λ the linaer isometries of the euclidian space $\mathbb{H}$ whose matrices with respect to the canonical basis are given, respectively, by diag $\{1, 1, 1, -1\}$, diag $\{1, 1, -1, -1\}$, diag$\{1, 1, -1, 1\}$. We define news multiplications on the space $\mathbb{H}$.

$$x *_1 y = \phi(x)\phi(y)$$

$$x *_2 y = \phi(x)y$$

$$x *_3 y = x\phi(y)$$

$$x *_4 y = \psi(x)\Lambda(y)$$

we get new class of algebras with the multiplication tables defined respectively by:

$Table.2.\quad \mathbb{M}_1$

	e	i	j	k
e	e	i	j	-k
i	i	-e	k	j
j	j	-k	-e	-i
k	-k	-j	i	-e

$Table.3.\quad \mathbb{M}_2$

	e	i	j	k
e	e	i	j	k
i	i	-e	k	-j
j	j	-k	-e	i
k	-k	-j	i	e

$Table.4.\quad \mathbb{M}_3$

	e	i	j	k
e	e	i	j	-k
i	i	-e	k	j
j	j	-k	-e	-i
k	k	j	-i	e

$Table.5.\quad \mathbb{M}_4$

	e	i	j	k
e	e	i	-j	k
i	i	-e	-k	-j
j	-j	k	-e	-i
k	-k	-j	-i	e

Lemma 3.1 . *The algebras $\mathbb{M}_1$,$\mathbb{M}_2$,$\mathbb{M}_3$, and $\mathbb{M}_4$ are absolute valued algebras with omnipresent idempotent e.*

Proof. All these algebras are trivially absolute valued. We have also:

1. e is central idempotent for $\mathbb{M}_1$, so it's an omnipresent.

2. e is left-unit for algebra $\mathbb{M}_2$ so the only non zero idempotent. It belongs to all subalgebras of $\mathbb{M}_2$, [10]. So e is omnipresent.

3. e is a right-unit for algebra $\mathbb{M}_3$ so it is omnipresent.

4. Let B be a two dimensional sub-algebra of $\mathbb{M}_4$, then there exist an nonzero idempotent f and t in B, such that $(f/t) = 0$ and $t^2 = \pm f$. Using the basis $\{e, i, j, k\}$ there exists $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2 \in \mathbb{R}$ such that $f = \alpha_1 e + \beta_1 i + \gamma_1 j + \delta_1 k$ and $t = \alpha_2 e + \beta_2 i + \gamma_2 j + \delta_2 k$ We have

$$i^2 = j^2 = -e, \;\; k^2 = e, \;\; ie = ei = i$$

$$je = ej = -j, \;\; ke = -k, \;\; ek = k$$

And

$$ik = ki = -j, \;\; ij = -ji = -k, \;\; jk = kj = -i$$

Since $f^2 = f$, then

$$\begin{aligned} \alpha_1^2 - \beta_1^2 - \gamma_1^2 + \delta_1^2 &= \alpha_1 & (1)\\ 2\alpha_1\beta_1 - 2\gamma_1\delta_1 &= \beta_1 & (2)\\ -2\alpha_1\gamma_1 - 2\beta_1\delta_1 &= \gamma_1 & (3)\\ \delta_1 &= 0 & (4) \end{aligned}$$

As $\|f\| = 1$, then $\alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2 = 1$ also $\alpha_1^2 - \beta_1^2 - \gamma_1^2 - \delta_1^2 = \alpha_1$ (1), we get

$$2\alpha_1^2 - \alpha_1 - 1 = 0$$

thus $\alpha_1 = 1$ or $\alpha_1 = -\frac{1}{2}$.

(a) If $\alpha_1 = 1$, therefore $e = f \in B$

(b) If $\alpha_1 = -\frac{1}{2}$, then the equalities (2) and (4) give $\beta_1 = \delta_1 = 0$. So

$$f = -\frac{1}{2}e \pm \frac{\sqrt{3}}{2}j \in A(e, j)$$

On the other hand, we know that $t^2 = \pm f$ then

$$\begin{aligned} \alpha_2^2 - \beta_2^2 - \gamma_2^2 + \delta_2^2 &= \pm\alpha_1 & (5)\\ 2\alpha_2\beta_2 - 2\gamma_2\delta_2 &= 0 & (6)\\ -2\alpha_2\gamma_2 - 2\beta_2\delta_2 &= \pm\gamma_1 & (7)\\ \delta_2 &= 0 & (8) \end{aligned}$$

Since $\|t\| = \|f\| = 1$, then $\alpha_2^2 + \beta_2^2 + \gamma_2^2 + \delta_2^2 = 1$. The equalities (5)and (8) give $2\alpha_2^2 - 1 = \pm\alpha_1$, thus $\alpha_2 \neq 0$ and $\gamma_2 \neq 0$. Hence the equalities (6) and (8) imply that $\beta_2 = \delta_2 = 0$, that is, $t = \alpha_2 e + \gamma_2 j \in A(e, j)$. Therefore $B = A(e, j)$, so $e \in B$. As a result e is a nonzero omnipresent idempotent of $\mathbb{M}_4$.

Remark 3.2 . *e is a nonzero omnipresent idempotent for the algebras $\mathbb{M}_2$, $\mathbb{M}_3$ and $\mathbb{M}_4$ which isn't a central idempotent.*

3.2 Construction of the standard isotope of $\mathbb{M}_1$, $\mathbb{M}_2$, $\mathbb{M}_3$ and $\mathbb{M}_4$

Let $\mathbb{M}$ denote one of absolute valued algebras $\mathbb{M}_1$, $\mathbb{M}_2$, $\mathbb{M}_3$ or $\mathbb{M}_4$. We constructon the vectorial space of $\mathbb{M}$ by the news multiplications given respectively, by $x * y = \bar{x}\bar{y}$, $x * y = \bar{x}y$, $x * y = x\bar{y}$, where $x \to \bar{x}$ is the standard conjugation of $\mathbb{M}$. The algebras obtained called the standard isotopes of $\mathbb{M}$, and denoted respectively by $\overset{*}{\mathbb{M}}$, $^*\mathbb{M}$, $\mathbb{M}^*$.

Since the conjugation is an isometry, $\overset{*}{\mathbb{M}}$, $^*\mathbb{M}$, $\mathbb{M}^*$ are absolute valued algebras, As any two dimensional sub-algebra of $\mathbb{M}$ is invariant under conjugation, then e is also an omnipresent idempotent of these news algebras.

4 Main results

In this section, we assume that A is a four dimensional absolute valued algebra with omnipresent idempotent e and having at least two different subalgebras B_1 and B_2 of dimension two.
We have the following studies.

4.1 B_1 and B_2 are isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$

Proposition 4.1 . *If B_1 and B_2 are isomorphic to $\mathbb{C}$. Then A is isomorphic to $\mathbb{H}$, $\mathbb{M}_1$, $\mathbb{M}_2$ or $\mathbb{M}_3$.*

Proof. Let $B_1 = A(e, i)$ and $B_2 = A(e, j)$ be a two subalgebras of A isomorphic to $\mathbb{C}$, then we have

$$i^2 = j^2 = -e, \;\; ie = ei = i \;\; and \;\; je = ej = j$$

We know also $(e/i) = (e/j) = 0$, so without loss of generality we may assume that $(i/j) = 0$. indeed, if $(i/j) \neq 0$ then $t = \frac{j-(i/j)i}{\|j-(i/j)i\|}$ is orthogonal to e. Since $te = et = t$ and $\|e\| = \|t\| = 1$, we get $t^2 = -e$. Which implies that $A(e, t)$ is isomorphic to $\mathbb{C}$.
Now in A there exists an orthonormal subset $\{e, i, j\}$ which can be extended to an orthonormal basis $\{e, i, j, k\}$ for A. Since $k \in \{e, i, j\}^\perp$, then $k^2 \in A(e, i) \cap A(e, j) = \{e\}$ (lemma 2.6.(1)). We get $k^2 = \pm e$. But since

$$(ek/e) = (ek/e^2) = (e/k) = 0$$

$$(ke/e) = (ke/e^2) = (k/e) = 0$$

$$(ek/i) = (ek/ei) = (k/i) = 0$$

$$(ke/i) = (ke/ie) = (k/i) = 0$$

$$(ek/j) = (ek/ej) = (k/j) = 0$$

$$(ke/j) = (ke/je) = (k/j) = 0$$

we obtain $ek = \varepsilon k$ and $ke = \zeta k$, where $|\varepsilon| = |\zeta| = 1$. We conclude that $A(e, k)$ is two-dimensional subalgebra of A, that is $A(e, k)$ is isomorphic to $\mathbb{C}$, $^*\mathbb{C}$, $\mathbb{C}^*$ or $\overset{*}{\mathbb{C}}$. We distinguish the following cases:

1. If $A(e, k)$ is isomorphic to $\mathbb{C}$.
Then e will be the unit element of A and, therefore the multiplication of A is given by Table.1, so A is isomorphic to the quaternion $\mathbb{H}$.

2. If $A(e, k)$ is isomorphic to $\overset{*}{\mathbb{C}}$.
So $ke = ek = -k$ and $k^2 = -e$, since

$$(ij/e) = (ij/-i^2) = -(i/j) = 0$$

$$(ij/i) = (ij/ie) = (i/j) = 0$$

and

$$(ij/j) = (ij/j) = (i/j) = 0$$

Hence $ij = k$ or $ij = -k$. In a similar manner, we can show that

$$ik = j \;\; or \;\; ik = -j$$

and

$$jk = i \;\; or \;\; jk = -i$$

Assume that $ij = k$, in this case we have $ik = j$ and $jk = -i$. Indeed, if $ik = -j$, then

$$i(j + k) = k - j = -ek - ej = -e(k + j)$$

Which gives $i = -e$ (A has no zero divisors), contradiction. Also if $jk = i$, then

$$(i + j)k = j + i = (j + i)e$$

which implies $k = e$, absurd. Moreover, by remark 2.7, we have

$$(ij/ji) = -(i^2/j^2) = -1$$

which means that

$$\|ij + ji\|^2 = 0$$

So $ij = -ji$, and by the same way we have $ik = -ki$, and $jk = -kj$. Therefore, the multiplication of A is given by the Table.2, which mean that A is isomorphic to $\mathbb{M}_1$.

3. $A(e, k)$ is isomorphic to *$\mathbb{C}$.
We have $ek = k$, $ke = -k$ and $k^2 = e$. Using remark 2.7, we have $(ik/ki) = -(i^2/k^2) = 1$ which means that

$$\|ik - ki\|^2 = 0$$

So $ik = ki$, similarly, we get

$$jk = kj \;\; and \;\; ij = -ji$$

By simple calculations, we show that

$$ij = k \;\; or \;\; ij = -k$$

,

$$ik = j \;\; or \;\; ik = -j$$

and

$$jk = i \;\; or \;\; jk = -i$$

Assume that $ij = k$, in this case we have $ik = -j$ and $jk = i$. Indeed, if $ik = j$, then

$$i(j + k) = k + j = ek + ej = e(k + j)$$

Which gives $i = e$ (A has no zero divisors), contradiction. Also if $jk = -i$, then

$$(i + j)k = -j - i = -je - ie = -(j + i)e$$

which implies $k = -e$, absurd. So the multiplication of A is given by the Table.3, and A is isomorphic to $\mathbb{M}_2$.

4. $A(e, k)$ is isomorphic to $\mathbb{C}^*$,
We have $ek = -k$, $ke = k$ and $k^2 = e$. By remark 2.7, we get

$$ik = ki, \;\; jk = kj \;\; and \;\; ij = -ji$$

And by simple calculations, we show that

$$ij = k \;\; or \;\; ij = -k$$

,

$$ik = j \;\; or \;\; ik = -j$$

and

$$jk = i \;\; or \;\; jk = -i$$

Assume that $ij = k$, in this case we have $ik = j$ and $jk = -i$. Indeed, if $ik = -j$, then

$$i(j + k) = k - j = -ek - ej = -e(k + j)$$

This implies that $i = -e$ (A has no zero divisors), contradiction. Also if $jk = i$, then

$$(i + j)k = j + i = je + ie = (j + i)e$$

which implies $k = e$, absurd. Then the product of A is given by Table.4, So A isomorphic to $\mathbb{M}_3$.

Proposition 4.2 . *If B_1 and B_2 are isomorphic to $\overset{*}{\mathbb{C}}$. Then A is isomorphic to $\overset{*}{\mathbb{H}}$, $\overset{*}{\mathbb{M}}_1$, $\overset{*}{\mathbb{M}}_2$ or $\overset{*}{\mathbb{M}}_3$.*

Proof.. we define a new multiplication on A by $x*y = \bar{x}\bar{y}$, we obtain an algebra $\overset{*}{A}$ which contains two different subalgebras isomorphic to $\mathbb{C}$. Therefore, applying proposition 4.1, $\overset{*}{A}$ is isomorphic to $\mathbb{H}$, $\mathbb{M}_1$, $\mathbb{M}_2$ or $\mathbb{M}_3$. Consequently, A is isomorphic to $\overset{*}{\mathbb{H}}$, $\overset{*}{\mathbb{M}}_1$, $\overset{*}{\mathbb{M}}_2$ or $\overset{*}{\mathbb{M}}_3$.

4.2 B_1 isomorphic to $\mathbb{C}$ and B_2 isomorphic to $\overset{*}{\mathbb{C}}$

We assume that $B_1 = A(e, i)$ isomorphic to $\mathbb{C}$ and $B_2 = A(e, j)$ isomorphic to $\overset{*}{\mathbb{C}}$, we have

$$\|i + j\|^2 = \|e\|^2\|i + j\|^2 = \|ei + ej\|^2 = \|i - j\|^2$$

That is

$$2 + 2(i/j) = 2 - 2(i/j)$$

Hence $(i/j) = 0$.

Proposition 4.3 . *If B_1 is isomorphic to $\mathbb{C}$ and B_2 is isomorphic to $\overset{*}{\mathbb{C}}$. Then A is isomorphic to $\mathbb{M}_1$, $\mathbb{M}_4$, $\overset{*}{\mathbb{M}}_1$, or $\overset{*}{\mathbb{M}}_4$.*

Proof.. We can form an orthonormal basis $\{e, i, j, k\}$ of A. Since $k \in \{e, i, j\}^{\perp}$, then $k^2 \in A(e,i) \cap A(e,j) = \{e\}$ (lemma 2.6.(1)). We get $k^2 = \pm e$. But since

$$(ek/e) = (ek/e^2) = (e/k) = 0$$

$$(ke/e) = (ke/e^2) = (k/e) = 0$$

$$(ek/i) = (ek/ei) = (k/i) = 0$$

$$(ke/i) = (ke/ie) = (k/i) = 0$$

$$(ek/j) = (ek/-ej) = (k/j) = 0$$

$$(ke/j) = (ke/-je) = (k/j) = 0$$

We obtain $ek = \varepsilon k$ and $ke = \zeta k$, where $|\varepsilon| = |\zeta| = 1$. We conclude that $A(e,k)$ is two-dimensional subalgebra of A, that is $A(e,k)$ is isomorphic to $\mathbb{C}$, $^*\mathbb{C}$, $\mathbb{C}^*$ or $\overset{*}{\mathbb{C}}$. We have the following cases:

1. If $A(e,k)$ is isomorphic to $\mathbb{C}$, or $\overset{*}{\mathbb{C}}$.
   Then A has two different subalgebras isomorphic to $\mathbb{C}$ or isomorphic to $\overset{*}{\mathbb{C}}$. Hence A is isomorphic to $\mathbb{M}_1$, $\overset{*}{\mathbb{M}}_1$, (Proposition 4.1, and Proposition 4.2.).
2. If $A(e,k)$ is isomorphic to $^*\mathbb{C}$.
   We have $ek = k$, $ke = -k$ and $k^2 = e$. According to remark 2.7, we have $(ik/ki) = -(i^2/k^2) = (e/e) = 1$ which means that

   $$\|ik - ki\|^2 = 0$$

   So $ik = ki$, and similarly, we get

   $$jk = kj \ \ and \ \ ij = -ji$$

   We can also show that

   $$ij = k \ \ or \ \ ij = -k$$

   $$ik = j \ \ or \ \ ik = -j$$

   and

   $$jk = i \ \ or \ \ jk = -i$$

   If $ij = -k$, then $ik = -j$ and $jk = -i$. Indeed, if $ik = j$, then

   $$i(j + k) = -k + j = -ek - ej = -e(k + j)$$

   So $i = -e$ (Absurde).
   Also if $jk = i$, then

   $$(i + j)k = -j + i = je + ie = (j + i)e$$

   which implies $k = e$, absurd. So the multiplication of A is given by Table.5, and A is isomorphic to $\mathbb{M}_4$.
3. If $A(e,k)$ is isomorphic to $\mathbb{C}^*$,
   On A we can define a new algebra $\overset{*}{A}$ by the multiplication $x * y = \bar{x}\bar{y}$. Then $\overset{*}{A}$ contains three different subalgebras isomorphic to $\overset{*}{\mathbb{C}}$, $\mathbb{C}$ and $^*\mathbb{C}$ respectively. Hence the last result imply that $\overset{*}{A}$ is isomorphic to $\mathbb{M}_4$. So A is isomorphic to $\overset{*}{\mathbb{M}}_4$.

4.3 B_1 and B_2 are isomorphic to $^*\mathbb{C}$ or $\mathbb{C}^*$

We have the following results

Proposition 4.4 . *If B_1 and B_2 are isomorphic to $^*\mathbb{C}$. Then A is isomorphic to $^*\mathbb{H}$, $^*\mathbb{M}_1$, $^*\mathbb{M}_2$ or $^*\mathbb{M}_3$.*

Proof.. We define a new multiplication on A by $x * y = \bar{x}y$, we obtain an algebra $^*A$ which contains two different subalgebras isomorphic to $\mathbb{C}$. Therefore, applying proposition 4.1, $^*A$ is isomorphic to $\mathbb{H}$, $\mathbb{M}_1$, $\mathbb{M}_2$ or $\mathbb{M}_3$. Consequently, A is isomorphic to $^*\mathbb{H}$, $^*\mathbb{M}_1$, $^*\mathbb{M}_2$ or $^*\mathbb{M}_3$.

Proposition 4.5 . *If B_1 and B_2 are isomorphic to $\mathbb{C}^*$. Then A is isomorphic to $\mathbb{H}^*$, $\mathbb{M}_1{}^*$, $\mathbb{M}_2{}^*$ or $\mathbb{M}_3{}^*$.*

Proof.. We change the product of A by $x*y = x\bar{y}$, we get the algebra noted A^* which contains two different subalgebras isomorphic to $\mathbb{C}$. So by Proposition 4.1. A^* is isomorphic to $\mathbb{H}$, $\mathbb{M}_1$, $\mathbb{M}_2$ or $\mathbb{M}_3$. Which mean that A is isomorphic to $\mathbb{H}^*$, $\mathbb{M}_1{}^*$, $\mathbb{M}_2{}^*$, or $\mathbb{M}_3{}^*$.

4.4 B_1 isomorphic to $\mathbb{C}$ and B_2 isomorphic to $^*\mathbb{C}$

We can pose $B_1 = A(e,i)$, and $B_2 = A(e,j)$ isomorphic to $^*\mathbb{C}$, we have $(i/j) = 0$.

Proposition 4.6 . *If B_1 isomorphic to $\mathbb{C}$ and B_2 isomorphic to $^*\mathbb{C}$. Then A is isomorphic to $\mathbb{M}_2$, $\mathbb{M}_4$, $^*\mathbb{M}_2$, or $^*\mathbb{M}_4$.*

Proof.. We can construct an orthonormal basis $\{e, i, j, k\}$ of A. and we have by the same argument in the precedent case, $A(e,k)$ is two-dimensional subalgebra of A.

1. If $A(e,k)$ is isomorphic to $\mathbb{C}$, or $^*\mathbb{C}$.
   Then A has two different subalgebras isomorphic to $\mathbb{C}$ or isomorphic to $^*\mathbb{C}$. Hence A is isomorphic to $\mathbb{M}_2$, or $^*\mathbb{M}_2$ (Propositions 4.1, and Proposition 4.4).
2. If $A(e,k)$ is isomorphic to $\overset{*}{\mathbb{C}}$.
   By Proposition 4.3. A is isomorphic to $\mathbb{M}_4$
3. If $A(e,k)$ is isomorphic to $\mathbb{C}^*$.
   We considere the product $x * y = \bar{x}y$, we obtain an algebra $^*A$ which contains three different subalgebras isomorphic to $\overset{*}{\mathbb{C}}$, $\mathbb{C}$ and $^*\mathbb{C}$ respectively. Therefore, applying the last result, $^*A$ is isomorphic to $\mathbb{M}_4$. Consequently, A is isomorphic to $^*\mathbb{M}_4$.

4.5 B_1 isomorphic to $\mathbb{C}$ and B_2 isomorphic to $\mathbb{C}^*$

We pose $B_1 = A(e,i)$, and $B_2 = A(e,j)$. we always have $(i/j) = 0$. the identity

$$\|i+j\|^2 = \|e\|^2\|i+j\|^2 = \|ei+ej\|^2 = \|i-j\|^2$$

Give

$$2 + 2(i/j) = 2 - 2(i/j)$$

So $(i/j) = 0$.

Proposition 4.7 . *If B_1 isomorphic to $\mathbb{C}$ and B_2 isomorphic to $\mathbb{C}^*$. Then A is isomorphic to $\mathbb{M}_3$, ${\mathbb{M}_3}^*$, $^*\mathbb{M}_4$ or $\overset{*}{\mathbb{M}}_4$.*

Proof.. We have $(i/j) = 0$ so $\{e,i,j\}$ is an orthonormal family which can be extend to an orthonormal basis $\{e,i,j,k\}$ of A. Since $A(e,k)$ is two-dimensional subalgebra of A. We have the following cases:

1. If $A(e,k)$ is isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$.
   Then A has two different subalgebras isomorphic to $\mathbb{C}$ or isomorphic to $\mathbb{C}^*$. Hence A is isomorphic to $\mathbb{M}_3$,or ${\mathbb{M}_3}^*$ (Proposition 4.1, and Proposition 4.5).
2. If $A(e,k)$ is isomorphic to $\overset{*}{\mathbb{C}}$.
   By Proposition 4.3. A is isomorphic to $\overset{*}{\mathbb{M}}_4$.
3. If $A(e,k)$ is isomorphic to $^*\mathbb{C}$, then A is isomorphic to $^*\mathbb{M}_4$ (Proposition 4.6).

Remark 4.8 .

1. *If A has two subalgebras $B_1 = A(e,i)$, isomorphic to $\overset{*}{\mathbb{C}}$, and $B_2 = A(e,j)$ isomorphic to $^*\mathbb{C}$. We can define a new algebra $\overset{*}{A}$, with product $x * y = \bar{x}\bar{y}$. So $\overset{*}{A}$ contains two different subalgebras isomorphic to $\mathbb{C}$ and $\mathbb{C}^*$ respectively. So $\overset{*}{A}$ is isomorphic to $\mathbb{M}_3$, ${\mathbb{M}_3}^*$, $^*\mathbb{M}_4$, or $\overset{*}{\mathbb{M}}_4$ (Proposition 4.7). Consequently A is isomorphic to $\mathbb{M}_4$, , $\overset{*}{\mathbb{M}}_3$, $^*\mathbb{M}_3$, or ${\mathbb{M}_4}^*$.*
2. *if A has two subalgebras $B_1 = A(e,i)$ isomorphic to $\overset{*}{\mathbb{C}}$, and $B_2 = A(e,j)$ isomorphic to $\mathbb{C}^*$. We define a new multiplication on A by $x * y = \bar{x}\bar{y}$, and we obtain an algebra $\overset{*}{A}$, which contains two different subalgebras isomorphic to $\mathbb{C}$, and $^*\mathbb{C}$ respectively. By Proposition 4.6, $\overset{*}{A}$ is isomorphic $\mathbb{M}_2$, $\mathbb{M}_4$, $^*\mathbb{M}_2$, or $^*\mathbb{M}_4$. Hence A is isomorphic to $\overset{*}{\mathbb{M}}_2$, $\overset{*}{\mathbb{M}}_4$, ${\mathbb{M}_2}^*$ or ${\mathbb{M}_4}^*$.*

4.6 B_1 isomorphic to $^*\mathbb{C}$ and B_2 isomorphic to $\mathbb{C}^*$

Let's $B_1 = A(e,i)$, and $B_2 = A(e,j)$. We have

$$\|i+j\|^2 = \|i+j\|^2\|e\|^2 = \|ie+je\|^2 = \|-i+j\|^2$$

So

$$2 + 2(i/j) = 2 - 2(i/j)$$

Hence $(i/j) = 0$.

Proposition 4.9 *If B_1 isomorphic to $^*\mathbb{C}$ and B_2 isomorphic to $\mathbb{C}^*$. Then A is isomorphic to $^*\mathbb{M}_1$, , $^*\mathbb{M}_4$, ${\mathbb{M}_1}^*$,or ${\mathbb{M}_4}^*$*

Proof. We construct an orthonormal basis $\{e,i,j,k\}$ of A.:

1. If $A(e,k)$ is isomorphic to $\mathbb{C}^*$, or $^*\mathbb{C}$.
   Then A has two different subalgebras isomorphic to $^*\mathbb{C}$ or isomorphic to $\mathbb{C}^*$. Hence A is isomorphic to $^*\mathbb{M}_1$,or ${\mathbb{M}_1}^*$ (Propositions. 4.4, and Proposition. 4.5).
2. If $A(e,k)$ is isomorphic to $\mathbb{C}$, the result is a consequence of the Proposition. 4.6, thus A is isomorphic to $^*\mathbb{M}_4$.
3. If $A(e,k)$ is isomorphic to $\overset{*}{\mathbb{C}}$, the result is a consequence of the remark. 4.8. Hence A is isomorphic to ${\mathbb{M}_4}^*$.

Remark 4.10 *If $ij = -k$, we subistitute $-k = t$ we obtain $ij = t$, so we use the basis $\{e,i,j,t\}$, we again get the same classifications.*

5 Conclusion

In this section, we have the following main result.

Theorem 5.1 *Let A be a four dimensional absolute valued algebra with a nonzero omnipresent idempotent e, and having two different subalgebras B_1 and B_2 of dimension two. The following table specifies the isomorphisms classes.*

$Table.6. \quad All \quad classifications$

B_1	B_2	A
$\mathbb{C}$	$\mathbb{C}$	$\mathbb{H}$, $\mathbb{M}_1$, $\mathbb{M}_2$, $\mathbb{M}_3$
$\overset{*}{\mathbb{C}}$	$\overset{*}{\mathbb{C}}$	$\overset{*}{\mathbb{H}}$, $\overset{*}{\mathbb{M}}_1$, $\overset{*}{\mathbb{M}}_2$, $\overset{*}{\mathbb{M}}_3$
$\mathbb{C}$	$\overset{*}{\mathbb{C}}$	$\mathbb{M}_1$, $\mathbb{M}_4$, $\overset{*}{\mathbb{M}}_1$, $\overset{*}{\mathbb{M}}_4$
$^*\mathbb{C}$	$^*\mathbb{C}$	$^*\mathbb{H}$, $^*\mathbb{M}_1$, $^*\mathbb{M}_2$, $^*\mathbb{M}_3$
$\mathbb{C}^*$	$\mathbb{C}^*$	$\mathbb{H}^*$, $\mathbb{M}_1{}^*$, $\mathbb{M}_2{}^*$, $\mathbb{M}_3{}^*$
$\mathbb{C}$	$^*\mathbb{C}$	$\mathbb{M}_2$, $\mathbb{M}_4$, $^*\mathbb{M}_2$, $^*\mathbb{M}_4$
$\mathbb{C}$	$\mathbb{C}^*$	$\mathbb{M}_3$, $\mathbb{M}_3{}^*$, $^*\mathbb{M}_4$, $\overset{*}{\mathbb{M}}_4$
$\overset{*}{\mathbb{C}}$	$^*\mathbb{C}$	$\mathbb{M}_4$, $\overset{*}{\mathbb{M}}_3$, $^*\mathbb{M}_3$, $\mathbb{M}_4{}^*$
$\overset{*}{\mathbb{C}}$	$\mathbb{C}^*$	$\overset{*}{\mathbb{M}}_2$, $\overset{*}{\mathbb{M}}_4$, $\mathbb{M}_2{}^*$, $\mathbb{M}_4{}^*$
$^*\mathbb{C}$	$\mathbb{C}^*$	$^*\mathbb{M}_1$, , $^*\mathbb{M}_4$, $\mathbb{M}_1{}^*$,*or* $\mathbb{M}_4{}^*$

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