On Orthogonality according to an Index in Semi–normed Spaces

ARTUR STRINGA, ARBËR QOSHJA
Department of Mathematics, Department of Applied Mathematics,
University of Tirana,
Sheshi "Nënë Tereza", Tirana,
ALBANIA

Abstract: - In a recent study, we introduce the concept of orthogonality and transversality according to an index, obtaining some results on linear dependence and independence in semi-normed spaces. In this paper, we discuss the concept of orthogonality in semi-normed spaces.

Key-Words: semi-norm, semi-pre-inner product, orthogonality according to an index, orthogonality

Received: December 15, 2022. Revised: June 2, 2023. Accepted: June 23, 2023. Published: July 24, 2023.

1 Introduction

This paper derives from our previous work, [1]. Specifically, to carry over Hilbert space type arguments to the theory of Banach spaces, [3], constructed on a vector space a type of inner product, named semi–inner product (s.i.p), [9], with a more general axiom system that of Hilbert space, [4].

Definition 1.1
Let X be a real vector space. We say that a real semi-inner product (in short s.i.p.) is defined on X if for every there corresponds a real number and the following properties hold, [3]:

(1)(i) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(ii) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$, for $x, y, z \in X$ and $\lambda \in R$

(2) $\langle x, x\rangle > 0$, for $x \neq 0$

(3) $\langle x, y\rangle^2 \leq \langle x, x\rangle \langle y, y\rangle$

The pair $(X, \langle \cdot , \cdot\rangle)$ is a semi-inner product space (in short, s.i.p.s.).

A s.i.p.s. is a normed vector space with $\|x\| = \langle x, x\rangle^{1/2}$, [3].

For Lumer the importance of an s.i.p. space is that every normed vector space can be represented as an s.i.p. space so that the theory of operators on Banach space can be penetrated by Hilbert space arguments, [4].

Aiming to generalize condition (2) in the definition of s.i.p., we have introduced in [1], the semi-pre-inner product function, which is a generalization of the s.i.p. function's concept.

Definition 1.2
Let X be a real vector space. Consider a function

defined on $X \times X$ as follows, [2]:

$$X \times X \to R$$

$$(x, y) \mapsto [x, y].$$

If $[x, y]$ satisfies the conditions:

(1) $[x, x] \geq 0, x \in X$

(2) $[\lambda x, y] = \lambda[x, y]$, $\lambda \in R$ and $x, y \in X$

(3) $[x + y, z] = [x, z] + [y, z]$ $x, y \in X$

(4) $[x,y]^2 \leq [x,x]\,[y,y]$

then, we say that is a semi-pre-inner product on X (in short s.p.i.p.).

The pair $(X,[\cdot\,,\cdot])$ is called a semi-pre-inner product space (in short s.p.i.p.s.).

In [2], it proved that for every semi-norm function p in the vector space X, there is a s.p.i.p. $[\cdot\,,\cdot]$, such that $p^2(x) = [x\,,x],\ x \in X$.

Let X be a real vector space and $(X,\{p_\alpha\}_{\alpha\in\mathcal{A}})$ be a semi-normed space, where $\{p_\alpha\}_{\alpha\in\mathcal{A}}$ is a family of semi-norms on X and $\mathcal{A}$ is an index set. For every $\alpha \in \mathcal{A}$, let us denote by $[\cdot\,,\cdot]_\alpha$ the s.p.i.p., corresponding to the semi-norm p_α.

In [1], we defined an orthogonality relation in s.p.i.p. spaces as follows:

Definition 1.3

Let be $x, y \in X$ and $\alpha \in \mathcal{A}$. The vector x is called orthogonal according to the index α over the vector y, if $[y,x]_\alpha = 0$. In this case, the vector y is called transversal according to the index α over the vector x, [1].

Definition 1.4

Let be $x, y \in X$. The vector x is called orthogonal over the vector y, if the vector x is orthogonal according to every index $\alpha \in \mathcal{A}$ over the vector y. In this case, the vector y is called transversal over the vector x, [1].

Definition 1.5

Let $(V,\|\cdot\|)$ be a normed linear space. Denote by S the unit sphere in V. The normed space $(V,\|\cdot\|)$ is called Gâteaux differentiable [4], if for all $x, y \in S$ and real λ:

$$\lim_{\lambda\to 0}\frac{\|x+\lambda y\|-\|x\|}{\lambda} \text{ exists.}$$

Definition 1.6

Let $(X,\langle\cdot,\cdot\rangle)$ be a s.i.p. space. Denote by S the unit sphere in X. The s.i.p. space $(X,\langle\cdot,\cdot\rangle)$ is called a continuous s.i.p. space if for all $x,y \in S$ and real λ, [4]:

$$\lim_{\lambda\to 0}\langle y\,,x+\lambda y\rangle=\langle y\,,x\rangle.$$

Theorem 1.1

An s.i.p. space is a continuous s.i.p. space if and only if the norm is Gâteaux differentiable, [4].

Theorem 1.2

In a continuous s.i.p. space $(X,\langle\cdot,\cdot\rangle)$ x is normal to y, which is equivalent to $\langle x\,,y\rangle=0$, if and only if $\|x+\lambda y\|\geq\|x\|$ for all scalar λ, [4].

2 Main Results

Our first main concern is to define the class of continuous s.p.i.p. spaces. We will show that in such spaces the orthogonality [5] according to an index is a generalization to the orthogonality relation as studied by J. Gilles in **Theorem 1.2**. Also, we show that in continuous s.p.i.p. spaces, it holds similar results compared with the results of **Theorem 1.1**.

In the end, we will get some good results for orthogonality according to an index on separable semi-normed spaces, [6].

Definition 2.1

Let be $(X,[\cdot,\cdot])$ an s.p.i.p. space and p the semi-norm, corresponding to the s.p.i.p. $[\cdot,\cdot]$.

The s.p.i.p. space $(X,[\cdot,\cdot])$ is called a continuous s.p.i.p. space if the s.p.i.p. $[\cdot,\cdot]$ has the property:

For every $x,y \in X$, such that $p(x)=p(y)=1$, $\lim_{\lambda\to 0}[y, x+\lambda y] = [y, x]$.

Let X be a real vector space and $(X,\{p_\alpha\}_{\alpha\in\mathcal{A}})$ be a semi-normed space, where $\{p_\alpha\}_{\alpha\in\mathcal{A}}$ is a family of semi-norms on X and $\mathcal{A}$ is an index set. For every $\alpha\in\mathcal{A}$, let us denote by $[\cdot,\cdot]_\alpha$ the s.p.i.p., corresponding to the semi-norm p_α.

Theorem 2.1

Let be $\alpha\in\mathcal{A}$, such that the s.p.i.p. space $(X,[\cdot,\cdot]_\alpha)$ is a continuous s.p.i.p. space.

Let be $x, y \in X$, such that $p_\alpha(x) = p_\alpha(y) = 1$. The vector x is orthogonal according to the index α over the vector y, if and only if for every $\lambda \in R$, $p_\alpha(x) \le p_\alpha(x+\lambda y)$.

Proof: Let be $\alpha\in\mathcal{A}$, such that the s.p.i.p. space $(X,[\cdot,\cdot]_\alpha)$ is a continuous s.p.i.p. space.

Let be $x, y \in X$, such that $p_\alpha(x) = p_\alpha(y) = 1$.

Assume that for every $\lambda\in R$, $p_\alpha(x) \le p_\alpha(x+\lambda y)$.

For all, $\lambda \in R$, it's true that

$p_\alpha(x)\cdot p_\alpha(x+\lambda y) \le (p_\alpha(x+\lambda y))^2$.

On the other hand, using **(3)** and **(4)** properties of semi–pre–inner product, for every $\lambda\in R$ we have:

$$([x,x+\lambda y]_\alpha)^2 \le (p_\alpha(x))^2(p_\alpha(x+\lambda y))^2 \Rightarrow$$

$$|[x,x+\lambda y]_\alpha| \le p_\alpha(x)\cdot p_\alpha(x+\lambda y) \Rightarrow$$

$$[x,x+\lambda y]_\alpha \le (p_\alpha(x+\lambda y))^2 = [x,x+\lambda y]_\alpha + \lambda[y,x+\lambda y]_\alpha \Rightarrow$$

$$\lambda[y,x+\lambda y]_\alpha \ge 0.$$

From here it follows that:

if $\lambda > 0$, then $[y,x+\lambda y]_\alpha \ge 0$ and if $\lambda < 0$, then

$$[y,x+\lambda y]_\alpha \le 0.$$

It's true that:

$$[y,x]_\alpha = \lim_{\lambda\to 0}[y,x+\lambda y]_\alpha = \lim_{\lambda\to 0^-}[y,x+\lambda y]_\alpha = \lim_{\lambda\to 0^+}[y,x+\lambda y]_\alpha.$$

From the above equations, the following inequalities hold:

$$[y,x]_\alpha \le 0 \text{ and } [y,x]_\alpha \ge 0,$$

So, $[y,x]_\alpha = 0$. As a result, the vector **x** is orthogonal according to the index α over the vector **y**.

Assume that the vector **x** is orthogonal according to the index α over the vector **y**. For every, $\lambda\in R$ we have that:

$$p_\alpha(x+\lambda y)\cdot p_\alpha(x) \ge |[x+\lambda y,x]_\alpha| = |[x,x]_\alpha + \lambda[y,x]_\alpha| = \left|(p_\alpha(x))^2\right| = (p_\alpha(x))^2,$$

from which, forever $\lambda \in R$, it's held the inequality $p_\alpha(x) \leq p_\alpha(x+\lambda y)$.

Corollary 2.1

Let's assume that for every $\alpha \in \mathcal{A}$, the s.p.i.p. space $(X, [\cdot,\cdot]_\alpha)$ is a continuous s.p.i.p. space.

Let be $x, y \in X$, such that for every index $\alpha \in \mathcal{A}, p_\alpha(x) = p_\alpha(y) = 1$.
The vector x is orthogonal over the vector y, if and only if for every index $\alpha \in \mathcal{A}$ and for every $\lambda \in R$, $p_\alpha(x) \leq p_\alpha(x+\lambda y)$.

Theorem 2.2

Let be $\alpha \in \mathcal{A}$. The s.p.i.p. space $(X, [\cdot,\cdot]_\alpha)$ is a continuous s.p.i.p. space if and only if for every $x, y \in X$, such that $p_\alpha(x) = p_\alpha(y) = 1$:

$$\lim_{\lambda \to 0} \frac{p_\alpha(x+\lambda y) - p_\alpha(x)}{\lambda}$$

exists.

Proof: Let be $\alpha \in \mathcal{A}$.

Assume that the s.p.i.p. space $(X, [\cdot,\cdot]_\alpha)$ is a continuous s.p.i.p. space.

Since p_α is a semi–norm on X, we have that $p_\alpha^{-1}\{0\} = \{x \in X / p_\alpha(x) = 0\}$ is a closed subspace of the vector space X. We note that the relation:

$$x \sim y \Leftrightarrow (x - y) \in p_\alpha^{-1}\{0\}$$

is an equivalent relation in X. Let denote by X_α the quotient set $X/_{p_\alpha^{-1}\{0\}}$, with respect to this equivalence relation and by $\hat{x}$ an equivalence class with a representative x. The function:

$$\widehat{p_\alpha} : X_\alpha \to \mathbb{R}^+, \text{ such that for every}$$

$$\hat{x} \in X_\alpha, \widehat{p_\alpha}(\hat{x}) = p_\alpha(x),$$

is a norm in X_α, [2]. Then, by [3], there exists a s.i.p. on X_α:

$$\langle \cdot,\cdot \rangle_\alpha : X_\alpha \times X_\alpha \to R,$$

such that $\langle \hat{x}, \hat{x} \rangle_\alpha = \left(\widehat{p_\alpha}(\hat{x})\right)^2$, for every $\hat{x} \in X_\alpha$.
Let us consider the function:

$$[\cdot,\cdot]_\alpha : X \times X \to \mathbb{R}, \text{ such that}$$

$$[x, y]_\alpha = \langle \hat{x}, \hat{y} \rangle_\alpha, \text{ for every } (x,y) \in X \times X.$$

The above function is a s.p.i.p. function, [2].

Let be $x, y \in X$, such that $p_\alpha(x) = p_\alpha(y) = 1$, so $\widehat{p_\alpha}(\hat{x}) = \widehat{p_\alpha}(\hat{y}) = 1$. Since:

$$\lim_{\lambda \to 0} \langle \hat{y}, \hat{x} + \lambda \hat{y} \rangle_\alpha = \lim_{\lambda \to 0} [y, x+\lambda y]_\alpha = [y,x]_\alpha = \langle \hat{y}, \hat{x} \rangle_\alpha,$$

by [4], we have that the norm function $\widehat{p_\alpha} : X_\alpha \to R^+$ is Gâteaux differentiable, i.e.,

$$\lim_{\lambda \to 0} \frac{\widehat{p_\alpha}(\hat{x} + \lambda \hat{y}) - p_\alpha(\hat{x})}{\lambda}$$

exists.

On the other hand, since for every $\lambda \in R$ the following equations are true:

$$\widehat{p_\alpha}(\hat{x} + \lambda \hat{y}) = p_\alpha(x + \lambda y) \text{ and}$$

$$\widehat{p_\alpha}(\hat{x}) = p_\alpha(x),$$

it follows that it exists $\lim_{\lambda \to 0} \frac{p_\alpha(x+\lambda y) - p_\alpha(x)}{\lambda}$.

Let us assume now that exists $\lim_{\lambda \to 0} \frac{p_\alpha(x+\lambda y) - p_\alpha(x)}{\lambda}$, for every two points $x, y \in X$, such that $p_\alpha(x) = p_\alpha(y) = 1$. From here, it follows that exists $\lim_{\lambda \to 0} \frac{\widehat{p_\alpha}(\hat{x}+\lambda\hat{y}) - p_\alpha(\hat{x})}{\lambda}$, for every two points $x, y \in X$, such that $\widehat{p_\alpha}(\hat{x}) = \widehat{p_\alpha}(\hat{y}) = 1$. So, the norm function $\widehat{p_\alpha} : X_\alpha \to \mathbb{R}^+$ is Gâteaux differentiable. By [4], we have that the s.i.p. space $(X_\alpha, \langle \cdot, \cdot \rangle_\alpha)$ is a continuous s.i.p. space, from where it follows that $\lim_{\lambda \to 0} \langle \hat{y}, \hat{x} + \lambda \hat{y} \rangle_\alpha = \langle \hat{y}, \hat{x} \rangle_\alpha$.

On the other hand, since for all $\lambda \in R$ hold equations:

$$[y, x + \lambda y]_\alpha = \langle \hat{y}, \hat{x} + \lambda \hat{y} \rangle_\alpha \text{ and}$$

$$[y, x]_\alpha = \langle \hat{y}, \hat{x} \rangle_\alpha,$$

we get that for every two points $x, y \in X$, such that $p_\alpha(x) = p_\alpha(y) = 1$,

$$\lim_{\lambda \to 0} [y, x + \lambda y]_\alpha = [y, x]_\alpha.$$

Thus, the s.p.i.p. space $(X, [\cdot, \cdot]_\alpha)$ is a continuous s.p.i.p. space.

As the semi–normed space $(X, \{p_\alpha\}_{\alpha \in \mathcal{A}})$ can be considered filtered, [2], we have that $X_\alpha \subset X_\beta$, where $\alpha, \beta \in \mathcal{A}$, if $p_\alpha(x) \le p_\beta(x)$, for all $x \in X$.

Definition 2.2

Let be $x \in X$, $B \subset X$ and $\alpha \in \mathcal{A}$. The vector x is called orthogonal according to the index α over the set B, if the vector x is orthogonal according to the index α overall vectors $b \in B$.

In this case, the set B is called transversality, [7], [8], according to the index α over the vector x.

Definition 2.3

Let be $x \in X$ and $B \subset X$. The vector x is called orthogonal over the set B if the vector x is orthogonal according to every index $\alpha \in \mathcal{A}$ over the set B. In this case, the set B is called transversality over the vector x.

Definition 2.4

Let be $B, C \subset X$ and $\alpha \in \mathcal{A}$. The set B is called orthogonal according to the index α over the set C, if all vectors $b \in B$ are orthogonal according to the index α over the set C. In this case, the set C is called transversality according to the index α over the set B.

Definition 2.5

Let be $B, C \subset X$. The set B is called orthogonal over the set C, if the set B is orthogonal according to every index $\alpha \in \mathcal{A}$ over the set C. In this case, the set C is called transversality over the set B.

Let be $\alpha \in \mathcal{A}$ and $B \subset X$. Denote by $B_\alpha^\perp$ the set of all vectors $x \in X$, which are orthogonal according to the index α over the set B and denote by $B^\perp$ the set of all vectors $x \in X$, which are orthogonal over the set B. It is clear that $B^\perp = \bigcap_{\alpha \in \mathcal{A}} B_\alpha^\perp$.

Theorem 2.3

Let's assume that semi–normed space $(X, \{p_\alpha\}_{\alpha \in \mathcal{A}})$ is separable, i.e., we assume that this set satisfies the condition:

Let be $x \in X$ and $x \neq 0$, then there is an index $\alpha \in \mathcal{A}$, such that $p_\alpha(x) \neq 0$.

Let be $\alpha \in \mathcal{A}$ and $B \subset X$. The following inclusions are true:

(i) $B \cap B_\alpha^\perp \subset p_\alpha{}^{-1}\{0\}$;

(ii) $B \cap B^\perp = B \cap \{0\}$;

Proof: **(i)** Let be $x \in B \cap B_\alpha^\perp$. Then, $[x, x]_\alpha = 0$. From here it follows that $p_\alpha(x) = 0$, thus $x \in p_\alpha{}^{-1}\{0\}$.

(ii) Let be $x \in B \cap B^\perp$. Then, for all index $\alpha \in \mathcal{A}$, $x \in B \cap B_\alpha^\perp$. From **(i)** it follows that for every index $\alpha \in \mathcal{A}$, $p_\alpha(x) = 0$. Since the semi–normed space$(X, \{p_\alpha\}_{\alpha \in \mathcal{A}})$ is separable, we get $x = 0$. So, $x \in B \cap \{0\}$. On the other hand, it is clear $B \cap \{0\} \subset B \cap B^\perp$.

We note that if B is a linear subset of X, then $B \cap B^\perp = \{0\}$.

Theorem 2.4

Let $(X, \{p_\alpha\}_{\alpha \in \mathcal{A}})$ be a separable semi–normed space.

The following statements are equivalent:

(i) If for points $x, y \in X$ there is a $\alpha_0 \in \mathcal{A}$, such that $[x, y]_{\alpha_0} = 0$, then for every $\alpha \in \mathcal{A}$, we have that $[x, y]_\alpha = 0$.

(ii) If for two points $x, y \in X$, where $y \neq 0$, there is only one $\lambda \in R$, such that for every $\alpha \in \mathcal{A}$, we have that $[x + \lambda y, y]_\alpha = 0$.

Proof: (i) ⇒ (ii)

Let be $x, y \in X$, where $y \neq 0$. Since the semi–normed space $(X, \{p_\alpha\}_{\alpha \in \mathcal{A}})$ is separable, there is a $\alpha_0 \in \mathcal{A}$, such that $p_{\alpha_0}(y) \neq 0$. Denote $\lambda = -\frac{[x,y]_{\alpha_0}}{\left(p_{\alpha_0}(y)\right)^2}$. We have that:

$$[x+\lambda y,y]_{\alpha_0} = \left[x - \frac{[x,y]_{\alpha_0}}{\left(p_{\alpha_0}(y)\right)^2} y, y\right]_{\alpha_0} = [x,y]_{\alpha_0} - \frac{[x,y]_{\alpha_0}}{\left(p_{\alpha_0}(y)\right)^2}[y,y]_{\alpha_0} = 0.$$

Thus, exists $\alpha_0 \in \mathcal{A}$, such that $[x+\lambda y, y]_{\alpha_0} = 0$. Since the statement **(i)** is true, we take that for every $\alpha \in \mathcal{A}$, $[x+\lambda y, y]_\alpha = 0$.

Let's suppose that except the $\lambda \in R$, exists also another scalar $\mu \in R$, such that for every $\alpha \in \mathcal{A}$, $[x+\lambda y, y]_\alpha = 0$. In particular:

$$[x+\lambda y, y]_{\alpha_0} = 0.$$

We have:

$$[x+\lambda y,y]_{\alpha_0} = [x+\mu y,y]_{\alpha_0} \Leftrightarrow [(\lambda-\mu)y,y]_{\alpha_0} = 0 \Leftrightarrow (\lambda-\mu)\left(p_{\alpha_0}(y)\right)^2 = 0 \Leftrightarrow \lambda = \mu,$$

so, the scalar λ is unique.

(ii) ⇒ (i)

Let's assume that for point $x, y \in X$ exists an $\alpha \in \mathcal{A}$, such that $[x,y]_\alpha = 0$.

If $y = 0$, then for all $\alpha \in \mathcal{A}$, $[x,y]_\alpha = 0$.

Let's suppose that $y \neq 0$. Since the semi–normed space $(X, \{p_\alpha\}_{\alpha\in\mathcal{A}})$ is separable, then exists a $\alpha_0 \in \mathcal{A}$, such that $p_{\alpha_0}(y) \neq 0$. On the other hand, since the statement **(ii)** is true, we get that exists a unique scalar $\lambda \in R$, such that for every $\alpha \in \mathcal{A}$, $[x+\lambda y, y]_\alpha = 0$. In particular, $[x+\lambda y, y]_{\alpha_0} = 0$. We have:

$$[x+\lambda y,y]_{\alpha_0} = 0 \Leftrightarrow [x,y]_{\alpha_0} + \lambda\left(p_{\alpha_0}(y)\right)^2 = 0 \Leftrightarrow \lambda = 0.$$

For all $\alpha \in \mathcal{A}$, we have $[x,y]_\alpha = [x+\lambda y, y]_\alpha = 0$.

Theorem 2.5

Let $(X, \{p_\alpha\}_{\alpha\in\mathcal{A}})$ be a separable semi–normed space.
The following statements are equivalent:

(i) For every two points $x, y \in X$, there $y \neq 0$, exists a unique scalar $\lambda \in R$, such that for every $\alpha \in \mathcal{A}$, we have that $[x+\lambda y, y]_\alpha = 0$.

(ii) For every two points $x, y \in X$ and for every two indexes $\beta, \gamma \in \mathcal{A}$, have:

$$[x,y]_\beta\left(p_\gamma(y)\right)^2 = [x,y]_\gamma\left(p_\beta(y)\right)^2.$$

Proof: (i) ⇒ (ii)

Let be $x, y \in X$ and $\beta, \gamma \in \mathcal{A}$.

If $y = 0$, then it is clear

$$[x,y]_\beta\left(p_\gamma(y)\right)^2 = [x,y]_\gamma\left(p_\beta(y)\right)^2.$$

Let's suppose $y \neq 0$. Since the semi–normed space $(X, \{p_\alpha\}_{\alpha\in\mathcal{A}})$ is separable, then exists $\alpha_0 \in \mathcal{A}$, such that $p_{\alpha_0}(y) \neq 0$. On the other hand, since the statement **(i)** is true, we get that exists a unique scalar $\lambda \in R$, such that for every $\alpha \in \mathcal{A}$, we have $[x+\lambda y, y]_\alpha = 0$. For all $\alpha \in \mathcal{A}$, we get:

$$[x+\lambda y, y]_\alpha = 0 \Leftrightarrow [x,y]_\alpha = -\lambda (p_\alpha(y))^2.$$

We have:

$$\underbrace{[x,y]_\beta = -\lambda \left(p_\beta(y)\right)^2}_{(1)}, \quad \underbrace{[x,y]_\gamma = -\lambda \left(p_\gamma(y)\right)^2}_{(2)}$$

and

$$[x,y]_{\alpha_0} = -\lambda \left(p_{\alpha_0}(y)\right)^2 \Leftrightarrow \lambda = -\frac{[x,y]_{\alpha_0}}{\left(p_{\alpha_0}(y)\right)^2}$$

If $\lambda = 0$, then $[x,y]_\beta = [x,y]_\gamma = 0$, from where it follows:

$$[x,y]_\beta \left(p_\gamma(y)\right)^2 = [x,y]_\gamma \left(p_\beta(y)\right)^2.$$

Let suppose that $\lambda \neq 0$. We will prove that for every $\alpha \in \mathcal{A}$, $[x,y]_\alpha \neq 0$. Assume the contrary, so we assume that exists an $\mu \in \mathcal{A}$, such that $[x,y]_\mu = 0$. Since the statement **(i)** is true, based on **Theorem 2.3,** we will get that for every $\alpha \in \mathcal{A}$, $[x,y]_\alpha = 0$. In particular, $[x,y]_{\alpha_0} = 0$, so $\lambda = 0$, which contradicts the assumption $\lambda \neq 0$. Thus, for every $\alpha \in \mathcal{A}$, $[x,y]_\alpha \neq 0$. From here it follows that $[x,y]_\beta \neq 0$ and $[x,y]_\gamma \neq 0$. Moreover, we have that:

$$\left(p_\beta(y)\right)^2 \neq 0 \text{ and } \left(p_\gamma(y)\right)^2 \neq 0.$$

Dividing side by side the equations **(1)** and **(2)** we will get:

$$\frac{[x,y]_\beta}{[x,y]_\gamma} = \frac{\left(p_\beta(y)\right)^2}{\left(p_\gamma(y)\right)^2} \Leftrightarrow$$

$$[x,y]_\beta \left(p_\gamma(y)\right)^2 = [x,y]_\gamma \left(p_\beta(y)\right)^2.$$

(ii) ⇒ (i)

Let be $x, y \in X$, where $y \neq 0$. Assume that exists $\beta_0 \in \mathcal{A}$, such that $[x,y]_{\beta_0} = 0$.

Since the statement **(ii)** is true, for all $\gamma \in \mathcal{A}$ and $\gamma \neq \beta_0$, we have:

$$[x,y]_\gamma \left(p_{\beta_0}(y)\right)^2 = 0,$$

from where it follows $[x,y]_\gamma = 0$, for every $\gamma \in \mathcal{A}$ and $\gamma \neq \beta_0$, i.e., $[x,y]_\gamma = 0$, for all $\gamma \in \mathcal{A}$.

By **Theorem 2.4** we get that exists a unique scalar $\lambda \in R$, such that for all $\alpha \in \mathcal{A}$,

$$[x+\lambda y, y]_\alpha = 0.$$

3 Conclusions

This research grew out of our earlier work and our goal was to see what new advancements may be achieved for classes of s.p.i.p. spaces formed by placing additional constraints on the s.p.i.p. function. We defined the class of continuous s.p.i.p. spaces. We demonstrated that orthogonality according to an index in such spaces is a continuation of the orthogonality relation studied by J. Gilles. We also demonstrated that the continuity limitation on the s.p.i.p. function is equal to the norm's Gâteaux differentiability. In continuous s.p.i.p. spaces, we obtained similar findings to other outcomes. Finally, using an index on separable semi-normed spaces, we confirmed robust orthogonality discoveries.

References:

[1] Stringa A., Qoshja A., (2022) On Semi–Normed Spaces, via Semi–Pre–Inner Product and Orthogonality According to an Index, Journal of Multidisciplinary Engineering Science and Technology (JMEST), Vol. 9, Issue 12, December 2022, ISSN: 2458-9403.

[2] Stringa A., (2021) On Strictly Convex and Strictly Convex according to an index Semi-Normed Vector Spaces, General Mathematics Notes, Vol. 4, No 2, June 2011, pp.10–22, ISSN 2219–7184, Copyright ICSRS Publication, 2011, www.i-crs.org Available free online at http://www.geman.in.

[3] Lumer G., (1961) Semi-Inner-Product Spaces, Transactions of the American Mathematical Society, Vol. 100, No 1, 1961, pp. 29-43.

[4] Giles J.R., (1967) Classes of Semi-Inner-Product Spaces, Transactions of the American Mathematical Society, Volume 129, Number 3, 1967, pp. 436-446.

[5] J. A. Wheeler; C. Misner; K.S. Thorne (1973). Gravitation. W.H. Freeman & Co. p. 58. ISBN 0-7167-0344-0.

[6] Simon, J. (2017). Semi-normed Spaces. In Banach, Fréchet, Hilbert and Neumann Spaces, J.Simon (Ed.).

[7] R. Thom, "Un lemma sur les applications différentiables" Bol. Soc. Mat. Mex. , 1 (1956) pp. 59–71.

[8] S. Lang, "Introduction to differentiable manifolds" , Interscience (1967).

[9] J. B. Conway. A Course in Functional Analysis. 2nd Edition, Springer-Verlag, New York, 1990, page 1.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare.