A Posteriori Improvement in Projection Method

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Abstract: - This work is devoted to the refinement of the approximate solution, obtained by the projection method. The proposed approach uses expanding the design space by adding new coordinate functions. As a result, it is possible to clarify previously obtained solution using small computer resources. Applying this approach to the finite element method allows produce a local refinement of the mentioned solution. Suggested Approach illustrated in the finite element method for a boundary value problem second order in one-dimensional and two-dimensional cases.

Key-Words: - a posteriori improvement, refinement calculations, variational-grid methods

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1 Introduction

The research in the field of artificial intelligence covers many areas of the development of science and practice. These studies give opportunities to optimize efforts aimed at the effective solution of difficult problems. Here we give examples of some studies in the mentioned area.

Paper, [1], predicts the future of excitation energy transfer with artificial intelligence-based quantum dynamics.

In the paper, [2], the authors propose algorithm with fast convergence rate for federated edge learning.

In paper, [3], the authors propose a new algorithm of machine learning techniques. Based on recent advancements in music structure analysis, the authors of paper, [4], automate the evaluation process by introducing a collection of metrics.

The development of a disagreement-based online learning algorithm is given in [5].

In [6], the authors develop a deep learning-based approach to model.

Let us mention some works in which itis possible to effectively use learning systems and means of artificial intelligence.

The paper, [7], is devoted to the solution of the nonstationary integro-differential equation with a degenerate elliptic differential operator.

It seems that adaptive methods developed with the help of artificial intelligence greatly simplify the solution of problems of mathematical physics.

The article, [8], investigate the approximate solution to a nonlinear Volterra integro-differential equation.

In the article, [9], the authors have proposed a highly efficient and accurate collocation method.

It can also be assumed that the mentioned means would be useful in solving the problems considered in the works, [10], [11].

This paper is devoted to the posteriori improvement of the approximation in the projection method for solving a linear equation with a self-adjoint positive definite operator. Improvement was obtained by expanding the projection space. The mentioned extension is a linear span, an original projection space and an added element energy space. The consideration of a given priori parameterized class of elements that allow you to build an adaptive method with the mentioned improvement. The proposed approach is applied to the methods finite elements for one-dimensional boundary second order tasks.

The consistent application of this approach allows us to optimize the process of refining the numerical solution of the boundary value problem without significantly increasing the requirements for computing system resources. The proposed approach leads to the construction of adaptive sequences of ambient spaces. These spaces form a telescopic system of nested spaces, on the basis of which an adaptive wavelet packet is built for the economical transmission of information.

2 Background

Consider a Hilbert space H with a scalar product (u, v). Let A be a positively self-adjoint defined operator with domain $\mathcal{D}(A)$. Consider the energy space H_A of the operator A. We denote the norm and the scalar product in the space Y_A $\mathbf{I}u\mathbf{I}$ and $[u, v]$ respectively, so that $\mathbf{I}u\mathbf{I} = \sqrt{[u, u]}$.

Here $u, v \in H_A$. In the discussed conditions the solution of the problem

$$Au = f, \qquad f \in H \tag{1}$$

is equivalent to the solution of the next problem

$$\min_{u \in H_A} F(u), \tag{2}$$

$$F(u) = \mathbf{I}u\mathbf{I}^2 - 2(u, f). \tag{3}$$

Denote u_* the solution to problem (2) (and hence also the solution to problem (1)). Let S be a subspace of H_A. Consider the solution $\widetilde{u}_*$ to the problem

$$\min_{\widetilde{u} \in S} F(\widetilde{u}). \tag{4}$$

Thanks to the well-known representation

$$F(u) = \mathbf{I}u_* - u\mathbf{I}^2 - \mathbf{I}u_*\mathbf{I}^2 \tag{5}$$

problem (4) is equivalent to problem

$$\min_{\widetilde{u} \in S} \mathbf{I}u_* - \widetilde{u}\mathbf{I}, \tag{6}$$

so

$$\mathbf{I}u_* - \widetilde{u}_*\mathbf{I} \le \mathbf{I}u_* - \widetilde{u}\mathbf{I} \quad \forall \widetilde{u} \in S. \tag{7}$$

It is clear from (4) that $\widetilde{u}_*$ satisfies the identity

$$[\widetilde{u}_*, \widetilde{v}] = (f, \widetilde{v}) \quad \forall \widetilde{v} \in S. \tag{8}$$

Consider the element φ of the energy space H_A, assuming that $\varphi \notin S$. Let S_1 be the linear span of the space S and element φ,

$$S_1 \stackrel{\text{def}}{=} \mathcal{L}\{S, \varphi\}, \tag{9}$$

where $\mathcal{L}\{\dots\}$ means the linear span of the objects, which are in curly brackets.

Consider the function

$$\mathcal{F}(t) \stackrel{\text{def}}{=} F(\widetilde{u}_* + t\varphi). \tag{10}$$

Let us pose the question of finding the point t_* of the minimum of this function,

$$\min_{t \in \mathbb{R}^1} \mathcal{F}(t). \tag{11}$$

Since $\mathcal{F}(t)$ is a quadratic function and $\lim_{t \to \pm\infty} \mathcal{F}(t) = +\infty$, a desired point exists and it is unique. We have

$$F(\widetilde{u}_* + t\varphi) = \mathbf{I}\widetilde{u}_* + t\varphi\mathbf{I}^2 - 2(\widetilde{u}_* + t\varphi, f) =$$

$$= \mathbf{I}\widetilde{u}_*\mathbf{I}^2 + 2t[\widetilde{u}_*, \varphi] + t^2\mathbf{I}\varphi\mathbf{I}^2 - 2(\widetilde{u}_*, f) - 2t(\varphi, f). \tag{12}$$

Lemma 1. Representation

$$\mathcal{F}(t) = At^2 + 2Bt + C, \tag{13}$$

is right. Here

$$A = \mathbf{I}\varphi\mathbf{I}^2,\ B = [\widetilde{u}_*, \varphi] - (\widetilde{u}_*, f), \quad C = \mathbf{I}\widetilde{u}_*\mathbf{I}^2 - 2(\widetilde{u}_*, f). \tag{14}$$

Proof. Formulas (13) – (14) follow from relations (3), (9), and (12).

Lemma 2. Relations

$$t_* = \frac{(f, \varphi) - [\widetilde{u}_*, \varphi]}{\mathbf{I}\varphi\mathbf{I}^2}, \tag{15}$$

$$\mathcal{F}(t_*) = -\frac{\{(f, \varphi) - [\widetilde{u}_*, \varphi]\}^2}{\mathbf{I}\varphi\mathbf{I}^2} + F(\widetilde{u}_*). \tag{16}$$

are fulfilled.

Proof. Obviously, the minimum point t_* of quadratic form (13) has the form $t_* = -B/A$, and the value of this quadratic form at the mentioned point is equal to $\Phi(t_*) = -B^2/A + C$. Taking into account formulas (14), we obtain the equalities (15) and (16).

Theorem 1. A formula

$$F(\widetilde{u}_* + t_*\varphi) = F(\widetilde{u}_*) - \frac{\{(f, \varphi) - [\widetilde{u}_*, \varphi]\}^2}{\mathbf{I}\varphi\mathbf{I}^2} \tag{17}$$

is correct.

Proof. Formula (17) is obtained from relations (9) and (16).

Formula (17) means that for $\varphi \notin S$ the energy initial approximation $\widetilde{u}_*$ can be reduced by the transition to a new approximation $\widetilde{u}_* + t_*\varphi$. The inequality

$$\mathbf{I}\widetilde{u}_* + t_*\varphi - u_*\mathbf{I}^2 = \mathbf{I}\widetilde{u}_* - u_*\mathbf{I}^2 - E, \tag{18}$$

follows from (17). Here

$$E \stackrel{\text{def}}{=} \frac{\{[\widetilde{u}_*, \varphi] - (f, \varphi)\}^2}{\mathbf{I}\varphi\mathbf{I}^2}. \tag{19}$$

The element φ is called qualifying element, and the number E is local energetic clarification.

Remark 1. By selecting the element $\varphi \in H_A$, the energy approximation will be greatly improved. Thus, it is necessary to find

$$M \stackrel{\text{def}}{=} \max_{\varphi \in H_A} \frac{\{(f, \varphi) - [\widetilde{u}_*, \varphi]\}^2}{\mathbf{I}\varphi\mathbf{I}^2}.$$

Due to the relation $(f, \varphi) - [u_*, \varphi]$ we have

$$M = \max_{\varphi \in H_A,\, \varphi \neq 0} \frac{\{[u_* - \widetilde{u}_*, \varphi]\}^2}{\mathbf{I}\varphi\mathbf{I}^2}.$$

It is clear that

$$M = \max_{\psi \in H_A, \mathbf{I}\psi\mathbf{I} = 1} \{[u_* - \widetilde{u}_*, \psi]\}^2. \tag{20}$$

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It can now be seen that the maximum occurs at

$$\psi = \psi_* \stackrel{\text{def}}{=} \frac{u_* - \widetilde{u}_*}{\mathbf{I}u_* - \widetilde{u}_*\mathbf{I}},$$

so $M = \mathbf{I}u_* - \widetilde{u}_*\mathbf{I}^2$. In this way, the energy approximation is maximally improved if, in formula (17) takes $\varphi = \varphi_* \stackrel{\text{def}}{=} u_* - \widetilde{u}_*$. According to formula (15) for $\varphi = \varphi^*$ we have

$$t_* = \frac{(f, \varphi_*) - [\widetilde{u}_*, \varphi_*]}{\mathbf{I}\varphi\mathbf{I}^2} = \frac{[u_* - \widetilde{u}_*, \varphi_*]}{\mathbf{I}\varphi_*\mathbf{I}^2} = 1,$$

for $\varphi = \varphi_*$.

From formulas (5) and (17) we obtain the minimum value $-\mathbf{I}u_*\mathbf{I}^2$ of the functional $F(u)$ in space H_A, $F(\widetilde{u}_* + t_*\varphi_*) = -\mathbf{I}u_*\mathbf{I}^2$ and $\widetilde{u}_* + t_*\varphi_* = u_*$. The result obtained indicates the logical integrity. The approach is under consideration.

3 Refinement space

Consider in the energy space H_A also one finite-dimensional subspace Φ with the properties $\Phi \cap S = \{0\}$. Let $\varphi_1, \dots, \varphi_N$ be the basis of this space. Denote S_1 is the linear span of the spaces S and Φ

$$S_1 = \mathcal{L}\{S, \Phi\}. \tag{21}$$

In what follows, t means a N-dimensional vector, $t \in \mathbb{R}^N$, $t = (t_1, t_2, \dots, t_N)$. The symbol $\cdot$ is used to denote the linear combination of elements $\varphi_1, \dots, \varphi_N$, $t \cdot \varphi = \sum_{i=1}^N t_i \varphi_i$. Consider the function

$$\mathcal{F}(t) = F(\tilde{u}_* + t \cdot \varphi), \tag{22}$$

where $F(u)$ is the energy functional (3). Since $\mathcal{F}(t)$ is a quadratic function with a positive definite principal part, then there exists and uniquely problem solution t_*

$$\min_{t \in \mathbb{R}^N} \mathcal{F}(t). \tag{23}$$

We have

$$\begin{aligned} &F(\tilde{u}_* + t \cdot \varphi) = \\ &= \mathbf{I}\tilde{u}_* + t \cdot \varphi\mathbf{I}^2 - 2(\tilde{u}_* + t \cdot \varphi, f) = \\ &= \mathbf{I}\tilde{u}_*\mathbf{I}^2 - 2(\tilde{u}_*, f) + 2\sum_{i=1}^N \{[\tilde{u}_*, \varphi_i] - (\tilde{f}, \varphi_i)\} t_i + \\ &+ \sum_{i,j=1}^N [\varphi_i, \varphi_j] t_i t_j. \end{aligned} \tag{24}$$

Consider matrix A_0, vector B_0 and constant C_0 defined by equalities

$$A_0 = ([\varphi_i, \varphi_j])_{i,j=1}^N, B_0 = ([\tilde{u}_*, \varphi_i] - (f, \varphi_i))_{i=1}^N, \tag{25}$$

$$C_0 = \mathbf{I}\tilde{u}_*\mathbf{I}^2 - 2(\tilde{u}_*, f). \tag{26}$$

Lemma 3.

$$\mathcal{F}(t) = (A_0 t, t) + 2(B_0, t) + C_0, \tag{27}$$

where parentheses mean scalar preproduction to $\mathbb{R}^N$.

Proof. Formula (27) follows from relations (22) and (24) – (26).

Lemma 4.

$$t_* = A_0^{-1} B_0, \;\; \mathcal{F}(t_*) = C_0 - (B_0, A_0^{-1} B_0). \tag{28}$$

Proof. From the positive definiteness of the quadratic form (27) follows the existence and uniqueness points of minimum t_*. Equating to zero the partial derivatives with respect to t_i, $i = 1, 2, \dots, N$, we get t_* in the first relation (28). Substituting t_* into (27) gives the second of the relation (28).

Theorem 2.

$$F(\tilde{u}_* + t_* \cdot \varphi) = F(\tilde{u}_*) - (B_0, A_0^{-1} B_0). \tag{29}$$

Proof. Formula (29) is obtained from relations (21) and (25) – (28).

Formula (29) means that under the conditions considered, the energy initial approximation $\tilde{u}_*$ can be reduced by transition to a new approximation $\tilde{u}_* + t_* \cdot \varphi$. From (29) follows the relation

$$\mathbf{I}\tilde{u}_* + t_* \cdot \varphi - u_*\mathbf{I}^2 = \mathbf{I}\tilde{u}_* - u_*\mathbf{I}^2 - E, \tag{30}$$

where

$$E = (B_0, A_0^{-1} B_0), \tag{31}$$

while the matrix A_0 and the vector B_0 are defined by relations (25).

The element $t_* \cdot \varphi$ is called qualifying element, and the number E is local energetic clarification.

4 Triangulation and its refinement with stable boundary

We consider regular rectilinear triangulations of polygonal domains on the plane $\mathbb{R}^2$ (otherwise called triangular sets). Triangulation is considered regular if each edge is a side of some triangle, and the end of the edge cannot be an interior point of another rib. Proper triangulation allows for refinement, which are again regular triangulations.

Let $\mathcal{M}$ be a closed polygonal domain of the plane $\mathbb{R}^2$. We denote its boundary by $\partial\mathcal{M}$. Consider some triangulation $\mathcal{T}$ of $\mathcal{M}$. Denote the s-skeleton of the triangulation $\mathcal{T}$ by $\mathcal{T}^s$, $s \in$

$\{0,1,2\}$ (i.e. $\mathcal{T}^0$ is the set of vertices, $\mathcal{T}^1$ is the set of edges, and $\mathcal{T}^2$ is the set of triangles). We will assume that $\mathcal{T}^0 = \{v_i \,|\, i = 1, 2, \ldots, K\}$, $\mathcal{T}^1 = \{l_s \,|\, s = 1, 2, \ldots, L\}$, $\mathcal{T}^2 = \{T_j \,|\, j = 1, 2, \ldots, N\}$.

Triangulation $\mathcal{T}$ is also called a two-dimensional (triangular) complex. A set-theoretic closure union of points of triangles in the triangular complex $\mathcal{T}$ is called the body of the triangular complex. The triangular body complex is denoted by $|\mathcal{T}|$. It matches with area $\mathcal{M}$. The triangular body border complex $\mathcal{T}$ is the set $\partial\mathcal{M}$.

Sets $\partial\mathcal{T}^0 \stackrel{\text{def}}{=} \{v_i \,|\, v_i \in \mathcal{T}^0, v_i \subset \partial\mathcal{M}\}$, $\partial\mathcal{T}^1 \stackrel{\text{def}}{=} \{l_s \,|\, l_s \in \mathcal{T}^1, l_s \subset \partial\mathcal{M}\}$ with induced incidence relations generate a one-dimensional complex $\partial\mathcal{T}$ subdividing the boundary $\partial\mathcal{M}$ of the region under consideration $\mathcal{M}$. Complex $\partial\mathcal{T}$ is called the induced boundary complex. Its body coincides with the boundary $\partial\mathcal{M}$ of the domain $\mathcal{M}$.

Refining $\overline{\mathcal{T}}$ triangulation $\mathcal{T}$ is a new simplicial complex. If at the same time the induced boundary complex is preserved,

$$\partial\overline{\mathcal{T}} = \partial\mathcal{T}, \tag{32}$$

then the triangulation of $\overline{\mathcal{T}}$ is called refinement with a stable boundary for the triangulation $\mathcal{T}$.

In what follows, we associate the vertex v_i with the vector $\mathbf{r}_i$, emanating from the origin and ending at the mentioned vertex. In the case under consideration, a topological triangulation structure is described by an incidence matrix $M_{\mathcal{T}}$ of the size $3 \times N$, whose i-th line contains the vertices of the i-th triangle.

Remark 3. Changing the order of matrix rows $M_{\mathcal{T}}$ matches the renumbered triangles, and changing the order of the elements in any row of the matrix does not change the incidence correspondences between the triangle and vertices. Thus, a matrix derived from the original one by the mentioned transformations describes the old topological structure of the triangulation. We also note, that the duplication of the rows in the incidence matrix does not violate the topological structure of triangulation (because the "multiplicity" triangles are not considered).

The barycentric star of vertex $\mathbf{r} \in \mathcal{T}^0$ is denoted by $\mathcal{Z}(\mathbf{r})$. The closure for the union of the points of its triangles is called the body of the barycentric star. The body of a barycentric star is denoted by $|\mathcal{Z}(\mathbf{r})|$.

Two triangles are considered adjacent if they have a common edge. We will assume that each triangle T of the triangulation under consideration has s neighboring triangles, where $s \in \{1, 2, 3\}$. If $s = 3$ then triangle T is called the inner triangle of the triangulation $\mathcal{T}$, and for $s \in \{1, 2\}$ the triangle T is called boundary triangle.

The set of neighboring triangles for T with the addition of the given triangle T, we call the a star triangle T. The star of the triangle T will be denoted by $\mathcal{Z}(T)$. The set-theoretic union triangles from $\mathcal{Z}(T)$ is called the body of the star $\mathcal{Z}(T)$, and is denoted by $|\mathcal{Z}(T)|$. Note that both the barycentric vertex star and the star triangles are regular triangulations (triangular complexes). Therefore, it is possible to refine these triangulations.

Consider an interior triangle T with vertices $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ and its star $\mathcal{Z}(T)$. Select the fragment $M_{\mathcal{Z}(T)}$ of the matrix $M_{\mathcal{T}}$ related to to the star $\mathcal{Z}(T)$. In view of Remark 2 above, without loss of generality, we have

$$M_{\mathcal{Z}(T)} = \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_1 \\ \mathbf{r}_2 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_3 \\ \mathbf{r}_3 & \mathbf{r}_5 & \mathbf{r}_6 & \mathbf{r}_4 \end{pmatrix}^T . \tag{33}$$

Consider vector functions of the variable $\xi \in [0,1]$

$$\mathbf{r}_{12}(\xi) \stackrel{\text{def}}{=} \xi\mathbf{r}_1 + (1-\xi)\mathbf{r}_2, \mathbf{r}_{23}(\xi) \stackrel{\text{def}}{=} \xi\mathbf{r}_2 + (1-\xi)\mathbf{r}_3, \tag{34}$$

$$\mathbf{r}_{32}(\xi) \stackrel{\text{def}}{=} \xi\mathbf{r}_3 + (1-\xi)\mathbf{r}_1. \tag{35}$$

Theorem 3. If T is an interior triangle with vertices $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, then for a fixed parameter $\xi \in (0,1)$ triangulation $\mathcal{T}_{(\xi)}$ defined by incidence matrix

$$M_{(\xi)} \stackrel{\text{def}}{=} (V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9, V_{10})^T, \tag{36}$$

where

$$V_1 = \begin{pmatrix} \mathbf{r}_{12}(\xi) \\ \mathbf{r}_{23}(\xi) \\ \mathbf{r}_{31} \end{pmatrix}, V_2 = \begin{pmatrix} \mathbf{r}_2 \\ \mathbf{r}_{23}(\xi) \\ \mathbf{r}_{12}(\xi) \end{pmatrix},$$

$$V_3 = \begin{pmatrix} \mathbf{r}_3 \\ \mathbf{r}_{31}(\xi) \\ \mathbf{r}_{23}(\xi) \end{pmatrix}, V_4 = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_{12}(\xi) \\ \mathbf{r}_{31}(\xi) \end{pmatrix},$$

$$V_5 = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_5 \\ \mathbf{r}_{12}(\xi) \end{pmatrix}, V_6 = \begin{pmatrix} \mathbf{r}_{12}(\xi) \\ \mathbf{r}_5 \\ \mathbf{r}_2 \end{pmatrix},$$

$$V_7 = \begin{pmatrix} \mathbf{r}_2 \\ \mathbf{r}_6 \\ \mathbf{r}_{23}(\xi) \end{pmatrix}, V_8 = \begin{pmatrix} \mathbf{r}_3 \\ \mathbf{r}_{23}(\xi) \\ \mathbf{r}_6 \end{pmatrix},$$

$$V_9 = \begin{pmatrix} \mathbf{r}_3 \\ \mathbf{r}_4 \\ \mathbf{r}_{31}(\xi) \end{pmatrix}, V_{10} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_{31}(\xi) \\ \mathbf{r}_4 \end{pmatrix}.$$

$M_{(\xi)}$ is a refinement with a stable boundary for triangulation $\mathcal{Z}(T)$. In addition, the relations

$$\lim_{\xi\to+0}\mathcal{T}_{(\xi)}=\lim_{\xi\to 1-0}\mathcal{T}_{(\xi)}=\mathcal{Z}(T). \tag{37}$$

Proof. For fixed $\xi\in(0,1)$ formulas (34) – (35) define points lying inside the edges (sides) of triangle T. Taking into account the structure of the matrix (36), we arrive at the conclusion that the simplicial complex $\mathcal{T}_{(\xi)}$ is the subdivision of the star $\mathcal{Z}(T)$. Relation (32) is obvious, and therefore the subdivision $\mathcal{T}_{(\xi)}$ is the refinement with a stable boundary for $\mathcal{Z}(T)$. Using Remark 1 and the formulas $\mathbf{r}_{12}(0)=\mathbf{r}_2$, $\mathbf{r}_{12}(1)=\mathbf{r}_1$, $\mathbf{r}_{23}(0)=\mathbf{r}_3$, $\mathbf{r}_{23}(1)=\mathbf{r}_2$, $\mathbf{r}_{32}(0)=\mathbf{r}_2$, $\mathbf{r}_{32}(1)=\mathbf{r}_3$, we derive at relations (37).

Remark 3. Similar assertions hold for boundary triangles.

5 Energy refinement for the finite elements in a two-dimensional problem

Triangulation refinement while maintaining less correctness can be obtained by the successive application of elementary grinding. Elementary grinding is the grinding in which one vertex with corresponding edges is added. Two types of elementary grindings exist.

The first type of elementary grinding is as follows. A pair of adjacent triangles with vertices $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$ after adding $\mathbf{r}_\xi$ to the edge $(\mathbf{r}_1,\mathbf{r}_3)$ and after adding edges $(\mathbf{r}_2,\mathbf{r}_\xi)$ and $(\mathbf{r}_4,\mathbf{r}_\xi)$ becomes four triangles according to display

$$\begin{pmatrix}\mathbf{r}_1 & \mathbf{r}_1\\ \mathbf{r}_2 & \mathbf{r}_3\\ \mathbf{r}_3 & \mathbf{r}_4\end{pmatrix}^T \longrightarrow \begin{pmatrix}\mathbf{r}_1 & \mathbf{r}_\xi & \mathbf{r}_\xi & \mathbf{r}_1\\ \mathbf{r}_2 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_\xi\\ \mathbf{r}_\xi & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_4\end{pmatrix}^T.$$

Let us describe the second type of elementary grinding. In this case the vertex $\mathbf{r}_\xi$ is placed inside the triangle and is connected to the edges with its vertices $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ according to the mapping

$$(\mathbf{r}_1\quad \mathbf{r}_2\quad \mathbf{r}_2)\longrightarrow\begin{pmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_\xi\\ \mathbf{r}_\xi & \mathbf{r}_2 & \mathbf{r}_3\\ \mathbf{r}_1 & \mathbf{r}_\xi & \mathbf{r}_3\end{pmatrix}.$$

Both types of elemental refinements retain triangulation correctness.

Let $\Omega_{\mathbf{r}_0}(\mathbf{r})$ be the Courant function for the vertex $\mathbf{r}_0=(x_0,y_0)$,

$$\Omega_{\mathbf{r}_0}(\mathbf{r})=\begin{cases}1-\Delta_T(\mathbf{r})/\Delta_T^0 \\ \qquad \text{for } \mathbf{r}\in|T|,\ \forall T\in\mathcal{Z}(\mathbf{r}_0),\mathbf{r}_0\in|T|,\\ 0\\ \qquad \text{for } \mathbf{r}\notin|\mathcal{Z}(\mathbf{r}_0)|.\end{cases}$$

Here $\mathbf{r}_0=(x_0,y_0)$, $\mathbf{r}_1=(x_1,y_1)$, $\mathbf{r}_2=(x_2,y_2)$ are vertices of triangle T, $\mathbf{r}=(x,y)$,

$$\Delta_T(\mathbf{r})=\begin{vmatrix}x-x_0 & y-y_0\\ x_2-x_1 & y_2-y_1\end{vmatrix},$$

$$\Delta_T^0=\begin{vmatrix}x_1-x_0 & y_1-y_0\\ x_2-x_0 & y_2-y_0\end{vmatrix}.$$

We associate each vertex $\mathbf{r}_i$ from the zero-dimensional spanning tree $\mathcal{T}^0$ with the Courant function $\Omega_{\mathbf{r}_i}(\mathbf{r})$. These functions are a linear independent system of functions. Consider the space $\mathcal{S}$ of the linear combinations of these functions,

$$\mathcal{S}\stackrel{\text{def}}{=}\{\widetilde{u}\mid \widetilde{u}=\sum_{\mathbf{r}_i\in\mathcal{T}^0}v_i\Omega_{\mathbf{r}_i},\ v_i\in\mathbb{R}^1\}.$$

$\{\Omega_{\mathbf{r}_i}\}_{\mathbf{r}_i\in\mathcal{T}^0}$ system we call the standard basis of Courant spaces.

Elementary triangulation refinement adds vertex $\mathbf{r}_\xi$ in triangulation, which allows us to consider a new function $\Omega_{\mathbf{r}_\xi}$ chime corresponding to the added top. It is clear that the added function is linearly independent with the previous ones.

Consider the linear span $\mathcal{S}_1$ of the functions $\{\Omega_{\mathbf{r}_i}\}_{\mathbf{r}_i\in\mathcal{T}^0}$ and $\Omega_{\mathbf{r}_\xi}$. It is clear that the result will be the extension of the space $\mathcal{S}$, $\mathcal{S}\subset\mathcal{S}_1$. It is easy to see that the set of the features just listed are not the standard Courant basis for the space $\mathcal{S}_1$. To go to the standard basis, you need to take into account the change structure of the barycentric stars, which is local in nature.

Consider an elementary refinement of triangulation. Let us add a new vertex $\mathbf{r}_\xi$ (together with the corresponding edges). We introduce the function $\varpi_{\mathbf{r}_\xi}$ chime corresponding to the added vertex. It is clear that this function is linearly independent with respect to system $\{\varpi_{\mathbf{r}_i}\}_{\mathbf{r}_i\in\mathcal{T}^0}$. Considering the linear span of the set of functions $\{\varpi_{\mathbf{r}_i}\}_{\mathbf{r}_i\in\mathcal{T}^0}\cup\{\varpi_{\mathbf{r}_\xi}\}$, we obtain a linear space $\mathcal{S}_1$ containing the space $\mathcal{S}$, $\mathcal{S}\subset\mathcal{S}_1$.

It is easy to see that the set of functions $\{\varpi_{\mathbf{r}_i}\}_{\mathbf{r}_i\in\mathcal{T}^0}\cup\{\varpi_{\mathbf{r}_\xi}\}$ are not the standard Courant basis for the space $\mathcal{S}_1$. The transition to the standard basis is obtained by a linear transformation that takes into account the change structures of the barycentric stars. The said transition is local in the sense that it affects only those Courant functions whose support has a non-empty intersection with the support of the added function. Later on, the transition to a standard basis is not required.

6 Energy refinement for the finite elements in a two-dimensional problem

Consider the quadratic functional

$$F(u) \overset{\text{def}}{=} \int_\Omega \Big(\sum_{i,j=1}^{2} p_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \\ + q(x)u^2 - 2u(x)f(x) \Big) dx, \tag{38}$$

where $x = (x_1, x_2)$, $q(x) > 0$, $a_{ij}(x) = a_{ji}(x)$,

$$\sum_{i,j=1}^{2} p_{ij}(x)\zeta_i\zeta_j \geq \gamma \sum_{i\in\{1,2\}} \zeta_i^2, \gamma = const > 0 \\ \forall \zeta_s \in \mathbb{R}^1, s = 1,2. \tag{39}$$

The functions $p_{ij}(x)$, $q(x)$ are measurable and bounded. Functional $F(u)$ is defined on the space $W_2^1(\Omega)$. As is known, the solution of a number of boundary value problems for differential equations

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(p_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + q(x)u = f(x), \quad x \in \Omega \tag{40}$$

is reduced to the minimization of the functional on the corresponding linear manifold of the space $W_2^1(\Omega)$. Its subspace is often called the energy space and is denoted by H_A, and the bilinear form

$$[u, v] \overset{\text{def}}{=} \int_\Omega \Big(\sum_{i,j=1}^{2} p_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + q(x)uv \Big) dx. \tag{41}$$

turns out to be an inner product in H_A. Denote u_* the solution of the problem

$$\min_{u\in H_A} F(u). \tag{42}$$

Let S be a subspace in H_A, P_S be orthoprojector of the space H_A onto the subspace S, and

$$\widetilde{u}_* \overset{\text{def}}{=} P_S u_* \tag{43}$$

be the approximate solution of problem (39).

Let us assume that the next condition is fulfilled (A). The closure of Ω contains a polygonal domain $\mathcal{M}$ with triangulation $\mathcal{T}$. The approximate solution (42) of problem (38) on polygon $\mathcal{M}$ is a linear combination Courant functions,

$$\widetilde{u}_*(x)\Big|_{x\in\mathcal{M}} = \sum_{\mathbf{r}_i\in\mathcal{T}^0} v_i^* \varpi_{\mathbf{r}_i}(\mathbf{r}). \tag{44}$$

Under condition (A) we have

$$F(\widetilde{u}^*) = \int_{\Omega\setminus\mathcal{M}} \mathcal{G}(x; \widetilde{u}^*(x))dx + \\ + \sum_{T\in\mathcal{T}^2} \int_{|T|} \mathcal{G}(x; \widetilde{u}^*(x))dx, \tag{45}$$

where

$$\mathcal{G}(x; u) = \sum_{i,j=1}^{2} p_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + q(x)u^2 - 2f(x)u. \tag{46}$$

Let T_1 and T_2 be two neighboring triangles, $T_1, T_2 \in \mathcal{T}^2$ defined by the matrix

$$\begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \\ \mathbf{r}_1 & \mathbf{r}_3 & \mathbf{r}_4 \end{pmatrix}. \tag{47}$$

Let us perform an elementary refinement (5.1). Consider the linear span of the space S and the coordinate span of the Courant function $\varpi_{\mathbf{r}_\xi}(\mathbf{r})$.

According to formulas (25) we have $N = 1$

$$A_0 = [\varphi_{\mathbf{r}_\xi}, \varphi_{\mathbf{r}_\xi}], B_0 = [\tilde{u}_*, \varphi_{\mathbf{r}_\xi}] - (f, \varphi_{\mathbf{r}_\xi}), \tag{48}$$

so that the decrease in energy in this case has the form

$$E = \frac{|[\tilde{u}_*, \varphi_{\mathbf{r}_\xi}] - (f, \varphi_{\mathbf{r}_\xi})|^2}{[\varphi_{\mathbf{r}_\xi}, \varphi_{\mathbf{r}_\xi}]}.$$

In this case we have

$$\varpi_{\mathbf{r}_\xi}(\mathbf{r}) = \begin{cases} \Delta_{\widetilde{T}_i,\mathbf{r}_\xi}(\mathbf{r})/\Delta^0_{\widetilde{T}_i,\mathbf{r}_\xi} \\ \qquad \text{for } \mathbf{r} \in |\widetilde{T}_i|, \ i = 1,2,3,4, \\ 0 \\ \qquad \text{for } \mathbf{r} \notin |\mathcal{Z}(\mathbf{r}_\xi)|, \end{cases} \tag{49}$$

where

$$\mathbf{r}_\xi = (x_\xi, y_\xi), \mathbf{r}_i = (x_i, y_i), \mathbf{r}_{i+1} = (x_{i+1}, y_{i+1})$$

are vertices of the triangle $\widetilde{T}_i$, and $\mathbf{r} = (x, y)$,

$$\Delta_{\widetilde{T}_i,\mathbf{r}_\xi}(\mathbf{r}) = \begin{vmatrix} x_i - x & y_i - y \\ x_{i+1} - x_i & y_{i+1} - y_i \end{vmatrix}, \\ \Delta^0_{\widetilde{T}_i,\mathbf{r}_\xi} = \begin{vmatrix} x_i - x_\xi & y_i - y_\xi \\ x_{i+1} - x_\xi & y_{i+1} - y_\xi \end{vmatrix}. \tag{50}$$

Hence, for $(x,y)\in|\widetilde{T}_i|$, $i=1,2,3,4$, we get

$$\frac{\partial\varpi_{\mathbf{r}_\xi}}{\partial x}=-\frac{y_{i+1}-y_i}{\Delta^0_{\widetilde{T}_i,\mathbf{r}_\xi}},\quad \frac{\partial\varpi_{\mathbf{r}_\xi}}{\partial y}=\frac{x_{i+1}-x_i}{\Delta^0_{\widetilde{T}_i,\mathbf{r}_\xi}}, \tag{51}$$

$$\Delta^0_{\widetilde{T}_i,\mathbf{r}_\xi}=\begin{vmatrix} x_i-x_\xi & y_i-y_\xi\\ x_{i+1}-x_i & y_{i+1}-y_i\end{vmatrix}.$$

According to formulas (25) we have $N=1$

$$A_0=[\varpi_{\mathbf{r}_\xi},\varpi_{\mathbf{r}_\xi}], B_0=[\tilde{u}_*,\varpi_{\mathbf{r}_\xi}]-(f,\varpi_{\mathbf{r}_\xi}), \tag{52}$$

so that the decrease in energy in this case has the form

$$E=\frac{|[\tilde{u}_*,\varpi_{\mathbf{r}_\xi}]-(f,\varpi_{\mathbf{r}_\xi})|^2}{[\varpi_{\mathbf{r}_\xi},\varpi_{\mathbf{r}_\xi}]}.$$

Our task is to evaluate the expressions

$$[\varpi_{\mathbf{r}_\xi},\varpi_{\mathbf{r}_\xi}],\quad [\tilde{u}_*,\varpi_{\mathbf{r}_\xi}],\quad (f,\varpi_{\mathbf{r}_\xi}).$$

For brevity, we confine ourselves to the case

$$q(x,y)\Big|_{(x,y)\in|T_1|\cup|T_2|}=0.$$

So we have

$$[\varpi_{\mathbf{r}_\xi},\varpi_{\mathbf{r}_\xi}]= =\sum_{i=1}^{4}\frac{1}{4mes^2|\widetilde{T}_i|}\int_{|\widetilde{T}_i|}\Big\{p_{11}(y_{i+1}-y_i)^2+ +p_{22}(x_{i+1}-x_i)^2\Big\}dxdy. \tag{53}$$

For $\mathbf{r}\in|T_j|$, $j=1,2$, we have

$$\varpi_{\mathbf{r}_1}(\mathbf{r})=\Delta_{T_j,\mathbf{r}_1}(\mathbf{r})/\Delta^0_{T_j,\mathbf{r}_1}, \tag{54}$$

where $\mathbf{r}_1=(x_1,y_1)$, $\mathbf{r}_{j+1}=(x_{j+1},y_{j+1})$, $\mathbf{r}_{j+2}=(x_{j+2},y_{j+2})$ are vertices of triangle T_j, $j=1,2$, $\mathbf{r}=(x,y)$,

$$\Delta_{T_j,\mathbf{r}_1}(\mathbf{r})=\begin{vmatrix} x_{j+1}-x & y_{j+1}-y\\ x_{j+2}-x_{j+1} & y_{j+2}-y_{j+1}\end{vmatrix},\quad \Delta^0_{T_j,\mathbf{r}_1}=\begin{vmatrix} x_{j+1}-x_1 & y_{j+1}-y_1\\ x_{j+2}-x_1 & y_{j+2}-y_1\end{vmatrix}. \tag{55}$$

Therefore, for $\mathbf{r}\in|T_j|$, $j=1,2$, we get

$$\frac{\partial\varpi_{\mathbf{r}_1}}{\partial x}(\mathbf{r})=-(y_{j+2}-y_{j+1})/\Delta^0_{T_j,\mathbf{r}_1},\quad \frac{\partial\varpi_{\mathbf{r}_1}}{\partial y}(\mathbf{r})=(x_{j+2}-x_{j+1})/\Delta^0_{T_j,\mathbf{r}_1}. \tag{56}$$

Similarly for $\mathbf{r}\in|T_1|$, we have

$$\varpi_{\mathbf{r}_2}(\mathbf{r})=\Delta_{T_1,\mathbf{r}_2}(\mathbf{r})/\Delta^0_{T_1,\mathbf{r}_2}. \tag{57}$$

Let $\mathbf{r}_1=(x_1,y_1)$, $\mathbf{r}_2=(x_2,y_2)$, $\mathbf{r}_3=(x_3,y_3)$ be vertex of the triangle T_1, $\mathbf{r}=(x,y)$, where $\mathbf{r}_1=(x_1,y_1)$, $\mathbf{r}_2=(x_2,y_2)$, $\mathbf{r}_3=(x_3,y_3)$ are vertices of triangle T_1, $\mathbf{r}=(x,y)$,

$$\Delta_{T_1,\mathbf{r}_2}(\mathbf{r})=\begin{vmatrix} x-x_1 & y-y_1\\ x_3-x_1 & y_3-y_1\end{vmatrix},\quad \Delta^0_{T_1,\mathbf{r}_2}=\begin{vmatrix} x_2-x_1 & y_2-y_1\\ x_3-x_2 & y_3-y_2\end{vmatrix}. \tag{58}$$

Therefore, for $\mathbf{r}\in|T_1|$, we get

$$\frac{\partial\varpi_{\mathbf{r}_2}}{\partial x}(\mathbf{r})=(y_3-y_1)/\Delta^0_{T_1,\mathbf{r}_2},\quad \frac{\partial\varpi_{\mathbf{r}_2}}{\partial y}(\mathbf{r})=-(x_3-x_1)/\Delta^0_{T_1,\mathbf{r}_2}. \tag{59}$$

For $\mathbf{r}\in|T_j|$, $j=1,2$, we have

$$\varpi_{\mathbf{r}_3}(\mathbf{r})=\Delta_{T_j,\mathbf{r}_3}(\mathbf{r})/\Delta^0_{T_j,\mathbf{r}_3}, \tag{60}$$

where $\mathbf{r}_1=(x_1,y_1)$, $\mathbf{r}_2=(x_2,y_2)$, $\mathbf{r}_3=(x_3,y_3)$ are vertices of T_1, $\mathbf{r}_1=(x_1,y_1)$, $\mathbf{r}_3=(x_3,y_3)$, $\mathbf{r}_4=(x_4,y_4)$ are vertices of T_2, $\mathbf{r}=(x,y)$,

$$\Delta_{T_1,\mathbf{r}_3}(\mathbf{r})=\begin{vmatrix} x_1-x & y_1-y\\ x_2-x_1 & y_2-y_1\end{vmatrix},\quad \Delta^0_{T_1,\mathbf{r}_3}=\begin{vmatrix} x_2-x_1 & y_2-y_1\\ x_3-x_2 & y_3-y_2\end{vmatrix}, \tag{61}$$

$$\Delta_{T_2,\mathbf{r}_3}(\mathbf{r})=\begin{vmatrix} x-x_1 & y-y_1\\ x_4-x_1 & y_4-y_1\end{vmatrix},\quad \Delta^0_{T_2,\mathbf{r}_3}=\begin{vmatrix} x_3-x_1 & y_3-y_1\\ x_4-x_3 & y_4-y_3\end{vmatrix}. \tag{62}$$

Therefore, for $\mathbf{r}\in|T_1|$ we get

$$\frac{\partial\varpi_{\mathbf{r}_3}}{\partial x}(\mathbf{r})=-(y_2-y_1)/\Delta^0_{T_1,\mathbf{r}_3},\quad \frac{\partial\varpi_{\mathbf{r}_3}}{\partial y}(\mathbf{r})=(x_2-x_1)/\Delta^0_{T_1,\mathbf{r}_3}, \tag{63}$$

and for $\mathbf{r}\in|T_2|$ we find

$$\frac{\partial\varpi_{\mathbf{r}_3}}{\partial x}(\mathbf{r})=(y_4-y_1)/\Delta^0_{T_2,\mathbf{r}_3},\quad \frac{\partial\varpi_{\mathbf{r}_3}}{\partial y}(\mathbf{r})=-(x_4-x_1)/\Delta^0_{T_2,\mathbf{r}_3}. \tag{64}$$

Similarly for $\mathbf{r}\in|T_2|$, we have

$$\varpi_{\mathbf{r}_4}(\mathbf{r})=\Delta_{T_2,\mathbf{r}_4}(\mathbf{r})/\Delta^0_{T_2,\mathbf{r}_4}, \tag{65}$$

where $\mathbf{r}_1 = (x_1, y_1)$, $\mathbf{r}_3 = (x_3, y_3)$, $\mathbf{r}_4 = (x_4, y_4)$ are vertices of triangle T_2, $\mathbf{r} = (x, y)$,

$$\Delta_{T_2,\mathbf{r}_4}(\mathbf{r}) = \begin{vmatrix} x_1 - x & y_1 - y \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}, \quad \Delta^0_{T_2,\mathbf{r}_4} = \begin{vmatrix} x_3 - x_1 & y_3 - y_1 \\ x_4 - x_3 & y_4 - y_3 \end{vmatrix}. \tag{66}$$

Therefore, for $\mathbf{r} \in |T_2|$, we get

$$\frac{\partial \varpi_{\mathbf{r}_4}}{\partial x}(\mathbf{r}) = -(y_3 - y_1)/\Delta^0_{T_2,\mathbf{r}_4}, \quad \frac{\partial \varpi_{\mathbf{r}_4}}{\partial y}(\mathbf{r}) = (x_3 - x_1)/\Delta^0_{T_2,\mathbf{r}_4}. \tag{67}$$

Let as proceed to the calculation of the expression $[\tilde{u}_*, \varpi_{\mathbf{r}_\xi}]$. We have

$$[\tilde{u}_*, \varpi_{\mathbf{r}_\xi}] = \sum_{j=1}^{4} v_j [\varpi_{\mathbf{r}_j}, \varpi_{\mathbf{r}_\xi}], \tag{68}$$

$$[\varpi_{\mathbf{r}_j}, \varpi_{\mathbf{r}_\xi}] = \sum_{i=1}^{4} [\varpi_{\mathbf{r}_j}, \varpi_{\mathbf{r}_\xi}]_{\widetilde{T}_i}. \tag{69}$$

Let us introduce the notation

$$A_{ji}(\xi) \stackrel{\text{def}}{=} [\varpi_{\mathbf{r}_j}, \varpi_{\mathbf{r}_\xi}]_{\widetilde{T}_i},$$
$$\widetilde{p}_k^{(i)} \stackrel{\text{def}}{=} \int_{|\widetilde{T}_i|} p_{kk}\,dxdy,$$
$$x_{ij} \stackrel{\text{def}}{=} x_i - x_j,$$
$$y_{ij} \stackrel{\text{def}}{=} y_i - y_j.$$

From relations (68) – (69) we have

$$[\tilde{u}_*, \varpi_{\mathbf{r}_\xi}] = \sum_{j=1}^{4} v_j \sum_{i=1}^{4} A_{ji}(\xi). \tag{70}$$

Using formulas (51), (56), (59), (63) – (64), (67), we successively find

$$A_{11}(\xi) = (mT_1 m\widetilde{T}_1)^{-1}[y_{32}y_{21}\widetilde{p}_1^{(1)} + x_{32}x_{21}\widetilde{p}_2^{(1)}],$$
$$A_{12}(\xi) = (mT_1 m\widetilde{T}_2)^{-1}[y_{32}^2\widetilde{p}_1^{(2)} + x_{32}^2\widetilde{p}_2^{(2)}],$$
$$A_{13}(\xi) = (mT_2 m\widetilde{T}_3)^{-1}[y_{43}^2\widetilde{p}_1^{(3)} + x_{43}^2\widetilde{p}_2^{(3)}],$$
$$A_{14}(\xi) = (mT_2 m\widetilde{T}_4)^{-1}[y_{43}y_{14}\widetilde{p}_1^{(4)} + x_{43}x_{14}\widetilde{p}_2^{(4)}],$$
$$A_{21}(\xi) = (mT_1 m\widetilde{T}_1)^{-1}[y_{13}y_{21}\widetilde{p}_1^{(1)} + x_{13}x_{21}\widetilde{p}_2^{(1)}],$$
$$A_{22}(\xi) = (mT_1 m\widetilde{T}_2)^{-1}[y_{13}y_{32}\widetilde{p}_1^{(2)} + x_{13}x_{32}\widetilde{p}_2^{(2)}],$$
$$A_{23}(\xi) = A_{24}(\xi) = 0.$$

7 Conclusion

A refinement of the approximate solution in the projection method is usually obtained by an orthogonal projection in H_A onto a wider space. This requires significant computer resources. The approach proposed in this paper leads to significant resource savings due to the use of a refined approximate solution. The peculiarity of this work lies in a fairly simple clarification of an obtained solution in a small subdomain of the considered area.

This paper also gives an energy estimate clarification. This paper is devoted to the posteriori improvement of the approximation in the projection method for solving a linear equation with self-adjoint positive definite operator. Improvement was obtained by expanding of the projection space.

The mentioned expanding is a linear span for original projection space and added element of energy space. Consideration of a priori given parameterized class of such elements allows the construction of an adaptive method for the mentioned improvement. In this work, the proposed approach is applied to the method finite elements for a two-dimensional boundary value problem of the second order. The use of this approach allows you to optimize the process refinement of the numerical solution of the boundary value problem without significant increase in computation time and resource requirements computing system.

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