The *r*-circulant Matrices Associated with *k*-Fermat and *k*-Mersenne Numbers

BAHAR KULOĞLU[1*], ENGİN ESER[2], ENGİN ÖZKAN[3]
[1]Department of Mathematics,
Graduate School of Natural and Applied Sciences,
Erzincan Binali Yıldırım University,
Yalnızbağ Campus, 24100, Erzincan,
TURKEY

[2]Department of Mathematics,
Erzincan Binali Yıldırım University,
Faculty of Art and Sciences,
Erzincan,
TURKEY

[3]Department of Mathematics,
Faculty of Arts and Sciences,
Erzincan Binali Yıldırım University,
Yalnızbağ Campus, 24100, Erzincan,
TURKEY

**Corresponding Author*

Abstract: - In this study, the main goal is to investigate the r-circulant matrices of k-Fermat and k-Mersenne numbers, then to find eigenvalues, determinants of these matrices, to evaluate their different norms (Spectral and Euclidean) and finally to find the right and skew-right circulant matrices.

Key-Words: - k-Fermat number, k-Merssenne number, Norm (Spectral and Euclidean), Eigenvalues, Circulant matrices

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1 Introduction

Prime numbers are numbers that have remained a mystery throughout human history. Since prime numbers are generator numbers, people find the formula for these numbers. The most well-known of these are Pierre de Fermat and Marin Mersenne. Fermat numbers included pseudoprimes. Firstly, Fermat conjectured that all numbers which are produced by $2^k + 1$ (k is a non-negative integer) are prime numbers (k must be the power of 2). One can see easily that the first 5 numbers are prime but when it comes to the 6th number there is a problem. Later Euler proved that this number has factors. So, it was a composite number. Fermat made a computational mistake. Now it is an open question that are there any other numbers like this?

Mersenne numbers have the form $2^k - 1$ (k is a positive integer). These numbers were studied in ancient times because of their connection to perfect numbers. Euclid-Euler theorem asserts this connection.

Later Francois Proth studied Fermat numbers. He found, [1], the numbers which are a generalized form of Fermat numbers. They are of form $k2^n + 1$, $k, n \in \mathbb{Z}^+$. Proth numbers are known as k-Fermat numbers. There are restrictions on these numbers as $2^n > k$, where $n, k \in \mathbb{N}$ and k is odd numbers. Here it can be easily seen that the general forms of Proth numbers without any restrictions are k -Fermat numbers. The first terms of these numbers are as follows:

3, 5, 13, 17, 41, 97, 113, 193, 241, 257, 353, 449, 577, 641, 673, 769, 929, 1153, 1217, 1409, 1601, 2113, 2689, 2753, 3137, 3329, 3457, 4481, 4993, 6529, 7297, 7681, 7937, 9473, 9601, 9857 (Proth numbers are referenced in the On-Line

Encyclopedia of Integer Sequences in OEIS as A080076, [2].)

In our study we examine k-Fermat numbers and k-Mersenne numbers, [3]. k-Fermat number sequence, denoted by $\{R_{k,n}\}$, is defined by

$$R_{k,n} = k2^n + 1, \quad k,n \in \mathbb{Z}^+$$

k-Mersenne number sequence denoted by $\{M_{k,n}\}$ is defined by

$$M_{k,n} = k2^n - 1, \quad k,n \in \mathbb{Z}^+$$

Recurrence relations of these numbers respectively,

$$R_{k,n} = 3kR_{k,n-1} - 2R_{k,n-2} \quad \text{with} \quad R_{k,0} = 2,\ R_{k,1} = 3 \quad (1.1)$$

$$M_{k,n} = 3kM_{k,n-1} - 2M_{k,n-2} \quad \text{with} \quad M_{k,0} = 0,\ M_{k,1} = 1$$

The first terms of these numbers are shown in Table 1.

Table 1. Some values for $R_{k,n}$ and $M_{k,n}$.

n	0	1	2	3	…
$R_{k,n}$	2	3	$9k^2 - 4$	$27k^3 - 12k - 6$	…
$M_{k,n}$	0	1	$3k$	$9k^2 - 2$	…

Now we introduce circulant and r-cirulant matrices. Among the subjects of algebra, especially in linear algebra, these matrices draw attention. These matrices are used in a number of operations such as signaling, coding, etc. They are also used in the field of chemistry, [4]. The study, [5], presented the visualization of breathing tense while reflecting atomic velocities on eigenvectors of the circulant matrix. In, [6], the author's circulant matrices are used in Hadamard transform spectroscopy and in the construction of the optimal chemical design. In [7], the authors presented the quantum optics effects in quasi-one-dimensional and two-dimensional carbon materials by the circulant matrix method. In [8], [9], the authors gave the lower and upper bounds for the spectral norms of r -circulant matrices and obtained some bounds related to the spectral norms of Hadamard and Kronecker products of these matrices with the Fibonacci and Lucas numbers. Also, they presented the spectral norms with k- Fibonacci and k-Lucas numbers. Circulant matrices are of great interest among the subjects of algebra, especially in the field of linear algebra as Algebraic Geometry, Number Theory, Topleitz operator, etc. $r-$circulant matrices are shown by $Circ(n)$ or C_r where n represents the size of matrix. Their inverse, conjugate transposes, sums, and multiplications can be malleable. For any r complex number without zero, we can define those matrices. We can examine its eigenvalues, Euclidean norm, spectral norm, determinants, and inverse. C_r is determined by its first-row element and r. Given $r = +1$ and $r = -1$ in recurrence relation we obtain its eigenvalues and determinant.

Circulant matrices and r-circulant matrices including Fibonacci, Pell, Pell-Lucas, etc. numbers have been of great interest. In several studies, eigenvalues, determinants, norms, bounds, and inverses for these matrices are found. For instance, [10], presented eigenvalues and the determinant of the right circulant matrices with Pell numbers and Pell-Lucas numbers. The study, [11], presented norms for circulant matrices including Fermat and Mersenne numbers. In, [12], the authors presented the exact inverse of circulant matrices with Fermat and Mersenne numbers. In, [13], the author solved the determinants of these matrices using matrix decomposition. In, [14], the authors studied the properties of the r-circulant matices involving Mersene and Fermat numbers. In this study, we present $r-$circulant matrices by using $k-$Fermat and $k-$Mersenne numbers. This study is constructed of three sections. In the first section, we introduce the Euclidean norm, Hadamard product, eigenvalues, and determinants of $r-$circulant matices. In second section we define the $r-$circulant matrices involving k- Fermat numbers. We find the eigenvalues, determinants, sum identities, norms, and the bound for the spectral norm for the k-Fermat r-circulant matrices. In the last section we study r-circulant matrices involving k-Mersenne numbers.

Lemma 1.1, [15], [16]. Let $X = (x_{ij})$ be a matrix. The Frobenius or Euclidean norm of X is defined as

$$\|X\|_{E=F} = \sqrt{\sum_{i=1}^{a}\sum_{j=1}^{b}|x_{ij}|^2}.$$

The column norm of X is defined as

$$\|X\|_1 = \max_{1 \le j \le n} \sum_{j=1}^{n}|x_{ij}|.$$

The row norm of X is defined as

$$\|X\|_\infty = \max_{1 \le j \le m} \sum_{j=1}^{n}|x_{ij}|.$$

The spectral norm of a matrix X defined as

$$\|X\|_2 = \sqrt{\gamma(X^*X)} = \mu_{max}(X).$$

where $\gamma(X^*X)$ denote the eigenvalues of (X^*X) and X^* is the conjugate transpose of X.

Matrix X has the relationship between the norm values given below:

$$\frac{1}{\sqrt{n}}\|X\|_F \leq \|X\|_2 \leq \|X\|_F \quad (1.2)$$

Lemma 1.2, [13]. Let $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$, $B = [b_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$, C is the Hadamard product of A and B, then we get

$$\|C\|_2 \leq m(A)n(B) \quad (1.3)$$

where $m(A) = max_{1\leq i\leq m}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2}$ and $n(B) = max_{1\leq j\leq n}\sqrt{\sum_{j=1}^{m}|b_{ij}|^2}$

Definition 1.3, [17]. For $r \in C - \{0\}$, a matrix C_r is said to be r-circulant matrix if is of the form and it is denoted by $C_r = Circ(r; c)$, where $\vec{c} = (c_0, c_1, \dots, c_{n-1})$ is the first-row vector. For $r = 1$ and $r = -1$, the right and skew-right circulant matrices are desired, respectively.

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \dots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \dots & rc_{n1} & c_0 \end{bmatrix}$$

Lemma 1.4, [17]. Let C_r be r-circulant matrices then its eigenvalues

$$\mu_i = \sum_{j=0}^{n-1} c_j \left(pw^{-i}\right)^j, \ i = 0,1,2, \dots., n-1$$

where w is the nth root of unity and p is the nth root of r.

Lemma 1.5, [16]. The Euclidean norm of r-circulant matrix C_r is given by

$$\|C_r\|_E = \sqrt{\sum_{j=0}^{n-1}|C_j|^2[n - j(1 - |r|^2)]} \quad (1.4)$$

Lemma 1.6, [10]. For any a and b , we get

$$\prod_{i=0}^{n-1}(a - bp_i w_{-i}) = a^n - rb^n \quad (1.5)$$

Lemma 1.7, [18]. Determinant of the circulant matrix C_n is

$$|C_n| = \prod_{i=0}^{n-1}\left(\sum_{j=0}^{n-1} a_j\left(t_l^j\right)\right)$$

where the entry $\{i, j\}$ is equal to the entry $\{i + l, j + l\}$ for $l = 1,2, \dots$ and $t_l = e^{\frac{2\pi l}{n}}$ are the n^th^ roots of unity. where p_i are n^{th} root of r.

Using the above lemmas we can calculate the eigenvalues, the determinant, Euclidean norms, and bounds for spectral norms of r-circulant matrices involving k-Fermat and k-Mersenne numbers with arithmetic indices. We present many new identities for k-Fermat and k-Mersenne numbers.

2 *k*-Fermat *r*-circulant Matrix

u and v be non-negative integers and $r \in \mathbb{C} - \{0\}$. The r-circulant matrices k-Fermat numbers are denoted by $R_{k,r}$ and defined as follows:

Definition 2.1 The k-Fermat r-circulant matrix is defined as $R_{k,r} = Circ(k, r; \vec{c})$ where first row vector is $\vec{c} = (R_{k,u}, R_{k,u+v}, R_{k,u+2v}, \dots, R_{u+(n-1)v})$ i.e., matrix of the form

$$R_{k,r}=\begin{bmatrix} R_{k,u} & R_{k,u+v} & R_{k,u+2v} & \cdots & R_{k,u+(n-2)v} & R_{k,u+(n-1)v} \\ rR_{k,u+(n-1)v} & R_{k,u} & R_{k,u+v} & \cdots & R_{k,u+(n-3)v} & R_{k,u+(n-2)v} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rR_{k,u+2v} & rR_{k,u+3v} & rR_{k,u+4v} & \cdots & R_{k,u} & R_{k,u+v} \\ rR_{k,u+v} & rR_{k,u+2v} & rR_{k,u+3v} & \cdots & rR_{k,u+(n-1)v} & R_{k,u} \end{bmatrix} \tag{2.1}$$

Theorem 2.2 The eigenvalues of k-Fermat r-circulant matrices $R_{k,r}$ are

$$\lambda_i(R_{k,r})=\begin{cases} -r(R_{k,vn+u})+rpw^{-i}2^v(R_{k,v(n-1)+u})+R_{k,u}-2^v pw^{-i}(R_{k,u-v}), & u>v \\ -r(R_{k,u(n+1)})+rpw^{-i}2^u(R_{k,un})+R_{k,u}-2^{u+1}pw^{-i}, & u=v \\ -r(R_{k,vn+u})+rpw^{-i}2^v(R_{k,v(n-1)+u})+R_{k,u}-2^u pw^{-i}(R_{k,v-u}), & v>u \end{cases}$$

where $i=0,1,2,\dots,n-1$.

Proof.

$$\begin{aligned} \lambda_i(R_{k,r}) &= \sum_{j=0}^{n-1} R_{k,u+jv}(pw^{-i})^j \\ &= \sum_{j=0}^{n-1}(k2^{u+jv}+1)(pw^{-i})^j \\ &= k2^u\sum_{j=0}^{n-1}(2^v pw^{-i})^j+\sum_{j=0}^{n-1}(pw^{-i})^j \\ &= k2^u\left(\frac{1-(2^v pw^{-i})^n}{1-2^v pw^{-i}}\right)+\left(\frac{1-(pw^{-i})^n}{1-pw^{-i}}\right) \\ &= k2^u\left(\frac{1-2^{vn}r}{1-2^v pw^{-i}}\right)+\left(\frac{1-r}{1-pw^{-i}}\right) \\ &= \frac{k2^u-k2^u pw^{-i}-k2^{vn+u}r+k2^{vn+u}rpw^{-i}+1-2^v pw^{-i}-r+2^v rpw^{-i}}{1-pw^{-i}-2^v pw^{-i}+2^v(pw^{-i})^2} \\ &= -r(1+k2^{vn+u})+rpw^{-i}2^v(k2^{vn+u-v}+1)+1+k2^u-2^v pw^{-i}-k2^u pw^{-i} \end{aligned}$$

for $u>v$

$$\lambda_i(R_{k,r})=-r(R_{k,vn+u})+rpw^{-i}2^v(R_{k,v(n-1)+u})+R_{k,u}-2^v pw^{-i}(R_{k,u-v})$$

for $u<v$

$$\lambda_i(R_{k,r})=-r(R_{k,vn+u})+rpw^{-i}2^v(R_{k,v(n-1)+u})+R_{k,u}-2^u pw^{-i}(R_{k,v-u}),$$

and for $u=v$

$$\lambda_i(R_{k,r})=-r(R_{k,u(n+1)})+rpw^{-i}2^u(R_{k,un})+R_{k,u}-2^{u+1}pw^{-i}$$

as desired.

Corollary 2.3 For $r=1$and $r=-1$, we get eigenvalues for the k-Fermat right and skew-right circulant matrices as follows:

$$\begin{aligned} \lambda_i(R_{k,1}) &\overset{u>v}{\triangleq} -(R_{k,vn+u})+pw^{-i}2^v(R_{k,v(n-1)+u}) + R_{k,u}-2^v pw^{-i}(R_{k,u-v}) \\ &\overset{u<v}{\triangleq} -(R_{k,vn+u})+pw^{-i}2^v(R_{k,v(n-1)+u})+R_{k,u} - 2^u pw^{-i}(R_{k,v-u}) \\ &\overset{u=v}{\triangleq} -(R_{k,u(n+1)})+pw^{-i}2^u(R_{k,un})+R_{k,u} - 2^{u+1}pw^{-i} \end{aligned}$$

and

$$\begin{aligned} \lambda_i(R_{k,-1}) &\overset{u>v}{\triangleq} (R_{k,vn+u})-\xi w^{-i}2^v(R_{k,v(n-1)+u}) + R_{k,u}-2^v\xi w^{-i}(R_{k,u-v}) \\ &\overset{u<v}{\triangleq} (R_{k,vn+u})-\xi w^{-i}2^v(R_{k,v(n-1)+u})+R_{k,u} - 2^u\xi w^{-i}(R_{k,v-u}) \\ &\overset{u=v}{\triangleq} (R_{k,u(n+1)})-\xi w^{-i}2^u(R_{k,un})+R_{k,u} - 2^{u+1}\xi w^{-i} \end{aligned}$$

where ξ is the nth. the root of -1.

Theorem 2.4 For a positive integer n, we have

$$\sum_{i=0}^{n-1} -r\left(R_{k,vn+u}\right) + rpw^{-i}2^{v}\left(R_{k,v(n-1)+u}\right) + R_{k,u} - 2^{v}pw^{-i}\left(R_{k,u-v}\right) = n(k2^{u}+1)$$

Proof. To get the desired result, we need to know that the trace of any given square matrix is equal to the sum of the eigenvalues of that matrix. In that case

$$\sum_{i=0}^{n-1} \lambda_i\left(R_{k,r}\right) = \sum_{i=0}^{n-1} -r\left(R_{k,vn+u}\right) + rpw^{-i}2^{v}\left(R_{k,v(n-1)+u}\right) + R_{k,u} - 2^{v}pw^{-i}\left(R_{k,u-v}\right)$$

according to the expression given above, this sum is equal to $nR_{k,u}$. In that case

$$\sum_{i=0}^{n-1} -r\left(R_{k,vn+u}\right) + rpw^{-i}2^{v}\left(R_{k,v(n-1)+u}\right) + R_{k,u} - 2^{v}pw^{-i}\left(R_{k,u-v}\right) = nR_{k,u} = n(k2^{u}+1)$$

as desired.

Theorem 2.5 Determinant of $R_{k,r}$ is given by

$$\det\left(R_{k,r}\right) = (-r)^{n}\left(R_{k,vn+u} + R_{k,u}\right)^{n} + r2^{vn}\left(r\left(R_{k,v(n-1)+u}\right) - \left(R_{k,u-v}\right)\right)^{n}$$

Proof. From Lemma 1.7, we get

$$\left|R_{k,r}\right| = \prod_{i=0}^{n-1}\left(\sum_{j=0}^{n-1} \lambda_j\left(R_{k,r}\right)\right).$$

For $u > v$,

$$= \prod_{i=0}^{n-1}\left(-r\left(R_{k,vn+u}\right) + rpw^{-i}2^{v}\left(R_{k,v(n-1)+u}\right) + R_{k,u} - 2^{v}pw^{-i}\left(R_{k,u-v}\right)\right)$$

$$= \prod_{i=0}^{n-1}\left(\left(-r\left(R_{k,vn+u}\right) + R_{k,u}\right) + pw^{-i}\left(r2^{v}\left(R_{k,v(n-1)+u}\right) - 2^{v}\left(R_{k,u-v}\right)\right)\right).$$

From Lemma 1.6

$$\left|R_{k,r}\right| = (-r)^{n}\left(R_{k,vn+u} + R_{k,u}\right)^{n} + r2^{vn}\left(r\left(R_{k,v(n-1)+u}\right) - \left(R_{k,u-v}\right)\right)^{n}.$$

Corollary 2.6 The determinants of the k-Fermat right circulant and skew-right circulant matrices are given as, respectively

$$\det\left(R_{k,1}\right) = (-1)^{n}\left(R_{k,vn+u} + R_{k,u}\right)^{n} + 2^{vn}\left(\left(R_{k,v(n-1)+u}\right) - \left(R_{k,u-v}\right)\right)^{n},$$

$$\det\left(R_{k,-1}\right) = \left(R_{k,vn+u} + R_{k,u}\right)^{n} - 2^{vn}\left(-\left(R_{k,v(n-1)+u}\right) - \left(R_{k,u-v}\right)\right)^{n}.$$

On setting $i = 0$, $r = 1$, and $p = 1$ in Theorem 2.2, the following sum identities are verified for the k-Fermat numbers.

$$\sum_{j=0}^{n-1} R_{k,u+jv}$$
$$= \begin{cases} R_{k,vn+u} + 2^v(R_{k,v(n-1)+u}) + R_{k,u} - 2^v R_{k,u-v}, & u > v) \\ -R_{k,u(n+1)} + 2^u\left(R_{k,un}\right) + R_{k,u} - 2^{u+1}, & u = v \\ -R_{k,vn+u} + 2^v(R_{k,v(n-1)+u}) + R_{k,u} - 2^u R_{k,v-u}, & u < v \end{cases}.$$

3 Norm of k-Fermat r-circulant Matrices

When we take $u = 0$ and $v = 1$, we get

$$R_{k,r}^* = \begin{bmatrix} R_{k,0} & R_{k,1} & R_{k,2} & \dots & R_{k,n-2} & R_{k,n-1} \\ rR_{k,n-1} & R_{k,0} & R_{k,1} & \dots & R_{k,n-3} & R_{k,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rR_{k,2} & rR_{k,3} & rR_{k,4} & \dots & R_{k,0} & R_{k,1} \\ rR_{k,1} & rR_{k,2} & rR_{k,3} & \dots & rR_{k,n-1} & R_{k,0} \end{bmatrix}.$$

From Lemma 3.1, the sum of the squares of the k-Fermat numbers will be used to obtain the norms of different matrices.

Lemma 3.1 The finite sum of squares of the k-Fermat numbers is given by

$$\sum_{j=0}^{n-1} (R_{k,j})^2 = \frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3} \qquad (3.1)$$

Proof.

$$\sum_{j=0}^{n-1} (R_{k,j})^2 = \sum_{j=0}^{n-1} (k2^j + 1)^2 = \sum_{j=0}^{n-1} k^2 2^{2j} + 1 + 2^{j+1}k$$
$$= k^2\left(\frac{4^n - 1}{3}\right) + n + 2k(2^n - 1)$$
$$= \frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}$$

as desired.

Theorem 3.2 The Euclidean norm for the k-Fermat r-circulant matrices is given by

$$\left\|R_{k,r}^*\right\|_E = \sum_{j=0}^{n-1} \left((R_{k,j} - 1)^2 + R_{k,j+1}\right)[n - j(1 - |r|^2)]$$

Proof. By Eq. (1.4) we get

$$\left\|R_{k,r}^*\right\|^2{}_E = \sum_{j=0}^{n-1} \left|R_{k,j}\right|^2 [n - j(1 - |r|^2)]$$
$$= \sum_{j=0}^{n-1} (k2^j + 1)^2 [n - j(1 - |r|^2)]$$

$$= \sum_{j=0}^{n-1} (k^2 2^{2j} + 1 + k2^{j+1})[n - j(1 - |r|^2)]$$
$$= \sum_{j=0}^{n-1} \left((R_{k,j} - 1)^2 + R_{k,j+1}\right)[n - j(1 - |r|^2)]$$

as desired.

Theorem 3.3 The bound for the spectral norm of the k-Fermat r-circulant matrices is:
for $|r| \geq 1$

$$\sqrt{\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}} \leq \left\|R_{k,r}^*\right\|_2 \leq .$$
$$\sqrt{\frac{12 + |r|^2\left(k^2(4^n - 1) + 3n + 6k(2^n - 1)\right)}{3}} . \sqrt{\frac{3 + (k^2(4^n - 1) + 3n + 6k(2^n - 1))}{3}},$$

and for $|r| < 1$

$$|r|. \sqrt{\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}} \leq \left\|R_{k,r}^*\right\|_2 \leq \sqrt{n\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}}$$

Proof. By Eq (1.4), the Euclidean norm is given as

$$\left\|R_{k,r}^*\right\|_E^2 = \sum_{j=0}^{n-1} \left|R_{k,j}\right|^2 [n - j(1 - |r|^2)].$$

Here, we need to examine the proof in 2 stages according to the state of r.

State 1. If $|r| \geq 1$, then from Lemma 3.1 we get

$$\left\|R_{k,r}^*\right\|_E^2 = \sum_{j=0}^{n-1} (n - j)\left|R_{k,j}\right|^2 + |r|^2 \sum_{j=0}^{n-1} j\left|R_{k,j}\right|^2 \geq \sum_{j=0}^{n-1} (n - j)\left|R_{k,j}\right|^2 + \sum_{j=0}^{n-1} j\left|R_{k,j}\right|^2$$

$$= \sum_{j=0}^{n-1} n\left|R_{k,j}\right|^2 = n\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}$$

which implies

$$\frac{\left\|R_{k,r}^*\right\|_E}{\sqrt{n}} \geq \sqrt{\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}}$$

and from Eq. (1.2) we get

$$\left\|R_{k,r}^*\right\|_2 \geq \sqrt{\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}}$$

Now to obtain the upper bound for the spectral norm, we write $R_{k,r}^*$ in the form of the Hadamard product of two matrices.

$$A = \begin{bmatrix} R_{k,0} & 1 & 1 & \dots & 1 & 1 \\ rR_{k,n-1} & R_{k,0} & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rR_{k,2} & rR_{k,3} & rR_{k,4} & \dots & R_{k,0} & 1 \\ rR_{k,1} & rR_{k,2} & rR_{k,3} & \dots & rR_{k,n-1} & R_{k,0} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & R_{k,1} & R_{k,2} & \dots & R_{k,n-2} & R_{k,n-1} \\ 1 & 1 & R_{k,1} & \dots & R_{k,n-3} & R_{k,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & R_{k,1} \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

then clearly $R_{k,r}^* = A \circ B$, where $\circ$ denotes the Hadamard product. Now,

$$s(A) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} \left|a_{ij}\right|^2}$$
$$= \sqrt{\left(R_{k,0}\right)^2 + |r|^2 \sum_{j=1}^{n-1} \left(R_{k,j}\right)^2}$$
$$= \sqrt{\frac{12 + |r|^2\left(k^2(4^n - 1) + 3n + 6k(2^n - 1)\right)}{3}}$$

$$t(B) = \max_{1 \le j \le n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{1 + \sum_{j=1}^{n-1} (R_{k,j})^2}$$

$$= \sqrt{\frac{3 + (k^2(4^n - 1) + 3n + 6k(2^n - 1))}{3}}.$$

Thus, by Lemma 1.2, we get

$$\|R^*_{k,r}\|_2 \le s(A)t(B)$$

$$= \sqrt{\frac{12 + |r|^2\left(k^2(4^n - 1) + 3n + 6k(2^n - 1)\right)}{3}} . \sqrt{\frac{3 + \left(k^2(4^n - 1) + 3n + 6k(2^n - 1)\right)}{3}} .$$

Hence, we have

$$\sqrt{\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}} \le \|R^*_{k,r}\|_2$$

$$\le \sqrt{\frac{12 + |r|^2\left(k^2(4^n - 1) + 3n + 6k(2^n - 1)\right)}{3}} \sqrt{\frac{3 + (k^2(4^n - 1) + 3n + 6k(2^n - 1))}{3}}$$

State 2. If $|r| < 1$, then from Eq. (1.4) and Lemma 3.1 we get

$$\|R^*_{k,r}\|_E^{\ 2} \ge \sum_{j=0}^{n-1} (n-j)|r|^2 |R_{k,j}|^2 + |r|^2 \sum_{j=0}^{n-1} j\, |R_{k,j}|^2$$

$$= n|r|^2 \frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}$$

$$\frac{\|R^*_{k,r}\|_E}{\sqrt{n}} \ge |r| \sqrt{\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}}$$

and from Eq. (1.2) we get

$$\|R^*_{k,r}\|_2 \ge |r| \sqrt{\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}}$$

We calculate the upper bound for the spectral norm of $R^*_{k,r}$.

Let

$$N = \begin{bmatrix} R_{k,0} & R_{k,1} & R_{k,2} & \dots & R_{k,n-2} & R_{k,n-1} \\ R_{k,n-1} & R_{k,0} & R_{k,1} & \dots & \dots & R_{k,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{k,2} & R_{k,3} & R_{k,4} & \dots & \dots & R_{k,1} \\ R_{k,1} & R_{k,2} & R_{k,3} & \dots & \dots & R_{k,0} \end{bmatrix}$$

and

$$M = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ r & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & r & \dots & r & 1 \\ r & r & r & \dots & r & 1 \end{bmatrix}$$

then clearly $R^*_{k,r} = M \circ N$, where $\circ$ denotes the Hadamard product. So,

$$s_1(M) = \max_{1 \le i \le n} \sqrt{\sum_{j=1}^{n} |m_{ij}|^2} = \sqrt{n}$$

$$t_1(N) = \max_{1 \le j \le n} \sqrt{\sum_{j=1}^{n} |n_{ij}|^2} = \sqrt{\sum_{j=1}^{n} R_{k,j}^{\ 2}}$$

$$= \sqrt{\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}}.$$

Hence, by Lemma 1.2, we have

$$\|R^*_{k,r}\|_2 \le s_1(M)t_1(N)$$

$$= \sqrt{n\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}}.$$

Thus

$$|r|. \sqrt{\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}} \le \|R^*_{k,r}\|_2$$

$$\le \sqrt{n\frac{k^2(4^n - 1) + 3n + 6k(2^n - 1)}{3}}$$

as desired.

Now we give the relations above for the k-Mersenne numbers.

4 k-Mersenne r-circulant matrix

u and v be non-negative integers and $r \in \mathbb{C} - \{0\}$. The r-circulant matrices k-Mersenne numbers are denoted by $M_{k,r}$ and defined as follows:

Definition 4.1 The k-Mersenne r-circulant matrix is defined as $M_{k,r} = Circ(k, r; \vec{c})$ where the first-row vector is
$\vec{c} = (M_{k,u}, M_{k,u+v}, M_{k,u+2v}, \dots, M_{u+(n-1)v})$ i.e., matrix of the form

$$M_{k,r} = \begin{bmatrix} M_{k,u} & M_{k,u+v} & M_{k,u+2v} & \cdots & M_{k,u+(n-2)v} & M_{k,u+(n-1)v} \\ rM_{k,u+(n-1)v} & M_{k,u} & M_{k,u+v} & \cdots & M_{k,u+(n-3)v} & M_{k,u+(n-2)v} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rM_{k,u+2v} & rM_{k,u+3v} & rM_{k,u+4v} & \cdots & M_{k,u} & M_{k,u+v} \\ rM_{k,u+v} & rM_{k,u+2v} & rM_{k,u+3v} & \cdots & rM_{k,u+(n-1)v} & M_{k,u} \end{bmatrix} \quad (4.1)$$

Theorem 4.2 The eigenvalues of k-Mersenne r-circulant matrices $M_{k,r}$ are

$$\lambda_i(M_{k,r}) = \begin{cases} -r(M_{k,vn+u}) + rpw^{-i}2^v(M_{k,v(n-1)+u}) + M_{k,u} - 2^v pw^{-i}(M_{k,u-v}), & u > v \\ -r(M_{k,u(n+1)}) + rpw^{-i}2^u(M_{k,un}) + M_{k,u} & u = v \\ -r(M_{k,vn+u}) + rpw^{-i}2^v(M_{k,v(n-1)+u}) + M_{k,u} - 2^u pw^{-i}(M_{k,v-u}), & v > u \end{cases}$$

where $i = 0,1,2, \dots, n-1$.

Proof. $\lambda_i(M_{k,r}) = \sum_{j=0}^{n-1} M_{k,u+jv}(pw^{-i})^j$

$$= \sum_{j=0}^{n-1}(k2^{u+jv} - 1)(pw^{-i})^j$$

$$= k2^u \sum_{j=0}^{n-1}(2^v pw^{-i})^j - \sum_{j=0}^{n-1}(pw^{-i})^j$$

$$= k2^u \left(\frac{1 - (2^v pw^{-i})^n}{1 - 2^v pw^{-i}}\right) - \left(\frac{1 - (pw^{-i})^n}{1 - pw^{-i}}\right)$$

$$= k2^u \left(\frac{1 - 2^{vn} r}{1 - 2^v pw^{-i}}\right) - \left(\frac{1 - r}{1 - pw^{-i}}\right)$$

$$= \frac{k2^u - k2^u pw^{-i} - k2^{vn+u} r + k2^{vn+u} rpw^{-i} - 1 + 2^v pw^{-i} - r.2^v rpw^{-i} + r}{1 - pw^{-i} - 2^v pw^{-i} + 2^v (pw^{-i})^2}$$

$$= -r(-1 + k2^{vn+u}) + rpw^{-i}2^v(k2^{vn+u-v} - 1) - 1 + k2^u + 2^v pw^{-i} - k2^u pw^{-i}$$

for $u > v$

$$\lambda_i(M_{k,r}) = -r(M_{k,vn+u}) + rpw^{-i}2^v(M_{k,v(n-1)+u}) + M_{k,u} - 2^v pw^{-i}(M_{k,u-v}),$$

for $u < v$

$$\lambda_i(M_{k,r}) = -r(M_{k,vn+u}) + rpw^{-i}2^v(M_{k,v(n-1)+u}) + M_{k,u} + k2^u pw^{-i}(M_{v-u}),$$

and for $u = v$

$$\lambda_i(M_{k,r}) = -r(M_{k,u(n+1)}) + rpw^{-i}2^u(M_{k,u}) + M_{k,u}$$

as desired.

Corollary 4.3 For $r = 1$and$r = -1$ we get eigenvalues for the k-Mersenne right and skew-right circulant matrices as follows:

$$\lambda_i(M_{k,1}) \overset{u>v}{\cong} -(M_{k,vn+u}) + pw^{-i}2^v(M_{k,v(n-1)+u}) + M_{k,u} + 2^v pw^{-i}(M_{k,u-v})$$
$$\overset{u<v}{\cong} -(M_{k,vn+u}) + pw^{-i}2^v(M_{k,v(n-1)+u}) + M_{k,u} + k2^u pw^{-i}(M_{k,v-u})$$
$$\overset{u=v}{\cong} -(M_{k,u(n+1)}) + pw^{-i}2^u(M_{k,un}) + M_{k,u}$$

and

$$\lambda_i(M_{k,-1}) \overset{u>v}{\cong} (M_{k,vn+u}) - \xi w^{-i}2^v(M_{k,v(n-1)+u}) + M_{k,u} - 2^v \xi w^{-i}(M_{k,u-v})$$
$$\overset{u<v}{\cong} (M_{k,vn+u}) - \xi w^{-i}2^v(M_{k,v(n-1)+u}) + M_{k,u} - k2^u \xi w^{-i}(M_{k,v-u})$$
$$\overset{u=v}{\cong} (M_{k,u(n+1)}) - \xi w^{-i}2^u(M_{k,un}) + M_{k,u}$$

where ξ is the nth. the root of -1.

Theorem 4.4 For a positive integer *n*, we have

$$\sum_{i=0}^{n-1} -r(M_{k,vn+u}) + rpw^{-i}2^v(M_{k,v(n-1)+u}) + M_{k,u} = n(k2^u - 1).$$

Proof. To get the desired result, we need to know that the trace of any given square matrix is equal to the sum of the eigenvalues of that matrix. In that case,

$$\sum_{i=0}^{n-1} \lambda_i(M_{k,r}) = \sum_{i=0}^{n-1} -r(M_{k,u(n+1)}) + rpw^{-i}2^u(M_{k,un}) + M_{k,u}.$$

According to the expression given above, this sum is equal to $nM_{k,u}$. So

$$\sum_{i=0}^{n-1} -r(M_{k,vn+u}) + rpw^{-i}2^v(M_{k,v(n-1)+u}) + M_{k,u} = nM_{k,u} = n(k2^u - 1)$$

as desired.

Theorem 4.5 Determinant of $M_{k,r}$ is given by

$$\det(M_{k,r}) = (-r)^n\left((M_{k,vn+u} + M_{k,u}\right)^n + r2^{vn}\left(r(M_{k,v(n-1)+u}) - (M_{k,u-v})\right)^n.$$

Proof. We can prove it like Theorem 2.5.

Corollary 4.6 The determinants of the k-Mersenne right circulant and skew-right circulant matrices are given as

$$\det(M_{k,1}) = (-1)^n\left((M_{k,vn+u} + M_{k,u}\right)^n + 2^{vn}\left((M_{k,v(n-1)+u}) - (M_{k,u-v})\right)^n.$$
$$\det(M_{k,-1}) = \left((M_{k,vn+u} + M_{k,u}\right)^n - 2^{vn}\left(-(M_{k,v(n-1)+u}) - (M_{k,u-v})\right)^n.$$

On setting $i = 0, r = 1,$ and $p = 1$in Theorem 4.2, the following sum identities are verified for the k-Mersenne numbers.

$$\sum_{j=0}^{n-1} M_{k,u+jv} = \begin{cases} -M_{k,vn+u} + 2^v\left(M_{k,v(n-1)+u}\right) + M_{k,u} - 2^v M_{k,u-v}, & u > v \\ -M_{k,u(n+1)} + 2^u\left(M_{k,un}\right) + M_{k,u} & u = v \\ -M_{k,vn+u} + 2^v\left(M_{k,v(n-1)+u}\right) + M_{k,u} + k2^u M_{k,v-u}, & u < v \end{cases}$$

5 Norm of k-Mersenne r-circulant matrices

When we take $u = 0$ and $v = 1$

$$M_{k,r}^{*} = \begin{bmatrix} M_{k,0} & M_{k,1} & M_{k,2} & \cdots & M_{k,n-2} & M_{k,n-1} \\ rM_{k,n-1} & M_{k,0} & M_{k,1} & \cdots & M_{k,n-3} & M_{k,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rM_{k,2} & rM_{k,3} & rM_{k,4} & \cdots & M_{k,0} & M_{k,1} \\ rM_{k,1} & rM_{k,2} & rM_{k,3} & \cdots & rM_{k,n-1} & M_{k,0} \end{bmatrix}.$$

From Lemma 5.1, the sum of the squares of the k-Mersenne numbers will be used to obtain the norms of different matrices.

Lemma 5.1 The finite sum of squares of the k-Mersenne numbers is given by

$$\sum_{j=0}^{n-1} (M_{k,j})^2 = \frac{k^2(4^n - 1) + 3n - 6k(2^n - 1)}{3}$$

Proof. The proof is similar to that of Lemma 3.1.

Theorem 5.2 The Euclidean norm for the k-Mersenne r-circulant matrices is given by

$$\left\|M_{k,r}^{*}\right\|_E = \sum_{j=0}^{n-1} \left((M_{k,j} + 1)^2 - M_{k,j+1}\right)[n - j(1 - |r|^2)].$$

Proof. By Eq. (1.4) we get

$$\begin{aligned} \left\|M_{k,r}^{*}\right\|^2{}_E &= \sum_{j=0}^{n-1} \left|M_{k,j}\right|^2 [n - j(1 - |r|^2)] \\ &= \sum_{j=0}^{n-1} (k2^j - 1)^2 [n - j(1 - |r|^2)] \\ &= \sum_{j=0}^{n-1} (k^2 2^{2j} - k2^{j+1} + 1)[n - j(1 - |r|^2)] \\ &= \sum_{j=0}^{n-1} \left((M_{k,j} + 1)^2 - M_{k,j+1}\right)[n - j(1 - |r|^2)]. \end{aligned}$$

Theorem 5.3 The bound for the spectral norm of the k-Mersenne r-circulant matrices is:

For $|r| \geq 1$

$$\sqrt{\frac{k^2(4^n - 1) + 3n - 6k(2^n - 1)}{3}} \leq \left\|M_{k,r}^{*}\right\|_2$$

$$\leq \sqrt{\frac{|r|^2\left(k^2(4^n-1)+3n-6k(2^n-1)\right)}{3}}\sqrt{\frac{3+(k^2(4^n-1)+3n-6k(2^n-1))}{3}}$$

and for $|r|<1$

$$|r|.\sqrt{\frac{k^2(4^n-1)+3n-6k(2^n-1)}{3}} \leq \left\|M_{k,r}^*\right\|_2$$

$$\leq \sqrt{n\frac{k^2(4^n-1)+3n-6k(2^n-1)}{3}}$$

Proof. We can prove it like Theorem 3.3.

6 Conclusion

Based on the k −Fermat and k −Mersenne number sequences with similar recurrences, which have been studied less, the r-circulant matrices of these sequences were created. Euclidean, row, and spectral norms, which are eigenvalues, determinants, and some special norm values, are discussed depending on this matrix. In addition, the lower and upper bounds of these matrices for the spectral norm were examined in closed form. In addition, right circulant and skew-right circulant matrices were examined for 1 and -1 values of r depending on eigenvalues. Finally, some interesting results and sum properties were given.

More interesting results can be expected by working with the lesser-known Fermat equation $2^{2^n}+1$ and the more general form of this equation, $a^{2^n}+b^{2^n}; a,b \in \mathbb{N}$.

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