On the divergence of the sum of prime reciprocals

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Abstract: The sum of the reciprocals of all prime numbers diverges but the divergence is very slow. We propose effective lower and upper bounds for partial sums under the Riemann hypothesis. We give an explicit error term for all Mertens' theorems.

Key-Words: Number Theory, arithmetic functions, Prime numbers, Mertens' theorems

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1 Introduction

In 14th-century, the study, [1], was considering the harmonic series

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

and proves for the first time the divergence of the sum of the reciprocals of the integers. Three centuries later, [2], would also consider this sum and used the relation

$$\sum_{n=1}^{\infty}\frac{1}{n} = \prod_{p\in\mathcal{P}}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right) = \prod_{p\in\mathcal{P}}\frac{1}{1-p^{-1}},$$

where the product is taken over the set $\mathcal{P}$ of all primes, to show once again the existence of infinitely many primes, the one appearing in Euclid's elements.

We can ask ourselves the question if you reduce the summation set only to the prime set, also an infinite set but a smaller set, the sum remains divergent or not?

The question was resolved by [2], who proved in 1744 the divergence of the sum of the reciprocals of all prime numbers

$$\sum_{p \text{ prime}}\frac{1}{p} = \tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}+\cdots = +\infty$$

The next step is to exhibit the growth of the series. You have to count the terms of the series up to a bound x

$$F(x) = \sum_{p \text{ prime}\le x}\frac{1}{p}, \qquad (1)$$

sum which becomes a function of x. By [3], the growth of this function is asymptotic to the "$\ln\ln$" function (iterate of ln function, the natural logarithm). Step by step, we can find the asymptotic expansion by studying the difference

$$\lim_{x\to+\infty}\left(\sum_{p \text{ prime}\le x}\frac{1}{p}-\ln\ln x\right), \qquad (2)$$

following the same idea on Euler's constant, [4],

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right).$$

One can establish that the limit of (2) exists (Mertens' second theorem, [5]) and converges to a constant M, the well-known Meissel-Lehmer constant. The value of M is approximately (sequence A077761 in the OEIS[1])

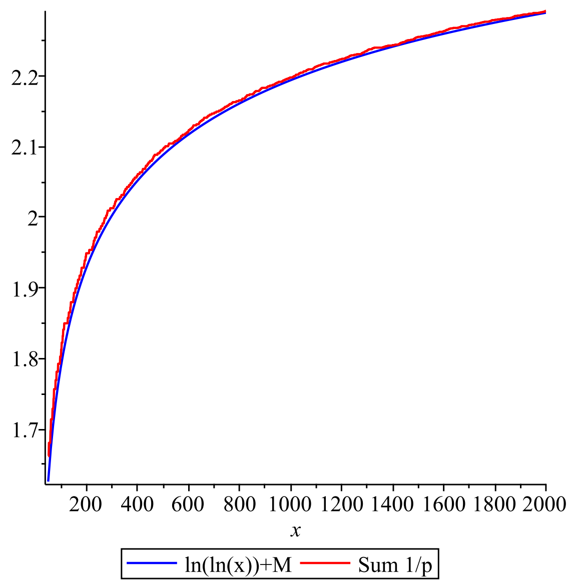
$$M \approx 0.2614972128476427837554268386...$$

Next, we can express completely the asymptotic expansion which expresses the behavior of the sum up to x and define

$$\sum_{p \text{ prime}\le x}\frac{1}{p} = \ln\ln x + M + O\left(\frac{1}{\ln x}\right). \qquad (3)$$

(See proof of [6])

[1] https://oeis.org/A077761



If you want to compute $F(x)$ for high values of x, an algorithm for fast computation of (1) is presented in [7].

The aim of this paper is to have a practical approximation for this sum, which follows the asymptotic expansion. We place ourselves under the Riemann hypothesis to have the best possible results.

We also put practical approximations for Mertens' first theorem about $\sum \ln p/p$ and Mertens' third theorem about $\prod(1-1/p)$.

2 Sum of the reciprocals of primes

Effective estimates are very useful to show other results. For example, if you use a classical upper bound for p_n (the n-th prime number)

$$p_n < n\ln n + n\ln\ln n \quad \text{for } n \geq 6$$

then showing the divergence of $F(x)$ is easy:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{p_n} &\geq \sum_{n=6}^{\infty}\frac{1}{p_n}\\ &\geq \sum_{n=6}^{\infty}\frac{1}{n\ln n + n\ln\ln n}\\ &\geq \sum_{n=6}^{\infty}\frac{1}{2n\ln n} = \infty\end{aligned}$$

by the integral test for convergence. We found, [1], result by another way (but not with the same tools). The best-known results about effective estimates of the sum (3) are from [8], [9], with an error term $O\left(\frac{\ln x}{\sqrt{x}}\right)$ instead of $O\left(\frac{1}{\ln x}\right)$.

2.1 First lemmas

Let's introduce ϑ and ψ as the first and second Chebyshev functions respectively.

Let's start with two results found by [10]:

Lemma 2.1 (Lemma 1, [10]). *Let, for $n \in \mathbb{N}^*$ and $\rho \in \mathbb{C}$,*

$$F_{\rho,n}(x) = \int_x^{\infty}\frac{x^{\rho}}{t^{n+1}}\frac{n(\ln t+1)}{\ln^2(t)}\mathrm{d}t.$$

If $Re(\rho) = 1/2$ then

$$F_{\rho,n}(x) = \frac{n}{n-\rho}\frac{x^{\rho-n}}{\ln n} + r_{\rho,n}(x)$$

with

$$|r_{\rho,n}(x)| \leqslant \frac{1}{|\rho|}\frac{1}{x^{n-1/2}\ln^2 x}\left(1+\frac{4}{(2n-1)\ln x}\right).$$

Proof. By integration by parts,

$$\int_x^{\infty}\frac{x^{\rho}}{t^{n+1}}\frac{n(\ln t+1)}{\ln^2(t)}\mathrm{d}t = \frac{n}{n-\rho}\frac{x^{\rho-n}}{\ln n} + \frac{\rho}{(\rho-n)^2}\left(-\frac{x^{\rho-n}}{\ln^2 x} + 2\int_x^{\infty}\frac{t^{\rho-n-1}}{\ln^3 t}\mathrm{d}t\right)$$

□

Lemma 2.2 (Lemma 4, [10]). *Under the Riemann hypothesis,*

$$\begin{aligned}&\int_x^{\infty}\frac{(\vartheta(t)-t)(\ln t+1)}{(t\ln t)^2}\mathrm{d}t\\ &\leqslant \frac{0.0462}{x^{1/2}\ln x}\left(1+\frac{1}{\ln x}+\frac{4}{\ln^2 x}\right)+\frac{\ln 2\pi}{x\ln x}\\ &+\frac{2}{\sqrt{x}\ln x}\left(1-\frac{1}{\ln x}+\frac{4}{\ln^2(x)}+\frac{1}{x^{1/6}}\right).\end{aligned}$$

Proof. Let $\vartheta(x) = x + \varepsilon_1(x)$ and $\psi(x) = x + \varepsilon_2(x)$. Since $\varepsilon_1(x) = \vartheta(x) - x = \vartheta(x) - (\psi(x) - \varepsilon_2(x))$, this last integral can be expressed by

$$\int_x^{\infty}\frac{\varepsilon_1(t)(t+1)}{(t\ln t)^2}\mathrm{d}t = I_1(x) - I_2(x)$$

where

$$I_1(x) = \int_x^{\infty}\frac{\varepsilon_2(t)(\ln t+1)}{(t\ln t)^2}\mathrm{d}t,$$

and

$$I_2(x) = \int_x^{\infty}\frac{(\psi(t)-\vartheta(t))(\ln t+1)}{(t\ln t)^2}\mathrm{d}t.$$

To obtain an upper bound for I_1, we use a classical explicit formula in number theory (Chapter 17 of [11])

$\psi(x) = x - \ln 2\pi - \sum_\rho \frac{x^\rho}{\rho} - \frac{1}{2}\ln(1-1/x^2)$ in the form

$$\int_x^\infty f_n(t)(\psi(t) - t + \ln 2\pi + \frac{1}{2}\ln(1-1/t^2))\mathrm{d}t$$
$$= -\int_x^\infty \sum_\rho f_n(t)\frac{t^\rho}{\rho}\mathrm{d}t$$

and taking $f_n(t) = \frac{n(\ln t+1)}{t^{n+1}\ln^2 t}$ (derivative of $-1/(t^n \ln t)$), we have $f_n'(t) = O(\frac{1}{t^3 \ln t})$ for $n \geqslant 1$ and thus inverting the sum and integral signs by the dominated convergence theorem. Thus

$$\begin{aligned} I_1(x) = & -\left(\sum_\rho \int_x^\infty f_1(t)\frac{t^\rho}{\rho}\mathrm{d}t\right) \\ & - \int_x^\infty f_1(t)(\ln 2\pi + \tfrac{1}{2}\ln(1-1/t^2))\mathrm{d}t. \end{aligned}$$

Moreover $\frac{1}{2}\ln(1-1/t^2) < 0$ and $|\frac{1}{2}\ln(1-1/t^2)| < \ln 2\pi$ hence

$$\begin{aligned} & -\int_x^\infty f_n(t)(\ln 2\pi)\mathrm{d}t \\ & < \ -\int_x^\infty f_n(t)(\ln 2\pi + \frac{1}{2}\ln(1-1/t^2))\mathrm{d}t \\ & < \ 0 \end{aligned}$$

Thus

$$0 < \int_x^\infty f_n(t)(\ln 2\pi + \frac{1}{2}\ln(1-1/t^2))\mathrm{d}t \leqslant \frac{\ln 2\pi}{x^n \ln x}.$$

On the other hand, according to Lemma 2.1,

$$\begin{aligned} & -\left(\sum_\rho \int_x^\infty f_n(t)\frac{t^\rho}{\rho}\mathrm{d}t\right) \\ & = \ \sum_\rho \frac{n}{\rho(\rho-n)}\frac{x^{\rho-n}}{\ln x} - \sum_\rho \frac{r_\rho(x)}{\rho} \end{aligned}$$

Then

$$\frac{n}{x^{n-1/2}\ln x}\sum_\rho \frac{x^{i\Im(\rho)}}{\rho(\rho-n)} - \sum_\rho \frac{r_\rho(x)}{\rho}$$

and $|\sum_\rho \frac{x^{i\Im(\rho)}}{\rho(\rho-n)}| \leqslant \sum_\rho \frac{1}{|\rho|^2} = \gamma + 2 - \ln 4\pi \leqslant 0.0462$. Thus, the formula applied for $n = 1$,

$$|I_1(x)| \leqslant \frac{0.0462}{x^{1/2}\ln x}\left(1 + \frac{1}{\ln x} + \frac{4}{\ln^2 x}\right) + \frac{\ln 2\pi}{x \ln x}.$$

To obtain an upper bound for I_2, a frame of the difference $\psi - \vartheta$ is used: according to Lemma 3 of [10], under the Riemann hypothesis, for $x \geqslant 121$,

$$\sqrt{x} < \psi(x) - \vartheta(x) < \sqrt{x} + \frac{4}{3}x^{1/3}.$$

Thus, with the notations of Lemma 2.1,

$$I_2(x) \leqslant F_{1/2,1}(x) + \frac{4}{3}F_{1/3,1}(x).$$

Moreover, $F_{1/2,1}(x) \leqslant \frac{2}{\sqrt{x}\ln x} - \frac{2}{\sqrt{x}\ln^2 x} + \frac{8}{\sqrt{x}\ln^3 x}$ and $F_{1/3,1}(x) \leqslant \frac{3}{2x^{2/3}\ln x}$. As a result,

$$I_2(x) \leqslant \frac{2}{\sqrt{x}\ln x}\left(1 - \frac{1}{\ln x} + \frac{4}{\ln^2(x)} + \frac{1}{x^{1/6}}\right).$$

□

2.2 Improvement of the error estimate.

In this section, we update the results, [8], [9] (Theorem 4.1 & 4.4).

Here and throughout the rest of the paper, $f(x) = \mathcal{O}^*(g(x))$ means $|f(x)| \leqslant g(x)$.

Theorem 2.3. *Let $M = 0.261497\cdots$, the Meissel-Mertens constant (sequence A077761 of OEIS). If the Riemann hypothesis is true, then we have for $x \geqslant 1628.6$,*

$$\sum_{p\leqslant x}\frac{1}{p} = \ln\ln x + M + \mathcal{O}^*\left(\frac{\ln(x/\ln x)}{8\pi\sqrt{x}}\right).$$

Proof. The sum of the reciprocals of the primes is related to $\vartheta(x)$ by (4.20) of [12],

$$\begin{aligned} \sum_{p\leqslant x}\frac{1}{p} = & \ \ln_2 x + M + \tfrac{\vartheta(x)-x}{x\ln x} \\ & - \int_x^\infty \tfrac{(\vartheta(y)-y)(1+\ln y)}{y^2\ln^2 y}\mathrm{d}y. \end{aligned}$$

Let's define Z_1 by

$$Z_1 = \left|\sum_{p\leqslant x}\frac{1}{p} - \ln_2 x - M\right|. \quad (4)$$

As

$$Z_1 \leqslant \frac{|\vartheta(x)-x|}{x\ln x} + \int_x^\infty \frac{|\vartheta(y)-y|(1+\ln y)}{y^2\ln^2 y}\mathrm{d}y. \quad (5)$$

we have for $x \geq 10^{11}$ by Proposition 2.5 of [9], and Lemma 2.2,

$$\begin{aligned} Z_1 \ & \leq \ \frac{1}{8\pi}\frac{\ln x - \ln\ln x - 2}{\sqrt{x}} + I_1(x) + I_2(x) \\ & \leq \ \frac{1}{8\pi}\frac{\ln(x/\ln x)}{\sqrt{x}}. \end{aligned}$$

with $I_1(x)$ and $I_2(x)$ defined in the proof of Lemma 2.2.

We check by direct calculation up to 10^{11} that the bounds remain valid for Z_1. □

3 Others sums or products of primes

We need to evaluate another integral for the other functions of primes.

Proposition 3.1. *Under the Riemann hypothesis,*

$$\int_x^\infty \frac{\vartheta(t)-t}{t^2}\mathrm{d}t \leqslant \frac{2}{\sqrt{x}}+\frac{0.0462}{x^{1/2}}+\frac{2}{x^{2/3}}+\frac{\ln 2\pi}{x}.$$

Proof. We need to evaluate

$$\int_x^\infty \frac{\vartheta(t)-t}{t^2}\mathrm{d}t = \int_x^\infty \frac{\vartheta(t)-\psi(t)+\psi(t)-t}{t^2}\mathrm{d}t = I_3(x)-I_4(x)$$

where

$$I_3(x)=\int_x^\infty \frac{\psi(t)-t}{t^2}\mathrm{d}t,$$

and

$$I_4(x)=\int_x^\infty \frac{\psi(t)-\vartheta(t)}{t^2}\mathrm{d}t.$$

To obtain an upper bound for I_3, we use a classical explicit formula in number theory (Chapter 17 of [11]) $\psi(x)=x-\ln 2\pi-\sum_\rho \frac{x^\rho}{\rho}-\frac{1}{2}\ln(1-1/x^2)$ in the form of

$$\int_x^\infty f(t)(\psi(t)-t+\ln 2\pi+\frac{1}{2}\ln(1-1/t^2))\mathrm{d}t$$

$$=-\int_x^\infty \sum_\rho f(t)\frac{t^\rho}{\rho}\mathrm{d}t$$

and taking $f(t)=\frac{1}{t^2}$, we have $f'(t)=O(\frac{1}{t^3})$ for $n\geqslant 1$ and thus invert the sum and integral signs by the dominated convergence theorem. Thus

$$I_3(x)= -\left(\sum_\rho \int_x^\infty f(t)\frac{t^\rho}{\rho}\mathrm{d}t\right) - \int_x^\infty f(t)(\ln 2\pi+\tfrac{1}{2}\ln(1-1/t^2))\mathrm{d}t.$$

For $t\geqslant 2$, $0<-\frac{1}{2}\ln(1-1/t^2)<\ln 2\pi$ so for $x\geqslant 2$,

$$-\int_x^\infty f(t)(\ln 2\pi)\mathrm{d}t < -\int_x^\infty f(t)(\ln 2\pi+\frac{1}{2}\ln(1-1/t^2))\mathrm{d}t<0$$

Thus

$$0<\int_x^\infty f(t)(\ln 2\pi+\frac{1}{2}\ln(1-1/t^2))\mathrm{d}t\leqslant \frac{\ln 2\pi}{x}.$$

On the other hand,

$$-\left(\sum_\rho \int_x^\infty f(t)\frac{t^\rho}{\rho}\mathrm{d}t\right)=\sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)}$$

Let, under the Riemann hypothesis

$$\frac{1}{x^{1/2}}\sum_\rho \frac{x^{i\Im(\rho)}}{\rho(\rho-1)}$$

and $|\sum_\rho \frac{x^{i\Im(\rho)}}{\rho(\rho-1)}|\leqslant \sum_\rho \frac{1}{|\rho|^2}=\gamma+2-\ln 4\pi\leqslant 0.0462$. Thus we have,

$$|I_3(x)|\leqslant \frac{0.0462}{x^{1/2}}+\frac{\ln 2\pi}{x}.$$

To obtain an upper bound for I_4, bounds of the difference $\psi-\vartheta$ are used: according to Lemma 3 of [10], under the Riemann hypothesis, for $x\geqslant 121$,

$$\sqrt{x}<\psi(x)-\vartheta(x)<\sqrt{x}+\frac{4}{3}x^{1/3}.$$

Thus,

$$\int_x^\infty \frac{\sqrt{t}+4/3\sqrt[3]{t}}{t^2}\mathrm{d}t=\left[-\frac{2}{\sqrt{t}}-\frac{2}{t^{2/3}}\right]_x^\infty.$$

As a result,

$$I_4(x)\leqslant \frac{2}{\sqrt{x}}+\frac{2}{x^{2/3}}.$$

□

Theorem 3.2. *Let B_3 (sequence A083343 in OEIS) the constant given by the infinite sum*

$$B_3=\gamma+\sum_{n=2}^\infty \sum_p (\ln p)/p^n \approx 1.33258\,22757\,33221.$$

Assuming the Riemann hypothesis, we have for $x\geqslant 1674.5$,

$$\sum_{p\leqslant x}\frac{\ln p}{p}=\ln x-B_3+\mathcal{O}^*\left(\frac{\ln(x/\ln x)}{8\pi\sqrt{x}}\ln x\right). \quad (6)$$

Proof. By [12], (4.21), which refines Mertens' first theorem,

$$\sum_{p\leq x}\frac{\ln p}{p}=\ln x-B_3+\frac{\vartheta(x)-x}{x}-\int_x^\infty \frac{\vartheta(y)-y}{y^2}\mathrm{d}y.$$

Let's define Z_2 by

$$Z_2=|\sum_{p\leq x}\frac{\ln p}{p}-\ln x+B_3|. \quad (7)$$

Hence, the remainder term can be bounded using the theta function

$$Z_2\leqslant \frac{|\vartheta(x)-x|}{x}+\int_x^\infty \frac{\vartheta(y)-y}{y^2}\mathrm{d}y$$

We use Proposition 2.5 of [9], for the first term, and Proposition 3.1 for the second, so we obtain

$$\begin{aligned} Z_2 &\leqslant \frac{\ln x}{8\pi\sqrt{x}}(\ln(x/\ln x)-2)+\int_x^\infty \frac{\vartheta(y)-y}{y^2}\mathrm{d}y \\ &\leqslant \frac{\ln x}{8\pi\sqrt{x}}(\ln(x/\ln x)-2)+\frac{2}{\sqrt{x}}+\frac{0.0462}{x^{1/2}} \\ &\quad +\frac{2}{x^{2/3}}+\frac{\ln 2\pi}{x}\le \frac{\ln x}{8\pi\sqrt{x}}\ln(x/\ln x), \end{aligned}$$

where the last inequality is only valid for $x \ge 2.04\cdot 10^{11}$.

We check (6) for $x \le 2.1\cdot 10^{11}$ by computer. □

Lemma 3.3. *For $x \ge 2$, we have*

$$\frac{1}{\zeta(2)} \le \prod_{p\le x}\left(1-\frac{1}{p^2}\right) < \frac{1}{\zeta(2)}\left(1+\frac{1}{x}\right).$$

Proof. We have by [6], (Proof of Theorem 280)

$$\left|\zeta(s)-\prod_{p\le q}\left(\frac{1}{1-p^{-s}}\right)\right| < \sum_{n=q+1}^{\infty}\frac{1}{n^\sigma},$$

where $\sigma = \Re(s)$. Hence

$$\frac{1}{\zeta(2)+\sum_{n>x}\frac{1}{n^2}} < \prod_{p\le x}\left(1-\frac{1}{p^2}\right)$$

and

$$\begin{aligned} \prod_{p\le x}\left(1-\frac{1}{p^2}\right) &< \frac{1}{\zeta(2)-\sum_{n>x}\frac{1}{n^2}} \\ &= \frac{1}{\zeta(2)}\frac{1}{1-\frac{1}{\zeta(2)}\sum_{n>x}\frac{1}{n^2}}. \end{aligned}$$

Let's take

$$u = \frac{1}{\zeta(2)}\sum_{n>x}\frac{1}{n^2}. \tag{8}$$

For $x \ge 2$, we have $u < \frac{1}{\zeta(2)}(\zeta(2)-1-1/4) < 0.25$ and

$$\frac{1}{1-u} < 1+u+\frac{4}{3}u^2 \quad \text{for } u < 0.25. \tag{9}$$

We have also $\sum_{n>x}\frac{1}{n^2} < \int_{x-1}^{\infty}\frac{\mathrm{d}t}{t^2} = \frac{1}{x-1}$. By combining definition (8) with inequation (9),

$$\frac{1}{1-u} < 1+\frac{1}{\zeta(2)}\frac{1}{x-1}+\frac{4}{3}\left(\frac{1}{\zeta(2)}\frac{1}{x-1}\right)^2 < 1+\frac{1}{x}$$

for $x \ge 5$. □

Theorem 3.4. *If the Riemann hypothesis is satisfied, we have*

for $x \ge 1628.0$,

$$\prod_{p\le x}\left(1-\frac{1}{p}\right) = \frac{e^{-\gamma}}{\ln x}\left(1+\mathcal{O}^*\left(\frac{\ln(x/\ln x)}{8\pi\sqrt{x}}\right)\right)$$

for $x \ge 1628.4$,

$$\prod_{p\le x}\left(\frac{p}{p-1}\right) = e^{\gamma}\ln x\left(1+\mathcal{O}^*\left(\frac{\ln(x/\ln x)}{8\pi\sqrt{x}}\right)\right)$$

for $x \ge 1629.2$,

$$\prod_{p\le x}\frac{p+1}{p} = \frac{6}{\pi^2}e^{\gamma}\ln x\left(1+\mathcal{O}^*\left(\frac{\ln(x/\ln x)}{8\pi\sqrt{x}}\right)\right).$$

Proof. Let $S = \sum_{p>x}(\ln(1-1/p)+1/p)$. By [12, (8.10)], we have

$$0 > S > S_0 = \frac{-1.02}{(x-1)\ln x} \text{ if } x > 1. \tag{10}$$

Then

$$\begin{aligned} P_1 &= \prod_{p\le x}\left(1-\frac{1}{p}\right) = \exp\left(\sum_{p\le x}\ln\left(1-\frac{1}{p}\right)\right) \\ &= \exp\left(M-\gamma-\sum_{p\le x}\frac{1}{p}-S\right) \end{aligned}$$

Proceeding in the same way as in the proof of Theorem 2.3, we obtain

$$Z_1 \le t(x) = \frac{1}{8\pi\sqrt{x}}\ln(x/\ln x)-\frac{2\ln\ln x}{8\pi\sqrt{x}\ln x} \tag{11}$$

Let $z(x) = \frac{1}{8\pi\sqrt{x}}\ln(x/\ln x)$. We have $\exp(t(x)+\frac{1.02}{(x-1)\ln x}) \le 1+z(x)$ and $(1+1/x)\exp(t(x)) \le 1+z(x)$ for $x \ge 10^{11}$.

Combining (11) and (10) with the previous result, we get for $x \ge 10^{11}$,

$$\begin{aligned} P_1 &\le \tfrac{e^{-\gamma}}{\ln x}\exp(Z_1-S) \le \tfrac{e^{-\gamma}}{\ln x}\exp(t(x)-S_0) \\ &\le \tfrac{e^{-\gamma}}{\ln x}\ln x(1+z(x)). \end{aligned}$$

Similarly, $P_2 = \frac{1}{P_1} = \prod_{p\le x}\left(\frac{p}{p-1}\right) \le e^{\gamma}(\ln x)\exp t(x) \le e^{\gamma}\ln x\,(1+z(x))$.
Moreover, $P_1 \ge \frac{1}{e^{\gamma}\ln x\,(1+z(x))} = \frac{e^{-\gamma}}{\ln x}\frac{1}{1+z(x)} \ge \frac{e^{-\gamma}}{\ln x}(1-z(x))$. We also have $P_2 = 1/P_1 \ge \frac{1}{e^{-\gamma}(1+z(x))/\ln x} \ge e^{\gamma}\ln x\frac{1}{1+z(x)} > e^{\gamma}\ln x(1-z(x))$.

The last product is closely related to the others. Since $1+1/p = (1-1/p^2)/(1-1/p)$, we write

$$P_3 = \prod_{p\le x}\frac{p+1}{p} = P_2\cdot\prod_{p\le x}\left(1-1/p^2\right).$$

Hence, by Lemma 3.3, $P_3 < P_2 \cdot \frac{1}{\zeta(2)}(1+1/x) < \frac{e^\gamma \ln x}{\zeta(2)}(1+1/x)\exp t(x) < \frac{e^\gamma \ln x}{\zeta(2)}(1+z(x))$ and $P_3 > P_2/\zeta(2) > \frac{e^\gamma \ln x}{\zeta(2)}(1-z(x))$.

We check by computer for $x \leq 10^{11}$ that the inequalities are still valid.

□

4 Conclusion

Analytic number theory, [13], studies the properties of functions on prime numbers using analytic objects (for example, here, the use of Riemann's zeta function). Under Riemann's hypothesis concerning the zeros of the zeta function, we obtain, in Section 2, an accurate estimate of the sum of the reciprocals of the primes. The result can be applied to other sums, as we saw in Section 3. This study can be developed for other functions using prime numbers.

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