On *K*-Chebsyhev Sequence

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Abstract: - In this study, firstly the k-Chebsyhev sequence is defined, and some terms of this sequence are given. Then, the relations between the terms of the k-Chebsyhev sequence are presented and the generating function of this sequence is obtained. In addition, the Catalan transformation of the sequence is given and the generating function of the Catalan k-Chebsyhev sequence is obtained. Finally, the Hankel transform is applied to the Catalan k-Chebsyhev transform.

Key-Words: - Fibonacci sequences, k-Chebsyhev sequences, Catalan Transformation, Cassini Identity, Binet Formula

Received: November 25, 2022. Revised: May 19, 2023. Accepted: June 12, 2023. Published: July 14, 2023.

1 Introduction

Many studies have been done on the Chebsyhev, Fibonacci, and Lucas sequence (see for details, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]). In [14], Chebsyhev polynomials of the first and second kind are defined as follows, respectively,

$T_{n+2} = 2xT_{n+1} - T_n \ n \geq 0,$ with $T_{k,0} = 1, T_{k,1} = x,$

$U_{n+2} = 2xU_{n+1} - U_n \ n \geq 0,$ with $U_{k,0} = 1, U_{k,1} = 2x.$

In [15], the relationship between Chebsyhev polynomials and matrices was examined. In [16], the authors defined the k-Fibonacci sequence and gave the properties related to the k-Fiboonacci sequence. Then the k-Pell, k-Jacobsthal, k-Pell-Lucas, Modified k-Pell and k-Jacobsthal-Lucas numbers were defined in [17], [18], [19], respectively,

$P_{k,n+2} = 2P_{k,n+1} + kP_{k,n} \ n \geq 0,$ with $P_{k,0} = 0, P_{k,1} = 1,$

$J_{k,n+2} = 2J_{k,n+1} + kJ_{k,n} \ n \geq 0,$ with $J_{k,0} = 0,$ $J_{k,1} = 1,$

$Q_{k,n+2} = 2Q_{k,n+1} + kQ_{k,n} \ n \geq 0,$ with $Q_{k,0} = 2, Q_{k,1} = 2,$

$q_{k,n+2} = 2q_{k,n+1} + q_{k,n} \ n \geq 0,$ with $q_{k,0} = 1, q_{k,1} = 1$

and

$S_{k,n+2} = S_{k,n+1} + 2kS_{k,n} \ n \geq 0,$ with $S_{k,0} = 2, S_{k,1} = 1.$

In addition, in [20], [21], k-Fibonacci, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, and Jacobsthal-Lucas numbers sequences which are a new generalization of the Fibonacci sequence were defined, and their properties were found. In [22], [23], [24], Hankel and Catalan's transformations were defined and some of their properties were brought to the literature.

In Chapter 2, we define the k-Chebsyhev sequence, then give the characteristic equation, the Binet formula, and some properties of the sequence. We also show the relationship between the positive and negative terms of the sequence, get a relationship between three consecutive terms, and examine the relationship between the terms in the limit infinite case. Finally in this chapter, we give the Cassini and the Honsberger identity for this sequence.

In Chapter 3, the Catalan transformation of the k-Chebsyhev sequence is defined, and some properties are given. Finally, the Hankel transform for the k-Chebsyhev sequence is calculated.

2 *k*-Chebsyhev Sequence

Let's take a positive real number k for the sequence we are going to define. The k-Chebsyhev sequences $C_{k,n}$ are as follows;

$$C_{k,n+1} = 2k.C_{k,n} - C_{k,n-1},\ n \geq 1 \quad (1)$$

with $C_{k,0} = 1$ and $C_{k,1} = k$.

Then, let's write the characteristic equation for the sequence, the roots of the characteristic equation, and the Binet formula.

The characteristic equation of the sequence is

$$r^2 - 2kr + 1 = 0.$$

The roots of the characteristic equation of the sequence are as follows;

$$\mathrm{r}_1 = k + \sqrt{k^2 - 1}$$

and

$$\mathrm{r}_2 = k - \sqrt{k^2 - 1}$$

where $r_1 + r_2 = 2k$, $r_1 - r_2 = 2\sqrt{k^2 - 1}$, $r_1^2 + r_2^2 = 4k^2 - 2$, $r_1.r_2 = 1$.

The Binet formula of the defined sequence $C_{k,n}$ is as follows.

$$C_{k,n} = \frac{r_1^n + r_2^n}{2}. \quad (2)$$

Now let's give the first elements of k-Chebyshev sequence.

- $C_{k,2} = 2k.C_{k,1} - C_{k,0} = 2k^2 - 1$
- $C_{k,3} = 2k.C_{k,2} - C_{k,1} = 4k^3 - 3k$

For some n, we give some values of the k-Chebyshev sequence in Table 1.

Table 1. k-Chebyshev sequence for $0 \leq n \leq 7$

n	$C_{k,n}$
0	1
1	k
2	$2k^2 - 1$
3	$4k^3 - 3k$
4	$8k^4 - 8k^2 + 1$
5	$16k^5 - 20k^3 + 5k$
6	$32k^6 - 48k^4 + 18k^2 - 1$
7	$64k^6 + 80k^5 + 32k^4 + 60k^3 + 8\mathrm{k}$

Theorem The Binet formula of the k-Chebsyhev sequence is as follows;

$$C_{k,n} = \frac{r_1^n + r_2^n}{2}.$$

Proof The general solution of a sequence is

$$C_{k,n} = A.{r_1}^n + B.{r_2}^n.$$

Here, the scalars A and B can be obtained by substituting the initial conditions. It is obtained by solving the given system of equations. For $n = 0$ it is $C_{k,0} = 1$ and for $n = 1$ it is $C_{k,1} = 1$. Thus $A = \frac{1}{2}$ and $B = \frac{1}{2}$ are obtained. From here

$$C_{k,n} = \frac{r_1^n + r_2^n}{2}. \quad \square$$

Theorem Let r be the positive and big root of the characteristic equation of k-Chebsyhev sequence.

$$\lim_{n\to\infty} \frac{C_{k,n+m}}{C_{k,n}} = r^m.$$

Proof

$$\lim_{n\to\infty} \frac{C_{k,n+m}}{C_{k,n}} = \lim_{n\to\infty} \frac{\frac{r_1^{n+m} + r_2^{n+m}}{2}}{\frac{r_1^n + r_2^n}{2}}$$

$$= \lim_{n\to\infty} \frac{r_1^{n+m} + r_1^{-n-m}}{r_1^n + r_1^{-n}} \quad (r_2 = r_1^{-1})$$

$$= r^m. \quad \square$$

Theorem The following relation is satisfied.

$$C_{k,n} = 2.C_{k,-n}.$$

Proof If the Binet formula of the sequence k-Chebsyhev is used

$$C_{k,-n} = \frac{r_1^{-n} + r_2^{-n}}{2}.$$

$$= \frac{\frac{1}{{\mathrm{r}_1}^{\mathrm{n}}} + \frac{1}{{\mathrm{r}_2}^{\mathrm{n}}}}{2}$$

$$= \frac{{\mathrm{r}_1}^{\mathrm{n}} + {\mathrm{r}_2}^{\mathrm{n}}}{2.(\mathrm{r}_1.\mathrm{r}_2)^{\mathrm{n}}} \quad ((r_1.r_2)^n = 1^n = 1)$$

$$= \frac{C_{k,n}}{2}.\ (C_{k,n} = \frac{r_1^n + r_2^n}{2})$$

Thus

$$C_{k,n} = 2.C_{k,-n}. \quad \square$$

Theorem (Cassini Identity)

$$C_{k,n+1}.C_{k,n-1} - C_{k,n}^2 = k^2 - 1.$$

Proof From the Binet formula of the sequence k-Chebsyhev, then we have

$$C_{k,n+1}.C_{k,n-1} - C_{k,n}^2 = \frac{({r_1}^{n+1} + {r_2}^{n+1})}{2}.\frac{({r_1}^{n-1} + {r_2}^{n-1})}{2} - (\frac{{r_1}^n + {r_2}^n}{2}).(\frac{{r_1}^n + {r_2}^n}{2})$$

$$= \frac{{r_1}^{2n} + {r_1}^{n+1}.{r_2}^{n-1} + {r_2}^{n+1}.{r_1}^{n-1} + {r_2}^{2n}}{4} - \frac{{r_1}^{2n} + 2.{r_1}^n.{r_2}^n + {r_2}^{2n}}{4}$$

$$= \frac{(r_1.r_2)^n.r_1}{4r_2} + \frac{(r_1.r_2)^n.r_2}{4r_1} - \frac{2.(r_1.r_2)^n}{4}$$

$$= \frac{r_1}{4r_2} + \frac{r_2}{4r_1} - \frac{2}{4}$$

$$= k^2 - 1. \quad \square$$

Theorem (Honsberger Identity)

$$C_{k,n+1}^2 - C_{k,n-1}^2 = \frac{C_{\mathrm{k,2n+2}} - C_{\mathrm{k,2n-2}}}{2}.$$

Proof By the Binet formula of the sequence k-Chebsyhev, we obtain

$$C_{k,n+1}^2 - C_{k,n-1}^2 = (C_{\mathrm{k,n+1}} - C_{\mathrm{k,n-1}})(C_{\mathrm{k,n+1}} + C_{\mathrm{k,n-1}})$$

$$= (\frac{r_1^{n+1} + r_2^{n+1}}{2} + \frac{r_1^{n-1} + r_2^{n-1}}{2})(\frac{r_1^{n+1} + r_2^{n+1}}{2} - \frac{r_1^{n-1} + r_2^{n-1}}{2})$$

$$=\frac{r_1^{2n+2}+r_1^{n+1}.r_2^{n+1}-r_1^{2n}-r_1^{n+1}.r_2^{n-1}}{4}+\frac{r_2^{n+1}.r_1^{n+1}+r_2^{2n+2}-r_2^{n+1}.r_1^{n-1}-r_2^{2n}}{4}+\frac{r_1^{2n}+r_1^{n-1}.r_2^{n+1}-r_1^{2n-2}-r_1^{n-1}.r_2^{n-1}}{4}+\frac{r_2^{n-1}.r_1^{n+1}+r_2^{2n}-r_2^{n-1}.r_1^{n-1}-r_2^{2n-2}}{4}$$
$$=\frac{r_1^{2n+2}+1+1+r_2^{2n+2}-r_1^{2n-2}-1-1-r_2^{2n-2}}{4}$$
$$=\frac{(r_1^{2n+2}+r_2^{2n+2})}{2.2}-\frac{(r_1^{2n-2}+r_2^{2n-2})}{2.2}$$
$$=\frac{C_{k,2n+2}-C_{k,2n-2}}{2}. \quad \square$$

Theorem There is the following relationship between the squares of consecutive terms of the sequence $C_{k,n}$.

$$C_{k,n+}^2+C_{k,n+1}^2=\frac{C_{k,2n}+C_{k,2n+2}}{2}+1.$$

Proof $C_{k,n}^2+C_{k,n+1}^2=(\frac{r_1^n+r_2^n}{2})^2+(\frac{r_1^{n+1}+r_2^{n+1}}{2})^2$
$$=\frac{r_1^{2n}+2r_1^n.r_2^n+r_2^{2n}}{4}+\frac{r_1^{2n+2}+2.r_1^{n+1}.r_2^{n+1}+r_2^{2n+2}}{4}$$
$$=\frac{r_1^{2n}+2+r_2^{2n}}{4}+\frac{r_1^{2n+2}+2+r_2^{2n+2}}{4}$$
$$=\frac{r_1^{2n}+r_2^{2n}}{2.2}+\frac{r_1^{2n+2}+r_2^{2n+2}}{2.2}+\frac{4}{4}$$
$$=\frac{C_{k,2n}+C_{k,2n+2}}{2}+1. \quad \square$$

Theorem (Generating Function)

$$C(x)=\frac{1-kx}{x^2-2kx+1}$$

Proof The following equations are written for the k-Chebsyhev sequence,

$$C(x)=\sum_{n=0}^{\infty}C_{k,n}.x^n=1+kx+\sum_{n=2}^{\infty}C_{k,n}.x^n$$
$$=1+kx+2k\sum_{n=2}^{\infty}C_{k,n-1}.x^n-\sum_{n=2}^{\infty}C_{k,n-2}.x^n$$
$$=1+kx+2kx(C(x)-1)-x^2.C(x).$$

Thus

$$C(x)=\frac{1-kx}{x^2-2kx+1}.$$

3 Catalan Number

In [23], [24], the n^{th} Catalan numbers where n is a positive integer are as follows.

$$C_n=\frac{1}{n+1}(2n,n) \text{ or } C_n=\frac{(2n)!}{(n+1)!.n!}$$

and the tensile function is given as

$$C(x)=\frac{1-\sqrt{1-4x}}{2x} \quad (3)$$

n for some natural numbers n. Catalan numbers are $\{1,1,2,5,14,132,429,1430,4862,...\}$ shaped.

3.1 Catalan Transformation for the *k*-Chebyshev Sequences

Using the Catalan transformation, we define the Catalan transformation of the k-Chebsyhev sequences $\{C_{k,n}\}$ as followed.

$$CC_{k,n}=\sum_{i=0}^{n}\frac{i}{2n-i}\binom{2n-i}{n-i}C_{k,i},\ n\geq 1 \quad (4)$$

with $CC_{k,0}=0$.

Now we can give the Catalan transformation of the first elements of the k-Chebsyhev sequence.

- $CC_{k,1}=\sum_{i=0}^{1}\frac{i}{2-i}\binom{2-i}{1-i}C_{k,i}=1.k=k$
- $CC_{k,2}=\sum_{i=0}^{2}\frac{i}{4-i}\binom{4-i}{2-i}C_{k,i}=2k^2+k-1$

Some values of the Catalan transformation of the k-Chebsyhev sequence is given Table 2.

Table 2. Catalan transformation of the k-Chebsyhev sequence $0\leq n\leq 6$

n	$CC_{k,n}$
0	0
1	k
2	$2k^2+k-1$
3	$4k^3+4k^2-k+2$
4	$8k^4+12k^3-18k^2-4k-4$
5	$16k^5+32k^4+16k^3-4k^2-8k-14$
6	$32k^6+80k^5-48k^4+124k^3-10k^2-17k-43$

We can show $\{C_{k,n}\}$ as the $n\ x\ 1$ matrix C_k and the product of the lower triangular matrix C as

$$\begin{bmatrix}CC_{k,1}\\CC_{k,2}\\CC_{k,3}\\CC_{k,4}\\\vdots\end{bmatrix}=\begin{bmatrix}1&&&&\cdots\\1&1&&&\cdots\\2&2&1&&\cdots\\5&5&3&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}\begin{bmatrix}C_{k,1}\\C_{k,2}\\C_{k,3}\\C_{k,4}\\\vdots\end{bmatrix}$$

So,

$$\begin{bmatrix}k\\2k^2+k-1\\4k^3+4k^2-k+2\\8k^4+12k^3-18k^2-4k-4\\\vdots\end{bmatrix}=\begin{bmatrix}1&&&&\cdots\\1&1&&&\cdots\\2&2&1&&\cdots\\5&5&3&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}\begin{bmatrix}k\\2k^2-1\\4k^3-3k\\8k^4-8k^2+1\\\vdots\end{bmatrix}$$

3.2 The Generating Function Catalan k-Chebsyhev Sequence

The generating function of k-Chebsyhev and Catalan sequences are as follows, respectively,

$$C(x) = \frac{1-kx}{x^2-2kx+1}$$

and

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Thus, the following equations are written for the generating function of the Catalan k-Chebsyhev sequence,

$$CC_{k,n}(x) = C_{k,n}\big(x * C(x)\big)$$

$$= \frac{1-k.\frac{1-\sqrt{1-4x}}{2}}{(\frac{1-\sqrt{1-4x}}{2})^2 - 2k.\frac{1-\sqrt{1-4x}}{2} + 1}$$

$$= \frac{4-2k-2k.(\sqrt{1-4x})}{6-4x-4k+(4k-2).\sqrt{1-4x}}.$$

3.3 Hankel Transform of the Catalan k-Chebsyhev Sequences

In [22], [23], let the set of the terms of a sequence be $A = \{a_1, a_2, a_3, \ldots\}$. The Hankel transform H_{n} of the terms of this sequence is defined as follows.

$$H_{\mathrm{n}} = det\begin{bmatrix} a_1\ a_2\ \ a_3\ a_4\ \ldots \\ a_2\ a_3\ \ a_4\ a_5\ldots \\ a_3\ a_4\ \ a_5\ a_6\ \ldots \\ a_4\ a_5\ \ a_6\ a_7\ \cdots \\ \vdots\ \ \vdots\ \ \ \vdots\ \ \vdots\ \ \ \ddots \end{bmatrix} \quad (5)$$

For example, the Hankel matrix of the 3rd Lucas sequence,

$$H_3 = det\begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 7 \\ 4 & 7 & 11 \end{bmatrix}.$$

The determinant of this matrix $H_3 = 0$.

If we apply Hankel's work to the Catalan k-Chebsyhev sequence, we finally get;

$$HCC_1 = det[CC_1]$$
$$= det[k] = k$$
$$HCC_2 = det\begin{bmatrix} CC_1 & CC_2 \\ CC_2 & CC_3 \end{bmatrix}$$
$$= det\begin{bmatrix} k & 2k^2+k-1 \\ 2k^2+k-1 & 4k^3+4k^2-k+2 \end{bmatrix}$$
$$= 2k^2+4k+1$$
$$HCC_3 = det\begin{bmatrix} CC_1 & CC_2 & CC_3 \\ CC_2 & CC_3 & CC_4 \\ CC_3 & CC_4 & CC_5 \end{bmatrix}$$
$$= -240k^5 - 88k^4 - 339k^3 + 86k^2 - 60k + 22.$$

4 Conclusions

In this paper, we first defined the k- Chebsyhev. We then gave the main features of this sequence. We also examined the relationships between the terms of this sequence. Finally, we introduced the Catalan and Hankel transformation of the sequence. This work can be further extended to Horadam numbers and Mersenne numbers.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

-Engin ÖZKAN carried out the introduction and the main result of the article.

-Hakan AKKUŞ has improved Chapter 2 and Chapter 3. All authors read and approved the final manuscript.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This study did not receive any funding in any form.

Conflict of Interest

The authors have no conflict of interest to declare.