Some Inequalities over the Eigenvalues of a Strongly Regular Graph

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Abstract: Let's consider a primitive strongly regular graph G. In this paper, we establish some inequalities over the spectrum of G in the environment of a real finite dimensional Euclidean Jordan algebra $\mathcal{A}$ associated with G recurring to a spectral analysis of some elements of $\mathcal{A}$ and recurring to a spectral analysis of the Generalized Krein parameters of G.

Key–Words:Euclidean Jordan Algebras, Theory of Graphs, Strongly regular graphs.

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1 Introduction

For a precise and hard description of Euclidean Jordan algebras, one must cite the monographs books, "Analysis on Symmetric Cones" , [2] and "Strongly Regular Graphs and Euclidean Jordan Algebras Revelations within an Unusual Relationship", [3].

Several mathematicians and scientists developed several applications of the theory of real Euclidean Jordan algebras in many areas of research, see for instance [4], [5], [6], [7], [8], [9], [10], [11], [12], [13] and [14] but our main goal is, recurring to this theory, to establish inequalities over the parameters of some symmetric discrete structures like the strongly regular graphs and the symmetric association schemes, see for instance [15] and [16].

This paper is organized as follows. In the section 2 we present the main results about real finite dimensional Euclidean Jordan algebras necessary for a good understanding of the following sections, and present in this section some examples. In the section 3 the more relevant results about strongly regular graphs are described. In section 4 some inequalities are established over the eigenvalues of a primitive strongly regular graph. Finally, in section 5 some conclusions are presented.

2 Some Concepts about Euclidean Jordan Algebras

In this section, we present some concepts about real finite-dimensional Euclidean Jordan algebras and follow the notation and the text presented in [1].

For a very readable text about the principal results and clear examples of Jordan algebras and properties of Euclidean Jordan algebras one must indicate the work "Strongly Regular Graphs and Euclidean Jordan Algebras Revelations within an Unusual Relationship", [3].

But we couldn't avoid citing the chapter "Symmetric Cones, Potential Reduction Methods and word by word Extensions", see [17], for a very good introductory text about Euclidean Jordan algebras.

A real finite-dimensional Euclidean Jordan algebra is a real finite-dimensional algebra, with an operation of vector multiplication $\bigstar$ such that for any of its elements, u, v and w we have $u\bigstar v = v\bigstar u$ and $u^{2\bigstar}\bigstar(u\bigstar v) = u\bigstar(u^{2\bigstar}\bigstar v)$, where $u^{2\bigstar} = u\bigstar u$, equipped with a scalar product $\bullet|\bullet$ such that $(u\bigstar v)|w = v|(u\bigstar w)$.

In the following text, we will designate a real finite-dimensional Euclidean Jordan algebra by RFEJA and, we will designate an Euclidean Jordan algebra only by EJA. And, the unit element of a RFEJA or of a EJA will be denoted by $\mathbf{e}$.

Example 1 *Let's consider a natural number m and the real vector space $\mathcal{B} = \mathbb{R}^m$ with the usual vector operations of addiction of vectors and the usual multiplication of a vector by a real scalar. Then if in $\mathcal{B}$ we* *consider a multiplication $\bigstar$ of two of it's elements u and v, such that:*

$$u\bigstar v = (u_1v_1, u_2v_2, \cdots, u_mv_m)$$

with $u = (u_1, u_2, \cdots, u_m)$ and $v = (v_1, v_2, \cdots, v_m)$, and if we consider the inner product $\bullet|\bullet$ such that $u|v = \sum_{j=1}^{m} u_jv_j$. then $\mathcal{B} = \mathbb{R}^m$ equipped with these two operations is a RFEJA. So, we must show that for any real numbers α and β, and for any vectors u, v and w of $\mathcal{B}$ we have

$$\begin{aligned}(\alpha u + \beta v)\bigstar w &= \alpha(u\bigstar w) + \beta(v\bigstar w),\\ u\bigstar v &= v\bigstar u,\\ u^{2\bigstar}\bigstar(u\bigstar v) &= u\bigstar(u^{2\bigstar}\bigstar v),\\ (u\bigstar v)|w &= v|(u\bigstar w),\end{aligned}$$

where $u^{2\bigstar} = u\bigstar u$. Let u, v and w be elements of $\mathcal{B}$ and let α and β be real numbers. Considering the notation $u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_m), w = (w_1, \ldots, w_m)$, we have the following calculations.

$$\begin{aligned}(\alpha u + \beta v)\bigstar w &= (\alpha u_1 + \beta v_1, \ldots, \alpha u_m + \beta v_m)\bigstar\\ &\bigstar\ w\\ &= (\alpha u_1w_1 + \beta v_1w_1, \ldots,\\ &\quad , \alpha u_mw_m + \beta v_mw_m)\\ &= (\alpha u_1w_1, \ldots, \alpha u_mw_m) +\\ &+\ (\beta v_1w_1, \ldots, \beta v_mw_m)\\ &= \alpha(u_1w_1, \ldots, u_mw_m) +\\ &+\ \beta(v_1w_1, \ldots, v_mw_m)\\ &= \alpha(u\bigstar w) + \beta(v\bigstar w),\end{aligned}$$

$$\begin{aligned}u\bigstar v &= (u_1, \ldots, u_m)\bigstar(v_1, \ldots, v_m)\\ &= (u_1v_1, \ldots, u_mv_m)\\ &= (v_1u_1, \ldots, v_mu_m)\\ &= v\bigstar u,\\ u^{2\bigstar} &= u\bigstar u,\\ &= (u_1, \ldots, u_m)\bigstar(u_1, \ldots, u_m)\\ &= (u_1^2, \ldots, u_m^2)\\ u^{2\bigstar}\bigstar(u\bigstar v) &= (u_1^2, \ldots, u_m^2)\bigstar\\ &\bigstar\ (u_1v_1, \ldots, u_mv_m)\\ &= (u_1^2(u_1v_1), \ldots, u_m^2(u_mv_m))\\ &= (u_1(u_1^2v_1), \ldots, u_m(u_m^2v_m))\\ &= (u_1, \ldots, u_m)\bigstar\\ &\bigstar\ (u_1^2v_1, \ldots, u_m^2v_m)\\ &= (u_1, \ldots, u_m)\bigstar(u^{2\bigstar}\bigstar v)\\ &= u\bigstar(u^{2\bigstar}\bigstar v),\\ (u\bigstar v)|w &= (u_1v_1, \ldots, u_mv_m)|(w_1, \ldots, w_m)\\ &= \sum_{k=1}^{m}((u_kv_k)w_k)\\ &= \sum_{k=1}^{m}(v_k(u_kw_k))\\ &= v|(u\bigstar w).\end{aligned}$$

So, we have proved that $\mathcal{B}$ is an EJA. Herein, we note that $e = (1, \cdots, 1)$, the vector with all coordinates equal to 1, is the unit vector of $\mathcal{B}$. Indeed, for any u in $\mathcal{B}$ we have

$$\begin{aligned}e\bigstar u &= (1, \ldots, 1)\bigstar(u_1, \ldots, u_m)\\ &= (u_1, \ldots, u_m)\\ &= (u_1, \ldots, u_m)\bigstar(1, \ldots, 1)\\ &= u\bigstar e\\ &= u.\end{aligned}$$

Example 2 *Consider the real vector space $\mathcal{B}$ of real symmetric matrices of order m, provided with the vector product of two of its vectors, $\bigstar$ such that $u\bigstar v = \frac{uv+vu}{2}$ for any elements u and v of $\mathcal{B}$, and with the inner product $\bullet|\bullet$ such that $u|v = trace(u\bigstar v)$ for any u and v of $\mathcal{B}$. Then $\mathcal{B}$ is a RFEJA and its unit is the identity matrix of order n. We will denote the RFEJA $\mathcal{B}$ only by Sym$(m, \mathbb{R})$, from now on.*

In the following text of this section let's consider a $m-$dimensional real EJA $\mathcal{B}$ provided with the vector product $\bigstar$ and with the inner product $\bullet|\bullet$, and being $\mathbf{e}$ the unit vector of $\mathcal{B}$. Since $\mathcal{B}$ is a power associative algebra, then the algebra spanned by u and $\mathbf{e}$ is associative for any $u \in \mathcal{B}$.

Let w be an element of $\mathcal{B}$. The rank of w is the smallest natural number l such that $\{\mathbf{e}, w^{1\bigstar}, \ldots, w^{l\bigstar}\}$ is a linearly dependent set of $\mathcal{B}$ and we write rank$(w) = l$. Since $\dim(\mathcal{B}) = m$ then rank$(w) \leq m$. The rank of the RFEJA $\mathcal{B}$ is the natural number r such that $r = \text{rank}(\mathcal{B}) = \max\{\text{rank}(w) : w \in \mathcal{B}\}$. One says that an element w of $\mathcal{B}$ such that rank$(w) = $ rank$(\mathcal{B})$ is a regular element of the RFEJA $\mathcal{B}$. Herein, we must say that the set of regular elements of the RFEJA $\mathcal{B}$ it is a dense set in $\mathcal{B}$.

Example 3 *Let's consider the EJA $\mathcal{B} = \mathbb{R}^m$ of example 1. Then an element $d = (d_1, d_2, \ldots, d_m)$ is an idempotent if and only if $d_i = 1$ or $d_i = 0$, for $i = 1, \cdots, m$. If $d = (d_1, d_2, \cdots, d_m)$ is such that all of it's coordinates are not zero and $d_i \neq d_j$ for $i \neq j$ and $1 \leq i, j \leq m$, then the set $\{e, d^{1\bigstar}, d^{2\bigstar}, \cdots, d^{m-1\bigstar}\}$ $= \{e, (d_1, d_2, \ldots, d_m), (d_1^2, d_2^2, \cdots, d_m^2), \cdots, (d_1^{m-1}, d_2^{m-1}, \ldots, d_m^{m-1})\}$ is a linearly independent set of the* *vector space $\mathcal{B}$, since we have:*

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ d_1 & d_2 & \dots & d_m \\ \vdots & \vdots & \dots & \vdots \\ d_1^{m-1} & d_2^{m-1} & \cdots & d_m^{m-1} \end{vmatrix} \neq 0.$$

But, since $\dim(\mathcal{B}) = m$ then the set $\{e, d^{1\bigstar}, d^{2\bigstar}, \cdots, d^{m-1\bigstar}, d^{m\bigstar}\}$ is a linearly dependent set of $\mathcal{B}$. So, m is the smallest natural number such that the set $\{e, d^{1\bigstar}, d^{2\bigstar}, \cdots, d^{m-1\bigstar}, d^{m\bigstar}\}$ is linearly dependent, then we conclude that rank$(d) = m$ and we conclude also that rank$(\mathcal{B}) = m$ and we must also say that any element $d = (d_1, d_2, \dots, d_m)$ such that $d_i \neq d_j$ for $i \neq j$ and $1 \leq i, j \leq m$ is a regular element of $\mathcal{B}$.

Let's consider a regular element w of a RFEJA $\mathcal{B}$ and the natural number $q = \text{rank}(w)$. Since $w^{q\,\bigstar}$ belongs to the real vector space spanned by the set $\{\mathbf{e}, w^{1\,\bigstar}, \dots, w^{q-1\,\bigstar}\}$ then there exists real scalars $\beta_1(w), \beta_2(w), \dots, \beta_{q-1}(w)$ and $\beta_q(w)$ such that

$$w^{q\bigstar} - \beta_1(w)w^{q-1\,\bigstar} + \cdots + (-1)^q\beta_q(w)e = 0, \quad (1)$$

being 0 is the zero element of $\mathcal{B}$. Taking into account (1) we call to the polynomial p such that

$$p(w, \lambda) = \lambda^q - \beta_1(w)\lambda^{q-1} + \cdots + (-1)^q\beta_q(w) \quad (2)$$

the characteristic polynomial of w and by construction the polynomial p is the minimal polynomial of w. The roots of the characteristic polynomial of w are called the eigenvalues of w. If u is not a regular vector of $\mathcal{B}$ then we conclude that its minimal polynomial has a degree less than q. And we define the characteristic polynomial of a non regular element u in the following way: we consider a succession u_n of regular elements converging to u and next we define $p(u, \lambda) = \lim_{n\to+\infty} p(u_n, \lambda)$, we note that the functions α_i are homogeneous functions of degree i over the coordinates of u on a fixed basis of $\mathcal{B}$. One calls to $\beta_1(u)$ the trace of u and write trace$(u) = \beta_1(u)$ and we call to $\beta_q(u)$ the determinant of u and we write det$(u) = \alpha_q(u)$.

Example 4 *Let's consider the RFEJA $\mathcal{B}$ of example 1 and let $u = (\alpha_1, \alpha_2, \dots, \alpha_m)$ supposing that all the α_is are distinct. Then u is a regular element of $\mathcal{B}$ and we have rank$(u) = m$. So, the set $S = \{(1, 1, \cdots, 1), u^{2\bigstar}, \dots, u^{m-1\,\bigstar}\}$ is a linearly independent set of $\mathcal{B}$ and since dim$(\mathcal{B}) = m$ then we conclude that S is a basis of $\mathcal{B}$. Now, let's consider the polynomial p defined through the equality (3).*

$$p(\lambda) = \prod_{i=1}^{m}(\lambda - \alpha_i). \quad (3)$$

Since $p(\alpha_i) = 0, \forall i = 1, \cdots, m$, then, we can write that $\alpha_j^m = \sum_{k=1}^{m}(-1)^{k+1}\sum_{1\leq i_1<i_2<\cdots<i_k\leq m}\prod_{l=1}^{k}\alpha_{i_l}\alpha_j^{m-k}$. And, therefore we have that $u^{m\bigstar} = \sum_{k=1}^{m}(-1)^{k+1}\sum_{1\leq i_1<i_2<\cdots<i_k\leq m}\prod_{l=1}^{k}\alpha_{i_l}u^{m-k\bigstar}$. Hence, we conclude that, $u^{m\bigstar} - \sum_{k=1}^{m}(-1)^{k+1}\sum_{1\leq i_1<i_2<\cdots<i_k\leq m}\prod_{l=1}^{k}\alpha_{i_l}u^{m-k\,\bigstar} = 0_m$, where 0_m is the zero vector of $\mathcal{B}$ and $u^{0\bigstar} = (1, 1, \cdots, 1)$. So, we conclude that the characteristic polynomial of u is the polynomial p defined by the equality (4).

$$p(u, \lambda) = \lambda^m + \sum_{k=1}^{m}(-1)^k\beta_k(u)\lambda^{m-k} \quad (4)$$

with $\beta_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_m, \beta_r = \alpha_1, \cdots, \alpha_m$. So, we have trace$(u) = \alpha_1 + \alpha_2 + \cdots + \alpha_m$ and det$(u) = \prod_{j=1}^{m}\alpha_j$. And, we have also that $\beta_j = \sum_{1\leq i_1<i_2<\cdots<i_j\leq m}\prod_{l=1}^{j}\alpha_{i_l}$. Next, let's consider the real finite Euclidean subalgebra of $\mathcal{B}$, $\mathbb{R}[u] = \{\alpha_1 e + \alpha_2 u^{1\bigstar} + \cdots + \alpha_m u^{m-1\,\bigstar}, \alpha_i \in \mathbb{R}, \forall i = 1, \cdots, m\}$ and the linear application $L(u)$ such that $L(u)y = u\bigstar y, \forall y \in \mathbb{R}[u]$. We have that:

$$\begin{aligned} L(u)\mathbf{e} &= u^{1\bigstar} = 0\mathbf{e} + u^{1\bigstar} + \sum_{j=2}^{m-1} 0u^{j\,\bigstar} \\ L(u)u^{1\,\bigstar} &= 0\mathbf{e} + 0u^{1\bigstar} + u^{2\bigstar} + \sum_{j=3}^{m-1} 0u^{j\bigstar} \\ \vdots \quad & \vdots \quad \vdots \\ L(u)u^{m-1\,\bigstar} &= (-1)^{m+1}\beta_m(u)\mathbf{e} + \cdots + \\ &+ (-1)^2\beta_1(u)u^{m-1\,\bigstar} \end{aligned}$$

So, the matrix of the linear application $L(u)$ on the basis $S =<\ , u^{1\bigstar}, \cdots, u^{m-1\,\bigstar} >$ is the matrix:

$$M_{L(u)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^{m+1}\beta_m(u) \\ 1 & 0 & \cdots & 0 & (-1)^m\beta_{m-1}(u) \\ 0 & 1 & \cdots & 0 & (-1)^{m-1}\beta_{m-2}(u) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & (-1)^2\beta_1(u) \end{bmatrix}$$

So have that Trace$(M_{L(u)}) = \beta_1(u)$ and Det$(M_{L(u)}) = \beta_m(u)$.

A vector w in $\mathcal{B}$ such that $w^{2\bigstar} = w$ is an idempotent of $\mathcal{B}$. Two idempotent u and v of $\mathcal{B}$ are orthogonal if $u\bigstar v = 0$, herein, we must say that $u|v = 0$ and this happen, since we have $u|v = (u\bigstar e)|v = e|(u\bigstar v) = e|0 = 0$. Let $t \in \mathbb{N} + 1$. The set $\mathcal{S} = \{d_1, d_2, \dots, d_t\}$ is a complete system of orthogonal idempotent of $\mathcal{B}$ if

$d_i^{2\bigstar} = d_i$ for $i = 1, \cdots, t$, $u \bigstar v = 0$ for $u \neq v$ and u and v in S, and finally if $\sum_{j=1}^{t} d_j = e$. An idempotent is primitive if is a nonzero idempotent of $\mathcal{B}$ and cannot be written as a sum of two nonzero orthogonal idempotent. Let $l \in \mathbb{N}+1$, the set $\{d_1, d_2, \ldots, d_l\}$ is a Jordan frame of $\mathcal{B}$ if $\{d_1, d_2, \ldots, d_l\}$ is a complete system of orthogonal idempotent of $\mathcal{B}$ such that each idempotent d_i for $i = 1, \ldots, l$ is primitive.

From now on we will designate a complete system of orthogonal idempotent and Jordan a frame, respectively by CSOI and by JF. An important property of a JF of a real finite-dimensional EJA $\mathcal{B}$ is that $\#\mathcal{B} = \text{rank}(\mathcal{B})$.

Example 5 *Consider the EJA of example 1, $\mathcal{B} = \mathbb{R}^5$. The set $\mathcal{S}_1 = \{e_1, e_2, e_3, e_4, e_5\} = \{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1)\}$ is a CSOI of $\mathcal{B}$. And the set $\mathcal{S}_1$ is also a JF of $\mathcal{B}$. But the set $\mathcal{S}_2 = \{g_1, g_2, g_3, g_4\} = \{e_1, e_2, e_3 + e_4, e_5\}$ is a CSOI of $\mathcal{B}$ but is not a JF of $\mathcal{B}$, since g_3 is not a primitive idempotent, indeed we have $g_3 = e_3 + e_4$ and e_3 and e_4 are idempotent and $e_3 \bigstar e_4 = 0$, where 0 is the zero vector of $\mathcal{B}$.*

Example 6 *Let's consider the RFEJA $\mathcal{B} = Sym(4, \mathbb{R})$. Then the set*

$$\begin{aligned} S &= \{d_1, d_2, d_3, d_4\} \\ &= \left\{ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right. \\ &= \left. \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\} \end{aligned}$$

is a JF of the RFEJA $\mathcal{B}$. But, the set $\{d_1 + d_2, d_3 + d_4\}$ is a CSOI of the RFEJA $\mathcal{B}$, but not a JF of $\mathcal{B}$.

Theorem 7 *([2], p. 43). Let $\mathcal{B}$ be a RFEJA. Then for x in $\mathcal{B}$ there exist unique real numbers $\gamma_1, \gamma_2, \ldots, \gamma_t$, all distinct, and a unique CSOI $\{d_1, d_2, \ldots, d_t\}$ of $\mathcal{B}$ such that*

$$x = \gamma_1 d_1 + \gamma_2 d_2 + \cdots + \gamma_t d_t. \tag{5}$$

The numbers γ_j's of (5) are the eigenvalues of x and the decomposition (5) is the first spectral decomposition of x.

Example 8 *Let $d = (d_1, d_2, \ldots, d_m)$ be an element of the RFEJA $\mathcal{B} = \mathbb{R}^m$ of example 1 then we have that there exists the JF $\{e_1, e_2, \ldots, e_m\}$, where the vectors e_i for $i = 1, \cdots, m$ are the vectors of the canonical basis of $\mathcal{B}$ such that*

$$d = d_1 e_1 + d_2 e_2 + \cdots, d_m e_m. \tag{6}$$

and (6) is the second spectral decomposition of d.

Example 9 *Let's consider the RFEJA $Sym(n, \mathbb{R})$. Then if A is a matrix with k distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{k-1}$ and λ_k then we have that decomposition (7).*

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k, \tag{7}$$

is the first spectral decomposition of A where $P_i = \frac{\prod_{j=1, j\neq i}^{k}(A - \lambda_j I_n)}{\lambda_i - \lambda_j}$ for $i = 1, \cdots, k$.

Theorem 10 *([2], p. 44). Let $\mathcal{B}$ be a RFEJA such that rank$(\mathcal{B}) = r$ and the vector $x \in \mathcal{B}$. Then, there exist a JF $\{d_1, d_2, \cdots, d_{r-1}, d_r\}$ of $\mathcal{B}$ and real numbers $\gamma_1, \gamma_2, \cdots, \gamma_{r-1}$ and γ_r such that we have:*

$$x = \gamma_1 d_1 + \gamma_2 d_2 + \cdots + \gamma_r d_r. \tag{8}$$

The decomposition (8) is called the second spectral decomposition of x.

Example 11 *Let's consider the RFEJA $\mathcal{B} = Sym(4, \mathbb{R})$ and let's consider the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.*

Then the decomposition (9) is a second spectral decomposition of the matrix A.

$$\begin{aligned} A &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} - 1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{aligned} \tag{9}$$

and -1 and 1 are the eigenvalues of A. And, the first spectral decomposition of the matrix is the decomposition presented on (10):

$$\begin{aligned} A &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &+ (-1) \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \end{aligned} \tag{10}$$

Remark 12 *Let's consider the RFEJA $\mathcal{B} = Sym(m,\mathbb{R})$. a matrix A with the distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$, and let's consider a orthonormal basis of $\mathbb{R}^n$, $S =< v_1, v_2, \cdots, v_m >$ where we suppose each v_i is a column vector of $\mathbb{R}^m$ for $i = 1, \cdots, m$. Then, considering $C_i = v_i v_i^T$ for $i = 1, \cdots, m$. then we obtain the first spectral decomposition of A is presented on inequality (11).*

$$A = \sum_{j=1}^{m} \lambda_{i_j} C_l \tag{11}$$

Where the eigenvalues $\lambda_{i_j} \in \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$.

Remark 13 *Let's consider the real finite Euclidean Jordan algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ with the operations of multiplication $\bigstar_1$ and $\bigstar_2$ respectively, and with the inner products $\bullet|_1\bullet$ and $\bullet|_2\bullet$ respectively. Then, $(\mathcal{B}_1;\mathcal{B}_2)$ equipped with the vector multiplication of vectors $(a;b)\bigstar(c;d) = (a\bigstar_1 c; b\bigstar_2 d)$ and the inner product $(a;b)|(c;d) = a|_1 c + b|_2 d$ is a RFEJA. And, if e_1 and e_2 are the units of $\mathcal{B}_1$ and of $\mathcal{B}_2$ respectively, then $(e_1;e_2)$ is the unit of RFEJA $(\mathcal{B}_1;\mathcal{B}_2)$.*

A Euclidean Jordan algebra is simple if it does not contain any non trivial ideal.

We are concerned on those RFEJA $\mathcal{B}$ such that rank$(\mathcal{B})$ = dim$(\mathcal{B}) = m$, since in those EJA any Jordan frame $S = \{d_1, d_2, \ldots, d_m\}$ is a basis of the Euclidean Jordan algebra $\mathcal{B}$ and we can define $||x||$ for $x = \lambda_1 d_1 + \lambda_2 d_2 + \ldots + \lambda_m d_m$ by the equality $||x|| = \sqrt{\text{trace}(x \circ x)} = \sqrt{\sum_{i=1}^m \lambda_i^2}$. Herein we must note that if u is a primitive idempotent then trace$(u) = 1$.

3 Some concepts about strongly regular graphs

In this section, we present the more important concepts about strongly regular graphs and we follow the text presented in the paper [1]. The works, see [18], " A course in Combinatorics" and "Algebraic Graph Theory" , see [19], are very good on the description of the algebraic properties of a strongly regular graph.

A graph G is a pair of sets $\{V(G), E(G)\}$ such that $V(G) = \{v_1, v_2, \cdots, v_m\}$ is the set of vertexes, nodes or points of G and $E(G)$ is the set of edges of G. Is natural to represent an edge between the vertexes v_k and v_l by $v_k v_l$. A graph G is called simple if it has non-parallel edges or loops. The number of vertexes of a graph G is called the order of G and the dimension of G is the number of edges of G.

A graph G of order m is called a null graph if $E(G) = \emptyset$ and $V(G) \neq \emptyset$. And, a simple graph G of order m is a complete graph if all pair of distinct vertexes of $V(G)$ are adjacent vertexes. And one denotes the complete graph of order m by K_m.

The complement graph of a simple graph G that is denoted by $\overline{G}$ is a simple graph with the same set of vertexes of G and such that two any of its vertexes are adjacent if and only if they are not adjacent vertexes of G.

From now on, we only consider non-empty, simple and non-complete graphs. One says that an edge of $E(G)$ is incident on a vertex $v \in V(G)$ if and only if v is an extreme vertex of this edge. The extreme points of an edge of the graph G are called adjacent vertexes or neighbors. The set of vertexes that are neighboring vertexes of a vertex v is called the neighborhood of the vertex v, one denotes this set by $N_G(v)$.

One defines the degree of a vertex v of a graph G the number of incident vertexes on v. A graph G is called $l-$regular if all of its vertexes have the same degree l.

A graph G, is called a $(m,l;c,d)$-strongly regular graph if G is as graph of order m, is a $l-$regular graph such that any pair of adjacent vertexes have c common neighbor vertexes and any pair of non adjacent vertexes have d common neighbor vertexes. In the following text of this section, we will designate a strongly regular graph by srg.

If G is a $(m,l;c,d)-$srg then the complement graph of $G, \overline{G}$ is a $(m, m-l-1; m-2l+d-2, m-2l+c)-$ srg.

A $(m,l;c,d)-$srg G is primitive if and only if G and $\overline{G}$ are connected. A $(m,l;c,d)-$srg is a non primitive srg if and only if $d = l$ or $d = 0$.

Consider a $(m,l;c,d)-$srg G and B_G its adjacency matrix. Then $B_G = [b_{ij}]$, where B_G is a matrix of order m such that $b_{ij} = 1$, if the vertex i is adjacent to j and 0 otherwise. The adjacency matrix of G satisfies the equation (12).

$$B_G^2 = l\,I_m + cB_G + d(J_m - B_G - I_m). \tag{12}$$

One defines the eigenvalues of G as being the eigenvalues of B_G. The eigenvalues of G, are l, θ and τ, see [19], where

$$\theta = (c - d + \sqrt{(c-d)^2 + 4(l-d)})/2, \tag{13}$$
$$\tau = (c - d - \sqrt{(c-d)^2 + 4(l-d)})/2. \tag{14}$$

Next, we refer to some feasibility conditions over the natural numbers c, d, l and m for the existence of a $(m,l;c,d)$ primitive srg G.

The multiplicities f_θ and f_τ are defined by the inequalities (15) and (16).

$$f_\theta = \frac{|\tau|n + z - k}{\theta - \tau}, \quad (15)$$

$$f_\tau = \frac{\theta n + k - \theta}{\theta - \tau}. \quad (16)$$

The conditions $f_\theta \in \mathbb{N}$ and $f_\tau \in \mathbb{N}$ are called the integrability conditions of a srg. Next we describe the called Krein conditions (17) and (18)

$$(\tau + 1)(l + \tau + 2\theta\tau) \leq (l + \tau)(\theta + 1)^2, (17)$$
$$(\theta + 1)(l + \theta + 2\theta\tau) \leq (l + \theta)(\tau + 1)^2, (18)$$

that have being deduced in the article [20].

Example 14 *Let's consider the parameter set* $(m, l; c, d) = (56, 22; 3, 12)$. *If this sequence would correspond to a strongly regular graph then we would have* $\theta = 1$ *and* $\tau = -10$. *But, then we would have:*

$$(\tau + 1)(1 + \tau + 2\theta\tau) = 261, \quad (19)$$
$$(l + \tau)(\theta + 1)^2 = -36. \quad (20)$$

Hence, these values of θ *and* τ *violate the Krein condition (17) and therefore doesn't exist any srg with the parameters* $(56, 22; 3, 12)$. *We must say that the Krein condition (18) is not violated by the values of* $\theta = 1$ *and* $\tau = -10$.

Next, we present the equality (21) that the parameters m, l, c and d of a $(m, l; c, d)-$ primitive srg satisfies .

$$l(l - 1 - c) = d(m - l - 1). \quad (21)$$

And, finally, we must say that if we consider the $(m, l; c, d)-$primitive srg then the multiplicities of the eigenvalues τ and θ must verify the inequalities (22) and (23).

$$n \leq \frac{f_\theta(f_\theta + 3)}{2}, \quad (22)$$

$$n \leq \frac{f_\tau(f_\tau + 3)}{2}. \quad (23)$$

The admissibility conditions (22) and (23) are known as the absolute bound feasibility conditions for the existence of a strongly regular graph.

Example 15 *Let's consider the parameter set* $(m, l; c, d) = (64, 30; 18, 10)$. *Then, recurring to (13) and (14) we get* $\theta = 10$ *and* $\tau = -2$. *Using the equalities (15) and (16) we have:*

$$f_\theta = 8$$
$$f_\tau = 55.$$

Since

$$n = 64$$
$$\frac{f_\theta(f_\theta + 3)}{2} = 44$$

then the absolute bound condition (22) is not satisfied therefore don't exist any strongly regular corresponding the parameter set $(64, 30; 18, 10)$. *We must say that the absolute bound condition (23) is not violated.*

4 Some inequalities over the eigenvalues of a strongly regular graph

In this section we will establish some inequalities over the parameters and the spectrum of a primitive srg. But firstly, we present some notation for the Schur product of matrices.

Given two real square matrices A and B of order n one considers the Schur product $\circ$ of these two matrices as being the matrix $C = A \circ B$ such that considering the notation $C = [c_{ij}]$, $A = [a_{ij}]$ and $B = [b_{ij}]$ then $c_{ij} = a_{ij}b_{ij}$. And, one defines the Schur powers of a square matrix A for a natural number j as being the matrix $A^{j\circ}$ such that $A^{\circ 0} = J_n, A^{1\circ} = A$ and $A^{j\circ} = AA^{j-1\circ}$ for $j \geq 2$.

Let's consider the $(m, l; c, d)-$primitive strongly regular graph G with the adjacency matrix A, such that $0 < d < l - 1$. Firstly, we will suppose that $l < \frac{m}{3}$ and that $c > d$. Next, let's $\mathcal{A}$ be the Euclidean Jordan subalgebra $\mathcal{A}$ of the RFEJA Sym$(m, \mathbb{R})$ spanned by the identity matrix of order m and the powers $A^i, i \in \mathbb{N}$. We have that rank$(\mathcal{A})$ = dim$(\mathcal{A}) = 3$. The eigenvalues of the matrix A are , $\lambda_1 = \frac{c-d+\sqrt{(c-d)^2+4(l-d)}}{2}$ and $\lambda_2 = \frac{c-d-\sqrt{(c-d)^2+4(l-d)}}{2}$ and $\mathcal{B} = \{F_1, F_2, F_3\}$ is a JF of $\mathcal{A}$ where:

$$F_1 = \frac{1}{m}I_m + \frac{1}{m}A + \frac{1}{m}(J_m - A - I_m) = \frac{J_m}{m},$$

$$F_2 = \frac{|\lambda_2|m + \lambda_2 - l}{n(\lambda_1 - \lambda_2)}I_n + \frac{m + \lambda_2 - l}{m(\lambda_1 - \lambda_2)}A + + \frac{\lambda_2 - l}{m(\lambda_1 - \lambda_2)}(J_m - A - I_m),$$

$$F_3 = \frac{\lambda_1 m + l - \lambda}{m(\lambda_1 - \lambda_2)}I_m + \frac{-m + l - \lambda_1}{m(\lambda_1 - \lambda_2)}A + + \frac{l - \lambda_1}{m(\lambda_1 - \lambda_2)}(J_m - A - I_m),$$

where J_m is the real symmetric matrix of order m where all entries are the real number 1 and I_m is the identity matrix of order m.

Let ϵ be a real positive number. Since the Schur power series $\sum_{k=0}^{+\infty} \frac{1}{l!}\left(\ln(1+\epsilon)\frac{A^2}{l+\epsilon}\right)^{k\circ}$ is convergent then let S be its sum. Using he fact that $A^2 = lI_m + cA + d(J_m - A - I_m)$ then we conclude that:

$$\begin{aligned} S &= \sum_{i=0}^{+\infty}\frac{1}{i!}\left(\ln(1+\epsilon)\frac{l}{l+\epsilon}\right)^i I_m + \\ &+ \sum_{i=0}^{+\infty}\frac{1}{i!}\left(\ln(1+\epsilon)\frac{c}{l+\epsilon}\right)^i A + \\ &+ \sum_{i=0}^{+\infty}\frac{1}{i!}\left(\ln(1+\epsilon)\frac{d}{l+\epsilon}\right)^i (J_m - A - I_m). \end{aligned}$$

Hence, we obtain the equality (24).

$$\begin{aligned} S &= (1+\epsilon)^{\frac{l}{l+\epsilon}} I_n + (1+\epsilon)^{\frac{c}{l+\epsilon}} A + \\ &+ (1+\epsilon)^{\frac{d}{l+\epsilon}} (J_m - A - I_m). \end{aligned} \tag{24}$$

Let's consider the spectral decomposition of the partial sum of order n, $S_n = \sum_{i=0}^{n}\frac{1}{i!}\left(\ln(1+\epsilon)\frac{A^2}{l+\epsilon}\right)^{i\circ}$, $S_n = q_{n1}F_1 + q_{n2}F_2 + q_{n3}F_3$. Since the spectral decomposition $S = q_1F_1 + q_2F_2 + q_3F_3$ is such that $q_i = \lim_{n\to+\infty} q_{ni}$ for $i = 1, \cdots, 3$ and $q_{ni} \geq 0$ for $i = 1, \cdots, 3$ then the q_is are all non negative. Next, let's consider the element $F_3 \circ S$ of $\mathcal{A}$. So, we can write the equality (25).

$$\begin{aligned} F_3 \circ S &= \frac{\lambda_1 m + l - \lambda_1}{m(\lambda_1 - \lambda_2)}(1+\epsilon)^{\frac{l}{l+\epsilon}} I_m + \\ &+ \frac{-m + l - \lambda_1}{m(\lambda_1 - \lambda_2)}(1+\epsilon)^{\frac{c}{l+\epsilon}} A + \\ &+ \frac{l - \lambda_1}{m(\lambda_1 - \lambda_2)}(1+\epsilon)^{\frac{d}{l+\epsilon}} (J_m - A - I_m). \end{aligned} \tag{25}$$

Recurring, to the spectral decomposition of $F_3 \circ S = q_{31}F_1 + q_{32}F_2 + q_{33}F_3$ we conclude that

$$\begin{aligned} q_{31} &= \frac{\lambda_1 m + l - \lambda_1}{m(\lambda_1 - \lambda_2)}(1+\epsilon)^{\frac{l}{l+\epsilon}} + \\ &+ \frac{-m + l - \lambda_1}{m(\lambda_1 - \lambda_2)}(1+\epsilon)^{\frac{c}{l+\epsilon}} l + \\ &+ \frac{l - \lambda_1}{m(\lambda_1 - \lambda_2)}(1+\epsilon)^{\frac{d}{l+\epsilon}} (m - l - 1) \end{aligned} \tag{26}$$

since we have the equality (27),

$$\begin{aligned} \frac{\lambda_1 m + l - \lambda_1}{m(\lambda_1 - \lambda_2)} &+ \frac{-m + l - \lambda_1}{m(\lambda_1 - \lambda_2)} l + \\ &+ \frac{l - \lambda_1}{m(\lambda_1 - \lambda_2)}(m - l - 1) = 0 \end{aligned} \tag{27}$$

then, after some algebraic manipulation of the expression of q_{31} recurring to (27) we obtain the equality (28).

$$\begin{aligned} q_{31} = \quad & \frac{\lambda_1 m + l - \lambda_1}{m(\lambda_1 - \lambda_2)}\left((1+\epsilon)^{\frac{l}{l+\epsilon}} - (1+\epsilon)^{\frac{d}{l+\epsilon}}\right) \\ - \quad & \frac{m - l + \lambda_1}{m(\lambda_1 - \lambda_2)}\left((1+\epsilon)^{\frac{c}{l+\epsilon}} - (1+\epsilon)^{\frac{d}{l+\epsilon}}\right) l. \end{aligned} \tag{28}$$

So, since $q_{31} \geq 0$ then we obtain the inequality (29).

$$\begin{aligned} & \frac{\lambda_1 m + l - \lambda_1}{m(\lambda_1 - \lambda_2)}\left((1+\epsilon)^{\frac{l}{l+\epsilon}} - (1+\epsilon)^{\frac{d}{l+\epsilon}}\right) \\ \geq \quad & \frac{m - l + \lambda_1}{m(\lambda_1 - \lambda_2)}\left((1+\epsilon)^{\frac{c}{l+\epsilon}} - (1+\epsilon)^{\frac{d}{l+\epsilon}}\right) l. \end{aligned} \tag{29}$$

Next, applying the Lagrange Theorem to the function f such that $f(x) = (1+\epsilon)^x, \forall x \in \mathbb{R}$ on the intervals $[\frac{d}{l+\epsilon}, \frac{l}{l+\epsilon}]$ and $[\frac{d}{l+\epsilon}, \frac{c}{l+\epsilon}]$, and majoring and minoring the function f on those intervals we obtain the inequality (30),

$$\begin{aligned} & \frac{\lambda_1 m + l - \lambda_1}{m(\lambda_1 - \lambda_2)}(1+\epsilon)^{\frac{l}{l+\epsilon}}\frac{l - d}{l+\epsilon} \\ \geq \quad & \frac{m - l + \lambda_1}{m(\lambda_1 - \lambda_2)}(1+\epsilon)^{\frac{c}{l+\epsilon}}\frac{c - d}{l+\epsilon} l \end{aligned} \tag{30}$$

this is we obtain the inequality (31).

$$\begin{aligned} & \frac{\lambda_1 m + l - \lambda_1}{m(\lambda_1 - \lambda_2)}(l - d) \\ \geq \quad & \frac{m - l + \lambda_1}{m(\lambda_1 - \lambda_2)}(c - d) l. \end{aligned} \tag{31}$$

and, finally we get (32).

$$\frac{\lambda_1 m + l - \lambda_1}{m(\lambda_1 - \lambda_2)} \geq \frac{m - l + \lambda_1}{m(\lambda_1 - \lambda_2)}\frac{c - d}{l - d} l. \tag{32}$$

Next, supposing that $l < \frac{m}{3}$, rewriting (32) and after some algebraic manipulation of (32) we deduce (33)

$$3\lambda_1 + 1 \geq 2\frac{c - d}{l - d} l. \tag{33}$$

And, finally, since $\lambda_1 > 1$ we conclude that the inequality (34) is verified.

$$4\lambda_1 \geq 2\frac{(c - d)l}{l - d} \tag{34}$$

and, so we have

$$\lambda_1 \geq \frac{(c - d)l}{2(l - d)}.$$

Then, we have established the Theorem 16.

Theorem 16 *Let's consider the* $(m, l; c, d)-$ *primitive srg* $\mathcal{U}$ *of order* m *such that* $0 < d < l-1$, $c > d$ *and* $l < \frac{m}{3}$. *Then we the equality (35) is verified.*

$$\lambda_1 \;\geq\; \frac{(c-d)l}{2(l-d)}. \qquad (35)$$

where λ_1 *is the positive eigenvalue of* $\mathcal{U}$ *distinct from the regularity of* $\mathcal{U}$.

Next, we will construct new inequalities over the parameters of a strongly regular graph G recurring to its Generalized Krein parameters. As, is known the Generalized Krein parameters of a strongly regular graph G, $q_{ijl;mn}$ are defined, see [15], as being the real numbers such that:

$$F_i^{m\circ} \circ F_j^{n\circ} \;=\; \sum_{l=1}^{3} q_{ijl;mn} F_l$$

where $i, j, l \in \{1, 2, 3\}$ and m and n are natural numbers such that at least one of them is greater than 1. And, the Generalized Krein parameters $q_{il;m}$, with i, l, m natural numbers such that $1 \leq i, l \leq 3$ and $m \geq 3$, are the unique real numbers such that:

$$F_i^{m\circ} \;=\; \sum_{l=1}^{3} q_{il;m} F_l.$$

Next, let's suppose that $l < \frac{m}{3}$. And, let's analyze the Generalized Krein parameter $q_{31;3}$. Firstly, since this parameters is non negative we have the following inequality (36).

$$\left(\frac{\lambda_1 m + l - \lambda_1}{m(\lambda_1 - \lambda_2)}\right)^3 + \left(\frac{-m + l - \lambda_1}{m(\lambda_1 - \lambda_2)}\right)^3 l + \\ + \left(\frac{l - \lambda_1)}{m(\lambda_1 - \lambda_2)}\right)^3 (m - l - 1) \geq 0. \qquad (36)$$

Since the equality (37) is verified

$$\frac{\lambda_1 m + l - \lambda_1}{m(\lambda_1 - \lambda_2)} + \frac{-m + l - \lambda_1}{m(\lambda_1 - \lambda_2)} l + \\ + \frac{l - \lambda_1}{m(\lambda_1 - \lambda_2)}(m - l - 1) = 0, \qquad (37)$$

after some algebraic manipulation of (36) and using the equality (37) we conclude that: $\frac{\lambda_1 m+l-\lambda_1}{m(\lambda_1-\lambda_2)}\left(\frac{(\lambda_1 m+l-\lambda_1)^2}{m^2(\lambda_1-\lambda_2)^2} - \frac{(l-\lambda_1)^2}{m^2(\lambda_1-\lambda_2)^2}\right) \geq \frac{m-l+\lambda_1}{m(\lambda_1-\lambda_2)}\left(\frac{(m-l+\lambda_1)^2}{m^2(\lambda_1-\lambda_2)^2} - \frac{(l-\lambda_1)^2}{m^2(\lambda_1-\lambda_2)^2}\right) l$ this is we deduce that: $\frac{\lambda_1 m+l-\lambda_1}{m(\lambda_1-\lambda_2)}\left(\frac{\lambda_1 m+2l-2\lambda_1}{m(\lambda_1-\lambda_2)}\right)\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right) \geq \frac{m-l+\lambda_1}{m(\lambda_1-\lambda_2)}\left(\frac{m-2l+2\lambda_1}{m(\lambda_1-\lambda_2)}\right)\frac{1}{\lambda_1-\lambda_2} l$. Next, supposing that $l < \frac{m}{3}$ we conclude that (38) is verified.

$$(3\lambda_1 + 1)(3\lambda_1 + 2)\lambda_1 \;\geq\; 2l. \qquad (38)$$

If $c > d$ then we conclude that $\lambda_1 > 1$ and therefore noting the following writing of (38),

$$\lambda_1^2 \left(\frac{3\lambda_1 + 1}{\lambda_1}\right)\left(\frac{3\lambda_1 + 2}{\lambda_1}\right)\lambda_1 \;\geq\; 2l \qquad (39)$$

we conclude from (39) that (40) is verified.

$$20\lambda_1^3 \;\geq\; 2l, \qquad (40)$$

and therefore in this case we have established the inequality (41).

$$\lambda_1^3 \;\geq\; \frac{l}{10}. \qquad (41)$$

Hence, we have established the Theorem 17.

Theorem 17 *Let's consider the* $(m, l; c, d)-$ *primitive srg* $\mathcal{U}$ *of order* m *such that* $0 < d < l - 1$ *and* $c > d, l < \frac{m}{3}$ *then the equality (42) is verified.*

$$\lambda_1^3 \;\geq\; \frac{l}{10}, \qquad (42)$$

where λ_1 *is the positive eigenvalue of* $\mathcal{U}$ *distinct from the regularity of* $\mathcal{U}$.

5 Conclusion

The results obtained in this paper are distinct from those obtained in the publication [1] and these inequalities over the eigenvalues of a primitive strongly regular graph are obtained recurring to methods distinct of those used on the paper [1]. On the future direction of research we will use other methods of spectral analysis to establish more general feasibility conditions for the existence of a strongly regular graph.

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