The Exponential Growth of Solution, Upper and Lower Bounds for the Blow-Up Time for a Viscoelastic Wave Equation with Variable-Exponent Nonlinearities

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Abstract: - This paper aims to study the model of a nonlinear viscoelastic wave equation with damping and source terms involving variable-exponent nonlinearities. First, we prove that the energy grows exponentially, and thus in L^{p_2} and L^{p_1} norms. For the case $2 \leq k(.) < p(.)$, we reach the exponential growth result of a blow-up in finite time with positive initial energy and get the upper bound for the blow-up time. For the case $k(.) = 2$, we use the concavity method to show a finite time blow-up result and get the upper bound for the blow-up time. Furthermore, for the case $k(.) \geq 2$, under some conditions on the data, we give a lower bound for the blow-up time when the blow-up occurs.

Key-Words: - Viscoelastic wave equations; Exponential Growth; Blow-up; Lower and upper bound; Sobolev spaces with variable exponents; variable nonlinearity.

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1 Introduction

In recent years, many researchers, in the different cases of the values of the memory kernel, specifically when $g = 0$ or $g > 0$, a nonlinear wave equation with a memory, damping, and source terms associated with the Laplace operator with Dirichlet type condition has been considered

$$\begin{aligned} &u_{tt} - \Delta u - \gamma \Delta u_t + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s \\ &+r|u_t|^{k-2}u_t(x,t) \\ &= |u|^{p-2}u(x,t) \ \text{ in } \Omega \times \mathbb{R}_+, \\ &u = 0 \text{ on } \Gamma \times (0,+\infty), \\ &u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), x \in \Omega, \end{aligned} \tag{1.1}$$

where $T > 0,\ p,\ k \geq 2,\ r,\ \gamma$ are positive constants, Δ stands for the Laplacian with respect to the spatial variables, Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$. The relaxation function g is a positive and uniformly decaying function, $u_0, and\ u_1$ are given functions belonging to suitable spaces.

Under suitable conditions on $g,\ p,\ k,\ u_0$, and u_1, using some known theorems in the mathematical literature, the global existence in time, blow up in finite time, the asymptotic behavior, and a lower bound for the blow-up time of the unique weak solution have been discussed.

From the physical point of view, this type of problem arises usually in viscoelasticity. It has been considered first by, [12], in 1970, where the general decay was discussed. Related problems to (1.1) have attracted a great deal of attention in the last two decades, and many results have appeared on the existence and long-time behavior of solutions. Look at this area, in, [6], [12], [17], [11], [18], [16], [19], and references therein.

In this work given positive measurable functions $p(.), k(.)$ On $\overline{\Omega}$ satisfying:

$$
\begin{aligned}
&2 \le p_1 = \operatorname*{essinf}_{x\in\Omega} p(x) \le p_2 = \operatorname*{esssup}_{x\in\Omega} p(x) < \infty,\\
&2 \le k_1 = \operatorname*{essinf}_{x\in\Omega} k(x) \le k(x)\\
&\qquad\qquad \le k_2 = \operatorname*{esssup}_{x\in\Omega} k(x) < \infty,
\end{aligned}
\tag{1.2}
$$

we consider the following semilinear generalized hyperbolic boundary value problem governed by partial differential equations that describe the evolution of viscoelastic materials with nonlinearities of variable exponent type under Dirichlet type condition:

$$
\begin{aligned}
&u_{tt} - \Delta u - \gamma \Delta u_t + \int_0^t g(t-s)\Delta u(x,s)ds\\
&\quad + r|u_t|^{k(x)-2}u_t(x,t)\\
&= |u|^{p(x)-2}u(x,t) \text{ in } \Omega \times \mathbb{R}_+, \qquad (1.3)\\
&u = 0 \text{ on } \Gamma \times (0,+\infty), \qquad (1.4)\\
&u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), x \in \Omega. \qquad (1.5)
\end{aligned}
$$

Problems with variable exponents of nonlinearity arise from many important mathematical models in engineering and physical sciences. For example, modeling of physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, thermoelasticity, nonlinear viscoelasticity, filtration processes through porous media, image processing, nuclear science, chemical reactions, heat transfer, population dynamics, biological sciences, etc., More details on these problems can be found in, [10], [13], [1], [2], [4], [8], [9], [3], [18], [5], [7], [15], and references therein.

As far as we have known, there is little information on the bounds for blow-up time to problem (1.3)-(1.5) when the initial energy is positive with $p(.)$ and $k(.)$ are not constants. So, it is natural to analyze the problem (1.3)-(1.5) and give further results on the behavior of solutions.

The contents of this paper are as follows. In Section 2 we give Preliminaries. In Section 3 we prove the exponential growth of the energy $E(t)$ of a solution. In Section 4, we consider an upper bound for the blow-up time in case $2 \le k_1 \le k_2 < p_1$. An upper bound for the blow-up time in case $k(x) = 2$, $\forall x$, is proved in Section 5. Section 6, is devoted to proving a lower bound for the blow-up time in case $k_1 \ge 2$.

2 Preliminaries

In the following section, we introduce some preliminaries and notations, which will be used throughout this paper.

Given a function $p: \overline{\Omega} \to [p_1, p_2] \subset (2, \infty)$, $p_{1,2} =$ const, we define the set

$$
L^{p(.)}(\Omega) = \begin{Bmatrix} v: \Omega \to \mathbb{R}: v \text{ measurablefunctionson}\Omega, \\ \varrho(v) = \varrho_{p(.)}(v) = \int_\Omega |v(x)|^{p(x)} dx < \infty. \end{Bmatrix}
$$

The variable-exponent space $L^{p(.)}(\Omega)$ equipped with the Luxemburg norm

$$
\|u\|_{p(.)} = \inf\left\{\lambda > 0, \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\},
$$

becomes a Banach space. In general, variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects, see the first discussed the $L^{p(x)}$ spaces and $W^{k,p(x)}$ spaces by Kovàcik and Rákosnik in [15].

We also assume that $p(x)$, $k(.)$ satisfies the following Zhikov–Fan uniform local continuity condition:

$$
|k(x) - k(y)| + |p(x) - p(y)| \le \frac{M}{|\log|x-y||},
$$

$$
\text{for all } x, y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2}, M > 0. \tag{2.1}
$$

Let us list some properties of the spaces $L^{p(.)}(\Omega)$ which will be used in the study of the problem (1.3)-(1.5).

- If $p(x)$ is measurable and

$$
1 < p_1 \le p(x) \le p_2 < \infty, \qquad \text{in } \Omega,
$$

then $L^{p(.)}(\Omega)$ is a reflexive and separable Banach space, and $C_0^\infty(\Omega)$ is dense in $L^{p(.)}(\Omega)$.

- If condition (2.1) is fulfilled, and Ω has a finite measure and p, q are variable exponents so that $p(x) \le q(x)$ almost everywhere in Ω, the inclusion

$$
L^{q(.)}(\Omega) \subset L^{p(.)}(\Omega)
$$

is continuous and

$$
\forall v \in L^{q(.)}(\Omega)\ \|u\|_{p(.)} \le C\|u\|_{q(.)}, C = C(|\Omega|, p_{1,2}) \tag{2.2}
$$

- It follows directly from the definition of the norm that

$$
\begin{aligned}
\min\left(\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right) &\le \varrho_{p(.)}(u)\\
&\le \max\left(\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right),
\end{aligned}
\tag{2.3}
$$

and

$$
\min\left(\varrho_{p(.)}(u)^{\frac{1}{p_1}}, \varrho_{p(.)}(u)^{\frac{1}{p_2}}\right) \le \|u\|_{p(.)} \le \max\left(\varrho_{p(.)}(u)^{\frac{1}{p_1}}, \varrho_{p(.)}(u)^{\frac{1}{p_2}}\right). \tag{2.4}
$$

- If $p: \Omega \to [p_1,\ p_2] \subset [1, +\infty)$ is a measurable function and $p_* > \underset{\{x\in\Omega\}}{ess\sup} p(x)$ with $p_* \le \frac{2n}{n-2}$, then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.

2.1 Mathematical Hypotheses

We begin this section by introducing some hypotheses and our main result. Throughout this paper, we use standard functional spaces, and denote that $(.,.)$,$\|.\|$ the inner products, and norms in $L^2(\Omega)$ and $H_0^1(\Omega)$ represented and they are given by:

$$(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x \text{ and } \|u\|_{L^2(\Omega)}^2 = \|u\|_2^2 = \int_\Omega u^2 \mathrm{d}x;$$

$$\|u\|_{H_0^1(\Omega)}^2 = \|u\|^2 = \int_\Omega |\nabla u|^2 \mathrm{d}x.$$

Next, we state the assumptions for the problem (1.3)-(1.5).

Let $k(.)$ and $p(.)$ are given measurable functions on $\overline{\Omega}$ satisfying the following conditions

$$\begin{aligned} &2 < p_1 \le p(x) \le p_2 \le \frac{2n}{n-2},\ n>2 \\ &\text{and } 2 \le p_1 \le p_2 < \infty \text{ if } n=2, \\ &2 < k_1 \le k(x) \le k_2 < \frac{2n}{n-2},\ n>2 \\ &\text{and } 2 \le k_1 \le k_2 < \infty \text{ if } n=2. \end{aligned} \quad (2.5)$$

$g: \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function satisfying

$1 - \int_0^\infty g(s)\mathrm{d}s = l > 0,\ \forall t \in \mathbb{R}^+$. (2.6)

By Corollary 3.3.4 in [13], we know $L^{p_2}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$. So, it is a consequence of the embedding $H_0^1(\Omega) \hookrightarrow L^{p_2}(\Omega)$ and Poincaré inequality that $\|u\|_{p(.)} \le B\|\nabla u\|_2$, (2.7)

where B is the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ determined by

$$B^{-1} = \inf\{|\nabla u| : u \in H_0^1(\Omega), \|u\|_{p(.)} = 1\}.$$

The following constants play a crucial role in the proof of our results. Let B_1, α_1, α_0, E_1 be constants satisfying

$$B_1 = \max(1,B),\ \alpha_1 = \left(\frac{l}{B_1^{p_1}}\right)^{\frac{1}{p_1-2}},$$
$$\alpha_0 = \|\nabla u_0\|_2, \qquad E_1 = l\left(\frac{1}{2} - \frac{1}{p_1}\right)\alpha_1^2. \quad (2.8)$$

3 Exponential Growth

In this section, will prove that the energy grows exponentially, and thus so the L^{p_1} and L^{p_2} norms to the problem (1.3)-(1.5) if the variable exponents $p(.)$, and $k(.)$ satisfy some conditions and the initial data are large enough (in the energy viewpoint). Firstly, we start with a local existence result for the problem (1.3)-(1.5), which can be obtained by the combination of the Faedo-Galerkin argument and the compactness method together with the Banach fixed point theorem. Hereafter, for simplicity, we take $a = 1$, and we have

Lemma 3.1 *Let* $2 \le p_1 \le p(x) \le p_2 \le \overline{q}$ *and* $max\left(2, \frac{\overline{q}}{\overline{q}+1-p_2}\right) \le k_1 \le k(x) \le k_2 \le \overline{q}$. *Then given* $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *there exists* $T>0$ *and a unique solution* u *of the problem (1.3)-(1.5) on* $(0,T)$ *such that*

$$u \in C\left(0,T;H_0^1(\Omega)\right) \cap C^1\left(0,T;L^2(\Omega)\right),$$
$$u_t \in L^2\left(0,T;H_0^1(\Omega)\right) \cap L^{k(.)}\left((0,T)\times\Omega\right).$$

The first main result of this paper reads as follows

Theorem 3.2 *Let* $k_2 < p_1 \le p(x) \le p_2$ *where* $2 \le p_1 \le p(x) \le p_2 \le \overline{q}$. *Assume the initial value* u_0 *is chosen to ensure that* $E(0) < E_1$ *and* $B_1^{-1} \ge \|\nabla u_0\|_2 > \alpha_1$ *hold. Then under the above conditions, the solution of problem (1.3)-(1.5) will grow exponentially in the* L^{p_1} *and* L^{p_2} *norms.*

For this purpose, we start with the following lemma defining the energy of the solution

Lemma 3.3 *The corresponding energy to the problem (1.3)-(1.5) is given by*

$$\begin{aligned} E(t) &= \frac{1}{2}(g \circ \nabla u) + \frac{1}{2}\|u_t\|_2^2 \\ &+ \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2 \\ &- \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x, \end{aligned} \quad (3.1)$$

furthermore, by easily verified formula

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \gamma\int_\Omega |\nabla u_t|^2\mathrm{d}x - r\int_\Omega |u_t|^{k(x)}\mathrm{d}x \le 0, \quad (3.2)$$

the inequality $E(t) \le E(0)$ is obtained, where

$$\begin{aligned} E(0) &= \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 \\ &- \int_\Omega \frac{1}{p(x)}|u_0|^{p(x)}\mathrm{d}x, \end{aligned} \quad (3.3)$$

and

$$(g \circ \nabla u)(t)$$

$$= \int_0^t g(t-s) \parallel \nabla u(t) - \nabla u(s) \parallel_2^2 \mathrm{d}s \geq 0.$$

We conclude from (2.3) and (3.1) that

$$\begin{aligned} E(t) &\geq \frac{1}{2}(g \circ \nabla u) + \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2 \\ &\quad - \frac{1}{p_1}\max\left(\|u\|_{p(.)}^{p_2}, \|u\|_{p(.)}^{p_1}\right) \\ &\geq \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2 \\ &\quad - \frac{1}{p_1}\max((B_1\|\nabla u\|_2)^{p_2}, (B_1\|\nabla u\|_2)^{p_1}) \\ &\geq \frac{1}{2}l\alpha^2 - \frac{1}{p_1}\max((B_1\alpha)^{p_2}, (B_1\alpha)^{p_1}) := h(\alpha) \forall \alpha \in [0, +\infty), \end{aligned} \tag{3.4}$$

where $\alpha = \|\nabla u\|_2$.

Lemma 3.4 *Let* $f: [0, +\infty) \to \mathbb{R}$ *be defined by*

$$f(\alpha) = \frac{1}{2}l\alpha^2 - \frac{1}{p_1}(B_1\alpha)^{p_1}. \tag{3.5}$$

Then the following claims hold under the hypotheses of Theorem 3.2: f is increasing for $0 < \alpha \leq \alpha_1$ and decreasing for $\alpha \geq \alpha_1$, $\lim_{\alpha \to +\infty} f(\alpha) = -\infty$ and $f(\alpha_1) = E_1$.

Proof. By the assumption that $B_1 > 1$ and $p_1 > 2$, one can see that $f(\alpha) = h(\alpha)$, for $0 < \alpha \leq B_1^{-1}$. Furthermore, $f(\alpha)$ is differentiable and continuous in $[0, +\infty)$.

$$f'(\alpha) = \alpha l - B_1^{p_1}\alpha^{p_1 - 1}, \qquad 0 \leq \alpha < B_1^{-1}.$$

Then **(i)** follows. Since $p_1 - 2 > 0$, we have $\lim_{\alpha \to +\infty} f(\alpha) = -\infty$. A usual computation yields $f(\alpha_1) = E_1$. Then **(ii)** holds valid.

Lemma 3.5 *Under the assumptions of Theorem 3.2, there exists a positive constant* $\alpha_2 > \alpha_1$ *such that*

$$\|\nabla u\|_2 \geq \alpha_2, t \geq 0, \tag{3.6}$$

$$\int_\Omega |u(x,t)|^{p(x)}\mathrm{d}x \geq (B_1\alpha_2)^{p_1}. \tag{3.7}$$

Proof. Since $E(0) < E_1$, it follows from Lemma 3.4 that there exists a positive constant $\alpha_2 > \alpha_1$, such that $E(0) = f(\alpha_2)$. By (3.4), we have $f(\alpha_0) = h(\alpha_0) \leq E(0) = f(\alpha_2)$, it follows from Lemma 3.4**(i)** that $\alpha_0 \geq \alpha_2$, so (3.6) holds for $t = 0$. Now we prove (3.6) by contradiction. Suppose that $\|\nabla u(t^*)\|_2 < \alpha_2$ for some $t^* > 0$. By the continuity of $\|\nabla u(.,t)\|_2$ and $\alpha_2 > \alpha_1$, we may take t^* such that $\alpha_2 > \|\nabla u(t^*)\|_2 > \alpha_1$, then it results from (3.4) that

$$E(0) = f(\alpha_2) < f(\|\nabla u(t^*)\|_2) \leq E(t^*),$$

which contradicts Lemma 3.3, and (3.6) holds.

By (3.1) and (3.2), we obtain

$$\begin{aligned} \frac{1}{p_1}\int_\Omega |u(x,t)|^{p(x)}\mathrm{d}x &\geq \int_\Omega \frac{1}{p(x)}|u(x,t)|^{p(x)}\mathrm{d}x \\ &\geq \frac{1}{2}l\|\nabla u\|_2^2 - E(0) \\ &\geq \frac{1}{2}l\alpha_2^2 - E(0) = \frac{1}{2}l\alpha_2^2 - f(\alpha_2) = \frac{1}{p_1}(B_1\alpha_2)^{p_1}, \end{aligned} \tag{3.8}$$

and (3.7) follows.

Let $H(t) = E_1 - E(t)$ for $t \geq 0$, we have the following lemma.

Lemma 3.6 *Under the assumptions of Theorem 3.2, the functional* $H(t)$ *illustrated above provides the following estimates:*

$$\begin{aligned} 0 < H(0) \leq H(t) &\leq \int_\Omega \frac{1}{p(x)}|u(x,t)|^{p(x)}\mathrm{d}x \\ &\leq \frac{1}{p_1}\varrho(u), \qquad t \geq 0. \end{aligned} \tag{3.9}$$

Proof. By Lemma 3.3, $H(t)$ is nondecreasing in t. Thus

$$H(t) \geq H(0) = E_1 - E(0) > 0, \qquad t \geq 0. \tag{3.10}$$

Combining (3.1), (2.8), (3.6) and $\alpha_2 > \alpha_1$, we have

$$\begin{aligned} &H(t) - \int_\Omega \frac{1}{p(x)}|u(x,t)|^{p(x)}\mathrm{d}x \\ &\leq E_1 - \tfrac{1}{2}l\|\nabla u\|_2^2 \\ &\leq l\left(\tfrac{1}{2} - \tfrac{1}{p_1}\right)\alpha_1^2 - \tfrac{1}{2}l\alpha_1^2 < 0, t \geq 0. \end{aligned} \tag{3.11}$$

and (3.9) follows from (3.10) and (3.11).

Based on the above three lemmas, we can give proof of Theorem 3.2.

Proof of Theorem 3.2. For $\varepsilon > 0$ small to be chosen later, we then define the auxiliary function

$$L(t) = H(t) + \varepsilon\int_\Omega u_t u\mathrm{d}x + \frac{1}{2}\varepsilon\gamma\int_\Omega |\nabla u|^2\mathrm{d}x. \tag{3.12}$$

Let us observe that L is a small perturbation of the energy. By taking the time derivative of (3.12), we obtain:

$$\begin{aligned} \frac{\mathrm{d}L(t)}{\mathrm{d}t} &= \gamma\|\nabla u_t\|_2^2 + \varepsilon\|u_t\|_2^2 \\ &\quad - \frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}g(t)\|\nabla u\|_2^2 \\ &\quad + \varepsilon\int_\Omega u_{tt}u\mathrm{d}x + \varepsilon\gamma\int_\Omega \nabla u\nabla u_t\mathrm{d}x. \end{aligned} \tag{3.13}$$

Using problem (1.3)-(1.5), the equation (3.13) becomes:

$$\begin{aligned} \frac{\mathrm{d}L(t)}{\mathrm{d}t} &= \gamma\int_\Omega |\nabla u_t|^2\mathrm{d}x + r\int_\Omega |u_t|^{k(x)}\mathrm{d}x + \varepsilon\|u_t\|_2^2 \\ &\quad - \varepsilon\|\nabla u\|_2^2 - \varepsilon r\int_\Omega |u_t|^{k(x)}u_t u\mathrm{d}x \\ &\quad + \varepsilon\int_\Omega |u(t)|^{p(x)}\mathrm{d}x \end{aligned} \tag{3.14}$$

$$+\varepsilon \int_\Omega \nabla u(t) \int_0^t g(t-s)\nabla u(s)\mathrm{d}s\mathrm{d}x.$$

To estimate the last term on the right-hand side of the previous equality, let $\delta > 0$ be determined later. Young's inequality drives to:

$$\int_\Omega |u_t|^{k(x)} u_t u \mathrm{d}x \leq \frac{1}{k_1}\int_\Omega \delta^{k(x)}|u|^{k(x)}\mathrm{d}x + \frac{k_2-1}{k_1}\int_\Omega \delta^{-\frac{k(x)}{k(x)-1}}|u_t|^{k(x)}\mathrm{d}x.$$

This yields by substitution in (3.14):

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} \geq \gamma \int_\Omega |\nabla u_t|^2\mathrm{d}x + r\int_\Omega |u_t|^{k(x)}\mathrm{d}x + \varepsilon\|u_t\|_2^2 + \varepsilon\int_\Omega |u(t)|^{p(x)}\mathrm{d}x$$
$$-\varepsilon\int_\Omega |\nabla u|^2\mathrm{d}x - \varepsilon r\frac{1}{k_1}\int_\Omega \delta^{k(x)}|u|^{k(x)}\mathrm{d}x - \varepsilon r\frac{k_2-1}{k_1}\int_\Omega \delta^{-\frac{k(x)}{k(x)-1}}|u_t|^{k(x)}\mathrm{d}x$$
$$+\varepsilon\|\nabla u\|_2^2\int_0^t g(s)\mathrm{d}s$$
$$+\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u(t)\big(\nabla u(s)-\nabla u(t)\big)\mathrm{d}x\mathrm{d}s \tag{3.15}$$

and for some positive number η to be determined later,

$$\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx$$
$$=-\int_0^t g(t-s)\int_\Omega \nabla u(t)\,\big(\nabla u(t)-\nabla u(s)\big)\mathrm{d}x\mathrm{d}s$$
$$+\|\nabla u\|_2^2\int_0^t g(s)\mathrm{d}s$$
$$\geq\left(1-\frac{1}{4\eta}\right)\|\nabla u\|_2^2\int_0^t g(s)ds-\eta(g\circ\nabla u)(t)$$

We want now to estimate the term involving $\int_\Omega |u|^{k(x)}\mathrm{d}x$ in (3.15), we have

$$\|u\|_{k_2} = \|u\|_{k_2}^{1-s}\|u\|_{k_2}^{s} \leq C\|u\|_{p(.)}^{1-s}\|\nabla u\|_2^s,$$

and $\|u\|_{k_1} = \|u\|_{k_1}^{1-s}\|u\|_{k_1}^{s} \leq C\|u\|_{p(.)}^{1-s}\|\nabla u\|_2^s$, which operates for:

$$\frac{2n}{n-2} \geq k_2 \geq k_1 > 2 \; and \; 0 < \frac{n}{2}-\frac{n}{k_2} \leq \frac{n}{2}-\frac{n}{k_1} \leq s < 1,$$

Thus, we have the following inequality:

$$\int_\Omega |u|^{k(x)}\mathrm{d}x \leq \|u\|_{k_2}^{k_2} + \|u\|_{k_1}^{k_1}$$
$$\leq C\max\left(\varrho_{p(.)}(u)^{\frac{k_1(1-s)}{p_1}}, \varrho_{p(.)}(u)^{\frac{k_1(1-s)}{p_2}}\right)\|\nabla u\|_2^{k_1 s}$$
$$+C\max\left(\varrho_{p(.)}(u)^{\frac{k_2(1-s)}{p_1}}, \varrho_{p(.)}(u)^{\frac{k_2(1-s)}{p_2}}\right)\|\nabla u\|_2^{k_2 s}$$

If $s < min\left(\frac{2}{k_2},\frac{2}{k_1}\right)$, using again Young's inequality, we get:

$$\max\left(\varrho_{p(.)}(u)^{\frac{k_1(1-s)}{p_1}}, \varrho_{p(.)}(u)^{\frac{k_1(1-s)}{p_2}}\right)\|\nabla u\|_2^{k_1 s}$$
$$\leq C\max\left(\varrho_{p(.)}(u)^{\frac{k_1(1-s)\mu}{p_1}}, \varrho_{p(.)}(u)^{\frac{k_1(1-s)\mu}{p_2}}\right) + C(\|\nabla u\|_2^2)^{\frac{k_1 s\theta}{2}}, \tag{3.16}$$

and

$$\max\left(\varrho_{p(.)}(u)^{\frac{k_2(1-s)}{p_1}}, \varrho_{p(.)}(u)^{\frac{k_2(1-s)}{p_2}}\right)\|\nabla u\|_2^{k_2 s}$$
$$\leq C\max\left(\varrho_{p(.)}(u)^{\frac{k_2(1-s)\mu}{p_1}}, \varrho_{p(.)}(u)^{\frac{k_2(1-s)\mu}{p_2}}\right) + C(\|\nabla u\|_2^2)^{\frac{k_2 s\theta}{2}}, \tag{3.17}$$

for $1/\mu + 1/\theta = 1$. Here we choose $(\theta,\mu) = \left(\frac{2}{k_1 s},\frac{2}{2-k_1 s}\right)$, and $(\theta,\mu) = \left(\frac{2}{k_2 s},\frac{2}{2-k_2 s}\right)$ in (3.16) and (3.17), respectively. Therefore the previous inequality becomes

$$\int_\Omega |u|^{k(x)}\mathrm{d}x \leq C\max\left(\varrho(u)^{\frac{2(1-s)k_1}{p_1(2-k_1 s)}}, \varrho(u)^{\frac{2(1-s)k_1}{p_2(2-k_1 s)}}\right)$$
$$+C\max\left(\varrho(u)^{\frac{2(1-s)k_2}{p_1(2-k_2 s)}}, \varrho(u)^{\frac{2(1-s)k_2}{p_2(2-k_2 s)}}\right) + C\|\nabla u\|_2^2. \tag{3.18}$$

Now, picking s such that:

$$0 < s \leq \min\begin{pmatrix}\frac{2(p_1-k_1)}{k_1(p_1-2)}, \frac{2(p_2-k_2)}{k_2(p_2-2)} \\ , \frac{2(p_2-k_1)}{k_1(p_2-2)}, \frac{2(p_1-k_2)}{k_2(p_1-2)}\end{pmatrix} < 1$$

we get

$$0 < max\begin{pmatrix}\frac{2(1-s)k_1}{p_1(2-k_1 s)}, \frac{2(1-s)k_2}{p_2(2-k_2 s)}, \\ \frac{2(1-s)k_1}{p_2(2-k_1 s)}, \frac{2(1-s)k_2}{p_1(2-k_2 s)}\end{pmatrix} \leq 1. \tag{3.19}$$

When the inequality (3.19) is satisfied, we apply the classical algebraic inequality:

$$z^d \leq (z+1) \leq \left(1+\frac{1}{\omega}\right)(z+\omega), \forall z \geq 0,$$
$$0 < d \leq 1, \omega \geq 0,$$

to get the following estimate:

$$\max\left(\varrho(u)^{\frac{2(1-s)k_1}{p_1(2-k_1 s)}}, \varrho(u)^{\frac{2(1-s)k_1}{p_2(2-k_1 s)}}\right)$$
$$+\max\left(\varrho(u)^{\frac{2(1-s)k_2}{p_1(2-k_2 s)}}, \varrho(u)^{\frac{2(1-s)k_2}{p_2(2-k_2 s)}}\right)$$
$$\leq C(1+H(0)^{-1})\big(\varrho(u)+H(0)\big)$$
$$\leq C\big(\varrho(u)+H(t)\big) \quad \forall t \geq 0. \tag{3.20}$$

Injecting the estimate (3.20) into (3.18) we obtain the following inequality

$$\int_{\Omega}|u|^{k(x)}\mathrm{d}x \leq C(\varrho(u)+2H(t)+l\|\nabla u\|_2^2), \tag{3.21}$$

which yields finally:

$$\int_{\Omega}|u|^{k(x)}\mathrm{d}x \leq C\begin{pmatrix}\varrho(u)+2E_1-\|u_t\|_2^2\\ +\int_{\Omega}\frac{2}{p(x)}|u|^{p(x)}\mathrm{d}x\end{pmatrix} \leq C\left(2E_1-\|u_t\|_2^2+\left(1+\frac{2}{p_1}\right)\int_{\Omega}|u|^{p(x)}\mathrm{d}x\right). \tag{3.22}$$

Therefore by injecting the inequality (3.22) into the inequality (3.15), we obtain:

$$\begin{aligned}\frac{\mathrm{d}L(t)}{\mathrm{d}t} &\geq \gamma\|\nabla u_t\|_2^2+\varepsilon\left(1+\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\right)\|u_t\|_2^2-2C\frac{\varepsilon r}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)E_1\\ &-\varepsilon\|\nabla u\|_2^2+\varepsilon\left(1-\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\left(1+\frac{2}{p_1}\right)\right)\int_{\Omega}|u(t)|^{p(x)}\mathrm{d}x\\ &+r\left(1-\frac{\varepsilon(k_2-1)}{k_1}\max\left(\delta^{-\frac{k_2}{k_1-1}},\delta^{-\frac{k_1}{k_2-1}}\right)\right)\int_{\Omega}|u_t|^{k(x)}\mathrm{d}x\\ &+\varepsilon\left(1-\frac{1}{4\eta}\right)\|\nabla u\|_2^2\int_0^t g(s)\mathrm{d}s-\varepsilon\eta(g\circ\nabla u),\end{aligned} \tag{3.23}$$

for some positive number η to be determined later. From the inequality

$$2H(t)=-\left(\|u_t\|_2^2-2E_1+(g\circ\nabla u)+\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2-\int_{\Omega}\frac{2}{p(x)}|u|^{p(x)}\mathrm{d}x\right),$$

we have

$$\begin{aligned}-l\|\nabla u\|_2^2 &\geq -\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2\\ &= 2H(t)+\|u_t\|_2^2+(g\circ\nabla u)-2E_1-\int_{\Omega}\frac{2}{p(x)}|u|^{p(x)}\mathrm{d}x\\ &\geq 2H(t)-2E_1+\|u_t\|_2^2+(g\circ\nabla u)-\frac{2}{p_1}\int_{\Omega}|u|^{p(x)}\mathrm{d}x\end{aligned} \tag{3.24}$$

Thus injecting it in (3.23), we get the following inequality:

$$\begin{aligned}\frac{\mathrm{d}L(t)}{\mathrm{d}t} &\geq \gamma\|\nabla u_t\|_2^2+\varepsilon\left(+\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\right)\|u_t\|_2^2\\ &+\varepsilon\left(1-\frac{2}{p_1}-\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\left(1+\frac{2}{p_1}\right)\right)\int_{\Omega}|u|^{p(x)}\mathrm{d}x\\ &+\varepsilon\left(2H(t)-2\left(1+\frac{r}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)C\right)E_1\right)\\ &+r\left(1-\frac{\varepsilon(k_2-1)}{k_1}\max\left(\delta^{-\frac{k_2}{k_1-1}},\delta^{-\frac{k_1}{k_2-1}}\right)\right)\int_{\Omega}|u_t|^{k(x)}\mathrm{d}x\\ &+\varepsilon\left(1-\frac{1}{4\eta}\right)\|\nabla u\|_2^2\int_0^t g(s)\mathrm{d}s\\ &+(1-\varepsilon\eta)(g\circ\nabla u)\end{aligned} \tag{3.25}$$

Using the definition of α_2 and E_1 (see equation (2.8) and the lemma 3.5), we have

$$\begin{aligned}&-2E_1-4\frac{r}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)CE_1\\ &-2E_1(B_1^2\alpha_2)^{\frac{-p_1}{2}}(B_1^2\alpha_2)^{\frac{p_1}{2}}\\ &-4\frac{r}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)CE_1(B_1^2\alpha_2)^{\frac{-p_1}{2}}(B_1^2\alpha_2)^{\frac{p_1}{2}}\\ &\geq\left(-2E_1(B_1^2\alpha_2)^{\frac{-p_1}{2}}-4C\frac{r}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)E_1(B_1^2\alpha_2)^{\frac{-p_1}{2}}\right)\int_{\Omega}|u|^{p(x)}\mathrm{d}x\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\frac{\mathrm{d}L(t)}{\mathrm{d}t} &\geq \gamma\|\nabla u_t\|_2^2+\varepsilon\left(2+\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\right)\|u_t\|_2^2\\ &+\varepsilon\begin{pmatrix}1-\frac{2}{p_1}-2E_1(B_1^2\alpha_2)^{\frac{-p_1}{2}}\\ -\frac{rC}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)\begin{bmatrix}\left(1+\frac{2}{p_1}\right)\\ +4E_1(B_1^2\alpha_2)^{\frac{-p_1}{2}}\end{bmatrix}\end{pmatrix}\times\int_{\Omega}|u|^{p(x)}\mathrm{d}x\\ &+2\varepsilon\left(H(t)+\frac{r}{k_1}\max\left(\delta^{k_2},\delta^{k_1}\right)CE_1\right)\\ &+\varepsilon\left(1-\frac{1}{4\eta}\right)\|\nabla u\|_2^2\int_0^t g(s)\mathrm{d}s\end{aligned}$$

$$+(1-\varepsilon\eta)(g\circ\nabla u) +r\left(1 -\frac{\varepsilon(k_2-1)}{k_1}\max\left(\delta^{-\frac{k_2}{k_1-1}},\delta^{-\frac{k_1}{k_2-1}}\right)\right)\int_\Omega |u_t|^{k(x)}\mathrm{d}x, \tag{3.26}$$

we have

$$1-\frac{2}{p_1}-2E_1(B_1^2\alpha_2)^{\frac{-p_1}{2}}>0$$

since $\alpha_2 > B_1^{-\frac{2p_1}{p_1-2}}$.

We choose now δ small enough such that

$$\begin{pmatrix} 1-\frac{2}{p_1}-2E_1(B_1^2\alpha_2)^{\frac{-p_1}{2}} \\ -\frac{rC}{k_1}\max(\delta^{k_2},\delta^{k_1})\begin{bmatrix}\left(1+\frac{2}{p_1}\right) \\ +4E_1(B_1^2\alpha_2)^{\frac{-p_1}{2}}\end{bmatrix}\end{pmatrix}>0$$

and taking $\eta > \frac{1}{4}$. Once δ and η are fixed, we choose ε small enough such that:

$$1-\varepsilon\eta>0,\left(1-\frac{\varepsilon(k_2-1)}{k_1}\max\left(\delta^{-\frac{k_2}{k_1-1}},\delta^{-\frac{k_1}{k_2-1}}\right)\right)>0 and L(0)>0$$

Therefore, the inequality (3.26) becomes

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t}\geq\varepsilon\kappa\left[H(t)+\|u_t\|_2^2+\int_\Omega|u|^{p(x)}\mathrm{d}x+E_1\right] \tag{3.27}$$

for some $\kappa > 0$.

Next, it is clear that, by Young's inequality and Poincaré's inequality, we get

$$L(t)\leq\lambda\begin{bmatrix}H(t)+\|u_t\|_2^2\\+\|\nabla u\|_2^2\end{bmatrix}$$

for some $\lambda > 0$. (3.28)

From (3.11), we have

$$\|\nabla u\|_2^2\leq\frac{2}{l}E_1+\frac{2}{lp_1}\int_\Omega|u(x,t)|^{p(x)}\mathrm{d}x,\qquad t\geq 0.$$

Thus, the inequality (3.28) becomes:

$$L(t)\leq\zeta\left[H(t)+\|u_t\|_2^2+\int_\Omega|u|^{p(x)}\mathrm{d}x+E_1\right] \text{ for some } \zeta>0. \tag{3.29}$$

From the two inequalities (3.27) and (3.29), we finally obtain the differential inequality:

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t}\geq\mu L(t) \text{ for some } \mu>0. \tag{3.30}$$

Integrating the previous differential inequality (3.30) on $(0,t)$ gives the following estimate for the function L:

$$L(t)\geq L(0)e^{\mu t}. \tag{3.31}$$

On the other hand, from the definition of the function L (and for small values of the parameter ε), it results:

$$\begin{aligned}L(0)e^{\mu t}\leq L(t)&\leq\frac{1}{p_1}\int_\Omega|u|^{p(x)}\mathrm{d}x\\&\leq\frac{1}{p_1}\max\left(\int_\Omega|u|^{p_2}\mathrm{d}x,\int_\Omega|u|^{p_1}\mathrm{d}x\right)\end{aligned} \tag{3.32}$$

From the two inequalities (3.31) and (3.32) we deduce the exponential growth of the solution in the L^{p_2} and L^{p_1}-norms.

Now, we state the blow-up results as follows.

4 An Upper Bound for the Blow-Up Time: The Case $2\leq k_1\leq k_2<p_1$

In this section, we prove the blow-up result under the condition of $2\leq k_1\leq k_2<p_1$ with positive initial energy and use C to denote a generic positive constant.

Theorem 4.1 *For any fixed $\delta<1$, assume that u_0, u_1 satisfy*

$$I(0)<0,\qquad E(0)<\delta E_1. \tag{4.1}$$

and (2.6) holds. Suppose that

$$\int_0^\infty g(s)\mathrm{d}s\leq\frac{p_1-2}{p_1-2+\left[(1-\hat{\delta})^2(p_1-2)+2(1-\hat{\delta})\right]^{-1}} \tag{4.2}$$

where $\hat{\delta}=\max\{0,\delta\}$. Under the condition of Lemma 3.1, if

$$2\leq k_1\leq k(x)\leq k_2<p_1\leq p(x)\leq p_2\leq\frac{2n}{n-2},$$

the solution of problem (1.3)-(1.5) blows up in a finite time T_1, in the sense that

$$\lim_{t\to T_1^-}(\|u_t(t)\|_2^2+\|\nabla u(t)\|_2^2)=+\infty. \tag{4.3}$$

Furthermore, the upper bound for T_1 can be estimated from above by

$$T_1\leq\frac{(1-a)C}{\chi\varepsilon a(L(0))^{\frac{a}{1-a}}}, \tag{4.4}$$

where the positive constants a, C, χ, and ε to be determined later.

Before the proof, we want to introduce some materials and lemmas firstly we provide the following functions:

$$E(t)=\frac{1}{2}(g\circ\nabla u)+\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2-\int_\Omega\frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x, \tag{4.5}$$

$$I(u(t)=\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2+(g\circ\nabla u)(t)-\int_\Omega|u|^{p(x)}\mathrm{d}x, \tag{4.6}$$

$$J\big(u(t)\big)=\frac{1}{2}\left[\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2+(g\circ\nabla u)(t)\right]-\int_\Omega\frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x, \tag{4.7}$$

$$E(t)=E(u(t),u_t(t))=J(t)+\frac{1}{2}\|u_t(t)\|_2^2 \tag{4.8}$$

Secondly, we need the following two lemmas.

Lemma 4.2 *Under the same conditions as in Theorem 4.1, one has*

$$I(t)<0$$

and

$$E_1<\frac{p_1-2}{2p_1}\left(\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2+(g\circ\nabla u)\right)<\frac{p_1-2}{2p_1}\int_\Omega|u|^{p(x)}\mathrm{d}x$$

for all $t\in 0,T)$. (4.9)

Proof. By (3.2) and (4.1), we have $E(t)\le\delta E_1$, for all $t\in 0,T)$. Besides, we can get $I(t)<0$ for all $t\in 0,T)$. If it is not true, then there exists some $t^*\in 0,T)$ such that $I(t^*)=0$. So $I(t)<0$ for all $0\le t<t^*$, i.e.

$$\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2+(g\circ\nabla u)(t)<\int_\Omega|u|^{p(x)}\mathrm{d}x<\|u\|_{p(.)}^{p_1},\quad 0\le t<t^* \tag{4.10}$$

By the proof of Lemma 3.4, we have

$$\begin{aligned}E_1&=\frac{p_1-2}{2p_1}l^{\frac{p_1}{p_1-2}}\frac{1}{B_1^{\frac{2p_1}{p_1-2}}}\le\frac{p_1-2}{2p_1}\left(\frac{l\|\nabla u\|_2^2}{\|u\|_{p(.)}^2}\right)^{\frac{p_1}{p_1-2}}\\&<\frac{p_1-2}{2p_1}\left(\frac{\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)}{\left(\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2+(g\circ\nabla u)(t)\right)^{\frac{2}{p_1}}}\right)^{\frac{p_1}{p_1-2}}\\&=\frac{p_1-2}{2p_1}\left[\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2+(g\circ\nabla u)(t)\right],\quad 0\le t<t^*\end{aligned} \tag{4.11}$$

Joined with (4.10) and (4.11), we obtain

$$\int_\Omega|u|^{p(x)}\mathrm{d}x>\frac{2p_1}{p_1-2}E_1>0,\qquad 0\le t<t^*$$

By the continuity of $t\mapsto\int_\Omega|u(t)|^{p(x)}\mathrm{d}x$, we get $u(t^*)\neq 0$. By (4.7), we get

$$E_1\le\frac{p_1-2}{2p_1}\int_\Omega|u(t^*)|^{p(x)}\mathrm{d}x\le J\big(u(t^*)\big)$$

which contradicts $J\big(u(t^*)\big)\le E(t^*)<E_1$. By repeating the previous step, we obtain (4.9). This completes the proof.

We set

$$H(t)=\hat\delta E_1-E(t), \tag{4.12}$$

then under the condition of theorem 4.1, we obtain

$$H'(t)=\gamma\int_\Omega|\nabla u_t|^2\,\mathrm{d}x-\frac{1}{2}(g'\circ\nabla u)+\frac{1}{2}g(t)\|\nabla u\|_2^2+r\int_\Omega|u_t|^{k(x)}\,\mathrm{d}x\ge r\int_\Omega|u_t|^{k(x)}\,\mathrm{d}x\ge 0 \tag{4.13}$$

and

$$0<H(0)\le H(t)<\hat\delta E_1+\int_\Omega\frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x<\left(\hat\delta\frac{p_1-2}{2p_1}+\frac{1}{p_1}\right)\int_\Omega|u|^{p(x)}\mathrm{d}x. \tag{4.14}$$

It's easy to examine the following lemma

Lemma 4.3 *Under the assumptions of Theorem 4.1, we have*

$$\|u(t)\|_{p_1}^s\le C\left(-H(t)-\|u_t\|_2^2-(g\circ\nabla u)(t)+\int_\Omega|u|^{p(x)}\mathrm{d}x\right) \tag{4.15}$$

for any $u\in H_0^1(\Omega)$ and $2\le s\le p_1$

Proof of Theorem 4.1. Assume by contradiction that (4.3) does not hold true. Then for all $T^* < +\infty$ and all $t \in 0, T^*]$, we have

$$\|u_t(t)\|_2^2 + \| \nabla u(t) \|_2^2 \leq C_1, \tag{4.16}$$

with C_1 is a positive constant. Motivated by [19], we set the function

$$L(t) = H^{1-a}(t) + \varepsilon \int_\Omega u_t u \mathrm{d}x + \varepsilon \frac{\gamma}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x, t \geq 0, \tag{4.17}$$

where $\varepsilon > 0$ is a positive constant to be chosen later, and

$$0 < a \leq \min\left(\frac{p_1 - 2}{2p_1}, \frac{p_1 - k_2}{p_1(k_2 - 1)}\right) < 1, \tag{4.18}$$

derivative the Eq (4.17) and using Eq. (1.3)-(1.5) we obtain

$$\begin{aligned} L'(t) &= (1-a)H^{-a}(t)H'(t) + \varepsilon \|u_t(t)\|_2^2 \\ &+\varepsilon \int_\Omega |u|^{p(x)} \mathrm{d}x - \varepsilon\gamma \|\nabla u(t)\|_2^2 \\ &-\varepsilon r \int_\Omega |u_t|^{k(x)-2} u_t u \mathrm{d}x \\ &+\varepsilon \int_\Omega \nabla u(t). \int_0^t g(t-s) \nabla u(s) \mathrm{d}s \mathrm{d}x. \end{aligned} \tag{4.19}$$

Applying the relation

$$\begin{aligned} p_1 H(t) = \quad & p_1 \hat{\delta} E_1 - \frac{p_1}{2} \|u_t(t)\|_2^2 \\ & - \frac{p_1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right) \| \nabla u(t) \|_2^2 \\ & - \frac{p_1}{2}(g \circ \nabla u)(t) + p_1 \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \mathrm{d}x \end{aligned}$$

and Young's inequality, we get from (4.19) that

$$\begin{aligned} L'(t) &= (1-a)H^{-a}(t)H'(t) \\ &+\varepsilon\left(1 + \frac{p_1}{2}\right)\|u_t(t)\|_2^2 - \varepsilon p_1 \hat{\delta} E_1 + \varepsilon p_1 H(t) \\ &+\varepsilon\left[\frac{p_1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right) - 1\right] \|\nabla u(t)\|_2^2 \\ &-\varepsilon r \int_\Omega |u_t|^{k(x)-2} u_t u \mathrm{d}x + \varepsilon \frac{p_1}{2}(g \circ \nabla u)(t) \\ &+\varepsilon\left(\int_\Omega |u|^{p(x)} \mathrm{d}x - p_1 \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \mathrm{d}x\right) \\ &+\varepsilon \int_0^t g(t-s) \|\nabla u(t)\|_2^2 \mathrm{d}s \\ &+\varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t).\left(\nabla u(s) - \nabla u(t)\right) \mathrm{d}x \mathrm{d}s \\ &\geq r(1-a)H^{-a}(t) \int_\Omega |u_t|^{k(x)} \mathrm{d}x \\ &+\varepsilon\left(1 + \frac{p_1}{2}\right)\|u_t(t)\|_2^2 - \varepsilon p_1 \hat{\delta} E_1 + \varepsilon p_1 H(t) \\ &+\varepsilon \begin{bmatrix} \left(\frac{p_1}{2} - 1\right) \\ -\left(\frac{p_1}{2} - 1 + \frac{1}{4\eta}\right) \int_0^t g(s)\mathrm{d}s \end{bmatrix} \| \nabla u(t) \|_2^2 \\ &+\varepsilon\left(\frac{p_1}{2} - \eta\right)(g \circ \nabla u)(t) - \varepsilon r \int_\Omega |u_t|^{k(x)-2} u_t u \mathrm{d}x \end{aligned} \tag{4.20}$$

By (4.9), estimate (4.20) becomes

$$\begin{aligned} L'(t) \geq \quad & r(1-a)H^{-a}(t) \int_\Omega |u_t|^{k(x)} \mathrm{d}x \\ &+\varepsilon\left(1 + \frac{p_1}{2}\right)\|u_t(t)\|_2^2 + \varepsilon p_1 H(t) \\ &+\varepsilon \left\{ \begin{array}{c} (1-\hat{\delta})\left(\frac{p_1}{2} - 1\right) \\ -\begin{bmatrix} (1-\hat{\delta})\left(\frac{p_1}{2} - 1\right) \\ +\frac{1}{4\eta} \end{bmatrix} \\ \int_0^t g(s)\mathrm{d}s \end{array} \right\} \| \nabla u(t) \|_2^2 \\ &+\varepsilon \begin{bmatrix} (1-\hat{\delta})\left(\frac{p_1}{2} - 1\right) \\ +(1-\eta) \end{bmatrix} (g \circ \nabla u)(t) \\ &-\varepsilon r \int_\Omega |u_t|^{k(x)-2} u_t u \mathrm{d}x \end{aligned} \tag{4.21}$$

Now, by using Young's inequality, we estimate the last term in (4.21) as follows

$$\begin{aligned} &\left| \int_\Omega |u_t|^{k(x)-2} u_t u \mathrm{d}x \right| \\ &\qquad \leq \frac{1}{k_1} \int_\Omega \delta^{k(x)} |u|^{k(x)} \mathrm{d}x \\ &\qquad + \frac{k_2 - 1}{k_2} \int_\Omega \delta^{-\frac{k(x)}{k(x)-1}} |u_t|^{k(x)} \mathrm{d}x, \\ &\qquad \forall \delta > 0 \end{aligned} \tag{4.22}$$

Therefore by taking δ so that

$$\delta^{-\frac{k(x)}{k(x)-1}} = M H^{-a}(t)$$

for a large constant M to be specified later, and substituted in (4.22)

$$\int_\Omega |u_t|^{k(x)-1}|u|\mathrm{d}x \le \frac{1}{k_1}\int_\Omega M^{1-k(x)}|u|^{k(x)}H^{a(k(x)-1)}(t)\mathrm{d}x + \frac{k_2-1}{k_2}MH^{-a}(t)\int_\Omega |u_t|^{k(x)}\mathrm{d}x$$

Then estimate (4.21) takes the form

$$L'(t) \ge r\left[(1-a) - \varepsilon\frac{k_2-1}{k_2}M\right.$$
$$\left.H^{-a}(t)\right]\int_\Omega |u_t|^{k(x)}\mathrm{d}x + \varepsilon\left(1+\frac{p_1}{2}\right)\|u_t(t)\|_2^2$$
$$+\varepsilon\left(p_1H(t) - \frac{M^{1-k_1}}{k_1}H^{a(k_2-1)}(t)\int_\Omega |u|^{k(x)}\mathrm{d}x\right)$$
$$+\varepsilon\left\{(1-\hat{\delta})\left(\frac{p_1}{2}-1\right)\right.$$
$$\left.-\left[(1-\hat{\delta})\left(\frac{p_1}{2}-1\right)+\frac{1}{4\eta}\right]\int_0^t g(s)\mathrm{d}s\right\}\|\nabla u\|_2^2$$
$$+\varepsilon\left[(1-\hat{\delta})\left(\frac{p_1}{2}-1\right)+(1-\eta)\right](g\circ\nabla u)(t)$$

Applying (4.2) and picking

$$\eta < (1-\hat{\delta})\left(\frac{p_1}{2}-1\right)+1,$$

we can get

$$L'(t) \ge r\left[(1-a) -\varepsilon\left(\frac{k_2-1}{k_2}\right)M\right]H^{-a}(t)\int_\Omega |u_t|^{k(x)}\mathrm{d}x$$
$$+\varepsilon\left(1+\frac{p_1}{2}\right)\|u_t(t)\|_2^2$$
$$+\varepsilon\left(p_1H(t)-\frac{M^{1-k_1}}{k_1}H^{a(k_2-1)}(t)\int_\Omega |u|^{k(x)}\mathrm{d}x\right)$$
$$+\varepsilon M_1\|\nabla u(t)\|_2^2 + \varepsilon M_2(g\circ\nabla u)(t), \tag{4.23}$$

where

$$M_1 = (1-\hat{\delta})\left(\frac{p_1}{2}-1\right) - \left[(1-\hat{\delta})\left(\frac{p_1}{2}-1\right) + \frac{1}{4\eta}\right]\int_0^t g(s)\mathrm{d}s > 0,$$

$$M_2 = \left[(1-\hat{\delta})\left(\frac{p_1}{2}-1\right)+(1-\eta)\right] > 0.$$

Using

$$H^{a(k_2-1)}(t)\int_\Omega |u|^{k(x)}\mathrm{d}x \le C\left(\hat{\delta}\frac{p_1-2}{2p_1}+\frac{1}{p_1}\right)^{a(k_2-1)} \| u(t)\|_{p(.)}^{k_2+p_1a(k_2-1)}$$

hence (4.23) yields

$$L'(t) \ge r\left[(1-a)-\varepsilon\frac{k_2-1}{k_2}M\right.$$
$$\left.H^{-a}(t)\right]\int_\Omega |u_t|^{k(x)}\mathrm{d}x + +\varepsilon\left(1+\frac{p_1}{2}\right)\|u_t(t)\|_2^2$$
$$+\varepsilon\left(p_1H(t)-\frac{M^{1-k_1}}{k_1}C\left(\hat{\delta}\frac{p_1-2}{2p_1}+\frac{1}{p_1}\right)^{a(k_2-1)}\right.$$
$$\left.\|u(t)\|_{p(.)}^{k_2+p_1a(k_2-1)}\right)$$
$$+\varepsilon M_1\|\nabla u(t)\|_2^2+\varepsilon M_2(g\circ\nabla u)(t). \tag{4.24}$$

We then use Lemma 4.3 and (4.18), for $s = k_2 + p_1a(k_2-1) \le p_1$, to deduce from (4.24)

$$L'(t) \ge r\begin{bmatrix}(1-a)\\ -\varepsilon\dfrac{k_2-1}{k_2}M\end{bmatrix}H^{-a}\int_\Omega |u_t|^{k(x)}\mathrm{d}x$$
$$+\varepsilon\left(1+\frac{p_1}{2}\right)\|u_t(t)\|_2^2$$
$$+\varepsilon M_1\|\nabla u(t)\|_2^2+\varepsilon M_2(g\circ\nabla u)(t)$$
$$+\varepsilon[p_1H(t)-C_2M^{1-k_1}$$
$$\times\left.\begin{pmatrix}-H(t)-\|u_t(t)\|_2^2-(g\circ\nabla u)(t)\\ +\varrho(u)\end{pmatrix}\right]$$
$$\ge r\begin{bmatrix}(1-a)\\ -\varepsilon\dfrac{k_2-1}{k_2}M\end{bmatrix}H^{-a}(t)\int_\Omega |u_t|^{k(x)}\mathrm{d}x$$
$$+\varepsilon\left(1+\frac{p_1}{2}+C_2M^{1-k_1}\right)\|u_t(t)\|_2^2$$
$$+\varepsilon M_1\|\nabla u(t)\|_2^2+\varepsilon\begin{pmatrix}M_2\\ +C_2M^{1-k_1}\end{pmatrix}(g\circ\nabla u)(t)$$
$$+\varepsilon\left(p_1+C_2M^{1-k_1}\right)H(t)-\varepsilon C_2M^{1-k_1}\varrho(u) \tag{4.25}$$

where $C_2 = \frac{C}{k_1}\left(\hat{\delta}\frac{p_1-2}{2p_1}+\frac{1}{p_1}\right)^{a(k_2-1)}$. By (3.1) and (4.12), we obtain

$$H(t) \ge \frac{1}{p_2}\varrho(u)-\frac{1}{2}\|u_t(t)\|_2^2-\frac{1}{2}\|\nabla u_t(t)\|_2^2 -\frac{1}{2}(g\circ\nabla u)(t),$$

writing $p_1 = 2M_3 + (p_1-2M_3)$, where $M_3 = \min\{M_1, M_2\}$, the estimate (4.25) yields

$$L'(t) \geq r\left[\begin{array}{c}(1-a)\\ -\varepsilon\left(\frac{k_2-1}{k_2}\right)M\end{array}\right]H^{-a}\int_\Omega |u_t|^{k(x)}\mathrm{d}x$$
$$+\varepsilon\left(1+\frac{p_1}{2}+C_2M^{1-k_1}-M_3\right)\|u_t(t)\|_2^2$$
$$+\varepsilon(M_1-M_3)\parallel \nabla u(t)\parallel_2^2$$
$$+\varepsilon\left(M_2+C_2M^{1-k_1}-M_3\right)(g\circ\nabla u)(t)$$
$$+\varepsilon\left(p_1-2M_3+C_2M^{1-k_1}\right)H(t)$$
$$+\varepsilon\left(\frac{2M_3}{p_1}-C_2M^{1-k_1}\right)\varrho(u). \tag{4.26}$$

We choose M large enough, (4.26) becomes

$$L'(t) \geq \quad r\left[\begin{array}{c}(1-a)-\\ \varepsilon\left(\frac{k_2-1}{k_2}\right)M\end{array}\right]H^{-a}(t)\int_\Omega |u_t|^{k(x)}\mathrm{d}x$$
$$+\chi\varepsilon\left(\begin{array}{c}H(t)+\|u_t(t)\|_2^2+\varrho(u)\\ +(g\circ\nabla u)(t)\end{array}\right), \tag{4.27}$$

for some positive constant χ. Once M is fixed, we choose ε small enough such that

$$(1-a)-\varepsilon\left(\frac{k_2-1}{k_2}\right)M>0,$$

and

$$L(0)=H^{1-a}(0)+\varepsilon\int_\Omega u_0u_1\mathrm{d}x+\frac{\varepsilon\gamma}{2}\|\nabla u_0\|_2^2>0$$

Hence, we have

$$L'(t)\geq\chi\varepsilon\left(\begin{array}{c}H(t)+\|u_t(t)\|_2^2+\varrho(u)\\ +(g\circ\nabla u)(t)\end{array}\right) \tag{4.28}$$

On the other hand, we have

$$L^{\frac{1}{1-a}}(t) \quad = \left(\begin{array}{c}H^{1-a}(t)\\ +\varepsilon\int_\Omega u_t(t)u(t)\mathrm{d}x\\ +\varepsilon\frac{\gamma}{2}\parallel\nabla u(t)\parallel_2^2\end{array}\right)^{\frac{1}{1-a}}$$
$$\leq C\left(\begin{array}{c}H(t)+\left|\int_\Omega u_t(t)u(t)\mathrm{d}x\right|^{\frac{1}{1-a}}\\ +\parallel\nabla u(t)\parallel_2^{\frac{1}{1-a}}\end{array}\right). \tag{4.29}$$

By Hölder's and Young's inequalities, (4.16), and Lemma 4.3, we get

$$\left|\int_\Omega u_t(t)u(t)\mathrm{d}x\right|^{\frac{1}{1-a}}\leq C(\|u(t)\|_2\|u_t(t)\|_2)^{\frac{1}{1-a}}$$
$$\leq C\|u(t)\|_{p(.)}^{\frac{1}{1-a}}\|u_t(t)\|_2^{\frac{1}{1-a}}$$
$$\leq\left(\|u(t)\|_{p(.)}^{\frac{2}{1-2a}}+\|u_t(t)\|_2^2\right)$$
$$\leq C\left(H(t)+\|u_t(t)\|_2^2+\varrho(u)+(g\circ\nabla u)(t)\right), \tag{4.30}$$

And

$$\parallel\nabla u(t)\parallel_2^{\frac{2}{1-a}}\leq C_1^{\frac{1}{1-a}}. \tag{4.31}$$

By Poincaré's inequality and (4.16), we have

$$\parallel u(t)\parallel_{p(.)}^{p_1}\leq B_1^{p_1}\parallel\nabla u(t)\parallel_2^{p_1}\leq B_1^{p_1}C_1^{\frac{p_1}{2}} \tag{4.32}$$

By virtue of (4.14) and (4.32), we get $H(t)$ is bounded. There exists a positive constant C_3 such that

$$H(t)+C_1^{\frac{1}{1-a}}\leq C_3H(t) \tag{4.33}$$

Therefore, we obtain

$$L^{\frac{1}{1-a}}(t)\leq C\left(\begin{array}{c}H(t)+\|u_t(t)\|_2^2+\varrho(u)\\ +(g\circ\nabla u)(t)\end{array}\right) \tag{4.34}$$

By joining (4.28) and (4.34), we reach that

$$L'(t)\geq\frac{\varepsilon\chi}{C}L^{\frac{1}{1-a}}(t) \tag{4.35}$$

A simple integration of (4.35) over $[0,t]$, yields that

$$L^{\frac{a}{1-a}}(t)\geq\frac{1}{L^{-\frac{a}{1-a}}(0)-\frac{a\varepsilon\chi}{C(1-a)}t},\quad\forall t\geq 0$$

This shows that $L(t)$ blows up in a finite time T_1, where

$$T_1\leq\frac{(1-a)C}{\chi\varepsilon a[L(0)]^{\frac{a}{1-a}}}$$

If we choose $T^*\geq\frac{(1-a)C}{\chi\varepsilon a[L(0)]^{\frac{a}{1-a}}}$, then we obtain $T_1\leq T^*$, which contradicts our assumption. This completes the proof.

5 An Upper Bound for the Blow-Up Time: The Case $k(x)=2,\forall x$

In this section, we prove a finite time blow-up result. We need the following lemma.

Lemma 5.1 *([14], Lemma1.1 and, [16], Logarithmic convexity methods) Assume that* $\varphi\in C^2([0,T))$ *satisfying:*

$$\varphi''\varphi-(1+\alpha)(\varphi')^2\geq 0,\qquad\alpha>0$$

and

$$\varphi(0)>0,\qquad\varphi'(0)>0,$$

then

$$\varphi\to\infty\text{ast}\to t_1\leq t_2=\frac{\varphi(0)}{\alpha\varphi'(0)}.$$

Theorem 5.2 *For any fixed* $\delta<1$, *suppose that (2.6) holds and* u_0,u_1 *satisfy*

$$I(0)<0,\ E(0)<\delta E_1. \tag{5.1}$$

Assume that

$$\int_0^\infty g(s)\mathrm{d}s \le \frac{p_1-2}{p_1-2+\left[(1-\hat{\delta})^2p_1+2\hat{\delta}(1-\hat{\delta})\right]^{-1}} \quad (5.2)$$

where $\hat{\delta}=\max\{0,\delta\}$, and suppose further that $\int_\Omega u_0u_1\mathrm{d}x>0$ for $0\le E(0)<E_1$. Under the assumption of Lemma 3.1, if

$$2=k_1=k(x)=k_2<p_1\le p(x)\le p_2\le\frac{2n}{n-2},$$

the solution of problem (1.3)-(1.5) blows up in a finite time T_2, in the sense that

$$\lim_{t\to T_2^-}\left[\| u(t) \|_2^2+\int_0^t \| \nabla u(s) \|_2^2 \,\mathrm{d}s+\int_0^t \| u(s) \|_2^2 \,\mathrm{d}s\right]=+\infty$$

Further, the upper bound for T_2 can be estimated by

$$T_2\le\frac{2(p_1-2)^2 \| u_0 \|_2^2+8(\gamma\|\nabla u_0\|_2^2+r\|u_0\|_2^2)^2}{(p_1-2)^3\int_\Omega u_1u_0\mathrm{d}x},$$

with some $t>0$ and φ is defined in (5.1).

Proof. Assume by contradiction that the solution u is global. Then for any $T>0$, we define the functional φ as follows

$$\begin{aligned}\varphi(t) &=\| u(t) \|_2^2+\gamma\int_0^t \| \nabla u(s) \|_2^2 \,\mathrm{d}s\\ &+r\int_0^t \| u(s) \|_2^2 \,\mathrm{d}s\\ &+(T_0-t)[\gamma\|\nabla u_0\|_2^2+r\|u_0\|_2^2]\\ &+(t+t_0)^2, \qquad t<T_0\end{aligned} \quad (5.3)$$

where t_0,T_0 and β are positive constants to be chosen later. Then using equation (1.3) and integration by parts, to get

$$\begin{aligned}\varphi'(t)&=2\int_\Omega u_t(t)u(t)\mathrm{d}x\\ &+2\gamma\int_0^t\int_\Omega \nabla u_s(s)\,\nabla u(s)\mathrm{d}x\mathrm{d}s\\ &\int_0^t\int_\Omega u_s(s)u(s)\mathrm{d}x\mathrm{d}s+2(t+t_0).\end{aligned} \quad (5.4)$$

And

$$\begin{aligned}\varphi''(t)&=2\|u_t(t)\|_2^2+2\int_\Omega u_{tt}(t)u(t)\mathrm{d}x\\ &-2\left(1-\int_0^t g(s)\mathrm{d}s\right)\| \nabla u(t) \|_2^2\\ &+2\gamma\int_\Omega \nabla u_t(\mathrm{t}).\nabla u(t)\mathrm{d}x\\ &+2r\int_\Omega u_t(t)u(t)\mathrm{d}x+2.\end{aligned} \quad (5.5)$$

Furthermore

$$\begin{aligned}\varphi''(t)&\ge 2\|u_t(t)\|_2^2\\ &-2\left(1-\int_0^t g\,(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2\\ &-\frac{1}{\varepsilon}\int_0^t g\,(s)\mathrm{d}s\|\nabla u(t)\|_2^2-\varepsilon(g\circ\nabla u)(t)+2\\ &+p_1(g\circ\nabla u)+p_1\|u_t\|_2^2\\ &+p_1\left(1-\int_0^t g\,(s)\mathrm{d}s\right)\|\nabla u\|_2^2-2p_1E(t)\\ &\ge(p_1+2)\|u_t(t)\|_2^2\\ &+\left((p_1-2)-\left(p_1-2+\frac{1}{\varepsilon}\right)\int_0^t g\,(s)\mathrm{d}s\right)\|\nabla u\|_2^2\\ &+(p_1-\varepsilon)(g\circ\nabla u)(t)-2p_1E(0)+2,\end{aligned} \quad (5.6)$$

where t_0,T_0 are constants to be determined later.

Case 1: If $\delta<0$, then $E(0)<0$, we choose $\varepsilon=p_1$ in (5.6). Then, by (5.3), (5.4), (5.5), (5.2), and (5.6), we have

$$\begin{cases}\varphi(0)=\int_\Omega u_0^2(x)\mathrm{d}x+T_0\left[\gamma\|\nabla u_0\|_2^2+r\|u_0\|_2^2\right]; \\ \qquad +t_0^2>0\\ \varphi'(0)=2\int_\Omega u_1u_0\mathrm{d}x+2t_0>0;\\ \varphi''(t)\ge 2-2p_1E(0)>0\forall t\ge 0.\end{cases}$$

Therefore φ and φ' are both positive. Thus, from (5.3)-(5.6) and (5.8), the following inequality, inferred for all $(\zeta,\eta)\in\mathbb{R}_+^2$, which implies that

$$\varphi(t)\varphi''(t)-\frac{p_1+2}{4}\left(\varphi'(t)\right)^2\ge 0. \quad (5.7)$$

Case2: If $0\le\delta<1$, then

$$0\le E(0)<\delta E_1<E_1,$$

we choose $\varepsilon=(1-\delta)p_1+2\delta$ in (5.6), using (5.2) Then, we have

$$\begin{aligned}
&\varphi''(t) \geq (p_1+2)\|u_t(t)\|_2^2 \\
&+\left((p_1-2)-\left(p_1-2+\frac{1}{(1-\delta)p_1+2\delta}\right)\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 \\
&+(p_1-2)\delta(g\circ\nabla u)(t)-2p_1E(0)+2 \\
&\geq (p_1+2)\|u_t(t)\|_2^2+\delta(p_1-2)\left(\left(1-\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2+(g\circ\nabla u)(t)\right) \\
&-2p_1E(0)+2 \\
&\geq (p_1+2)\|u_t(t)\|_2^2+2p_1(\delta E_1-E(0))+2>0.
\end{aligned} \tag{5.8}$$

Then, by (5.3), (5.4), (5.5), (5.2), and (5.6), we have

$$\begin{cases}\varphi(0)=\int_\Omega u_0^2(x)\mathrm{d}x+t_0^2 \\ +T_0[\gamma\|\nabla u_0\|_2^2+r\|u_0\|_2^2]>0; \\ \varphi'(0)=2\int_\Omega u_1u_0\mathrm{d}x+2t_0>0; \\ \varphi''(t)\geq 2-2p_1E(0)>0\forall t\geq 0.\end{cases}$$

Therefore φ and φ' are both positive.

Then using Lemma5.1, to infer

$$\varphi(t)\to\infty$$

as $t\to T^*$, where,

$$T^*\leq\frac{2\parallel u_0\parallel_2^2+2T_0[\gamma\|\nabla u_0\|_2^2+r\|u_0\|_2^2]+2t_0^2}{(p_1-2)\left(\int_\Omega u_1u_0\mathrm{d}x+t_0\right)}.$$

Now we go to choose appropriate t_0 and T_0. Let t_0 be any number that depends only on p_1, $E_0-E(0)$ and $\|u_0\|_{L^2(\Omega)}$ as

$$t_0\geq\frac{2(\gamma\|\nabla u_0\|_2^2+r\|u_0\|_2^2)}{(p_1-2)}$$

Fix t_0, then T_0 can be picked as

$$T_0=\frac{2\parallel u_0\parallel_2^2+2T_0[\gamma\|\nabla u_0\|_2^2+r\|u_0\|_2^2]+2t_0^2}{(p_1-2)\left(\int_\Omega u_1u_0\mathrm{d}x+t_0\right)}$$

so that

$$T_0=\frac{2\parallel u_0\parallel_2^2+2t_0^2}{(p_1-2)t_0+(p_1-2)\int_\Omega u_1u_0\mathrm{d}x-2\begin{pmatrix}\gamma\|\nabla u_0\|_2^2\\+r\|u_0\|_2^2\end{pmatrix}}$$

Therefore the lifespan of the solution $u(x,t)$ is bounded by

$$T^*\leq\inf_{t\geq t_0}\frac{2\parallel u_0\parallel_2^2+2t^2}{(p_1-2)t+(p_1-2)\int_\Omega u_1u_0\mathrm{d}x-2(\gamma\|\nabla u_0\|_2^2+r\|u_0\|_2^2)}$$

$$=\frac{2(p_1-2)^2\parallel u_0\parallel_2^2+8(\gamma\|\nabla u_0\|_2^2+r\|u_0\|_2^2)^2}{(p_1-2)^3\int_\Omega u_1u_0\mathrm{d}x}.$$

6 A Lower Bound for the Blow-Up Time: The Case $k_1\geq 2$.

In this section, by using a first-order differential inequality technique for a suitably defined auxiliary function and some Sobolev-type inequalities, we give a lower bound for the blow-up time T for the solution $u(x,t)$ of the problem (1.3)-(1.5) if

$$\begin{aligned}&2\leq k_1\leq k(x)\leq k_2<p_1\\&\leq p(x)\leq p_2\leq\tfrac{2n}{n-2},\end{aligned} \tag{6.1}$$

holds.

Theorem 6.1 *Under the condition of Lemma3.1, assume that (6.1) holds, then the solution of problem (1.3)-(1.5) will blow up in finite time T. Moreover, the blow-up time T can be estimated from above by $\hat{T}$, where* $\hat{T}$

$$=\max\left(\frac{2\ln\left[\begin{matrix}2^{\frac{3p_1-4}{2}}l^{p_1-1}(\|u_1\|_2^2+\parallel\nabla u_0\parallel_2^2)^{2-p_2}\\+|\Omega|^{\frac{p_1-2}{2}}\end{matrix}\right]}{p_1-2},\frac{2\ln\left[\begin{matrix}2^{\frac{3p_2-4}{2}}l^{p_2-1}(\|u_1\|_2^2+\parallel\nabla u_0\parallel_2^2)^{2-p_2}\\+|\Omega|^{\frac{p_2-2}{2}}\end{matrix}\right]}{p_2-2}\right) \tag{6.2}$$

and $|\Omega|=\int_\Omega\mathrm{d}x$.

Proof. We assume that $u(x,t)$ blows up at time T and define the auxiliary functional

$$\varphi(t)=\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\parallel\nabla u(t)\parallel_2^2+(g\circ\nabla u)(t) \tag{6.3}$$

Taking a derivative of $\varphi(t)$, and using (1.3), we get

$$\begin{aligned}\varphi'(t)=&\int_\Omega u_{tt}(t)u_t(t)\mathrm{d}x\\&+\left(1-\int_0^t g(s)\mathrm{d}s\right)\int_\Omega\nabla u(t).\nabla u_t(t)\mathrm{d}x\\&-g(t)\parallel\nabla u(t)\parallel_2^2+(g'\circ\nabla u)(t)\\&+\int_\Omega\nabla u_t(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x\\=&-\gamma\|\nabla u_t(t)\|_2^2-g(t\parallel\nabla u(t)\parallel_2^2+(g'\nabla u)(t)\\&+\int_\Omega u_tu|u|^{p(x)-2}\mathrm{d}x-r\int_\Omega|u_t|^{k(x)}\mathrm{d}x\\\leq&\int_\Omega u_tu|u|^{p(x)-2}\mathrm{d}x\end{aligned} \tag{6.4}$$

Using Young's inequality, we have

$$\left|\int_\Omega |u|^{p(x)-2}uu_t\mathrm{d}x\right|$$
$$\leq \frac{1}{2}\int_\Omega u_t^2\mathrm{d}x+\frac{1}{2}\int_\Omega |u|^{2p(x)-2}\mathrm{d}x$$
$$\leq \frac{1}{2}\int_\Omega u_t^2\mathrm{d}x$$
$$+\frac{1}{2}\max\left\{\int_\Omega |u|^{2p_2-2}\mathrm{d}x,\int_\Omega |u|^{2p_1-2}\mathrm{d}x\right\}$$
$$\leq \frac{1}{2}\int_\Omega u_t^2\mathrm{d}x$$
$$+\frac{1}{2}\max\left\{C_1\parallel \nabla u\parallel_2^{2p_1-2},C_2\parallel \nabla u\parallel_2^{2p_2-2}\right\}$$
$$\leq \frac{1}{2}\varphi+\frac{1}{2}\max\left\{\frac{C_1}{l^{p_1-1}}\varphi^{p_1-1},\frac{C_2}{l^{p_2-1}}\varphi^{p_2-1}\right\} \tag{6.5}$$

where

$$C_1=\frac{|\Omega|^{\frac{p_1-2}{2}}}{2^{\frac{p_1}{2}}},\qquad C_2=\frac{|\Omega|^{\frac{p_2-2}{2}}}{2^{\frac{p_2}{2}}}.$$

By (2.6), (6.4), and (6.5), we can obtain that

$$\varphi'(t)\leq \frac{1}{2}\varphi+\frac{1}{2}\max\left\{\frac{C_1}{l^{p_1-1}}\varphi^{p_1-1},\frac{C_2}{l^{p_2-1}}\varphi^{p_2-1}\right\} \tag{6.6}$$

Integrating inequality (6.6), we have

$$\varphi(t)$$
$$\leq \max\left(\begin{array}{c}\left(\left[(\varphi(0))^{2-p_1}+\frac{C_1}{l^{p_1-1}}\right]e^{-(p_1-2)\frac{1}{2}t}-\frac{C_1}{l^{p_1-1}}\right)^{\frac{-1}{p_1-1}}\\ ,\left(\left[(\varphi(0))^{2-p_2}+\frac{C_2}{l^{p_2-1}}\right]e^{-(p_2-2)\frac{1}{2}t}-\frac{C_2}{l^{p_2-1}}\right)^{\frac{-1}{p_2-1}}\end{array}\right)$$

Let

$$0<T_*:=\max\left(\frac{2}{p_1-2}\ln\left[\frac{l^{p_1-1}}{C_1}(\varphi(0))^{2-p_1}+1\right],\frac{2}{p_2-2}\ln\left[\frac{l^{p_2-1}}{C_2}(\varphi(0))^{2-p_2}+1\right]\right)<\infty. \tag{6.7}$$

then $\varphi(t)$ blows up at time T^*. Hence, $u(x,t)$ discontinues at some finite time $T\leq T^*$, that is to means, $u(x,t)$ blows up at a finite time T.

Next, we estimate T. By the values of C_1, C_2, we have

$$\frac{l^{p_1-1}}{C_1}(\varphi(0))^{2-p_1}+1\leq \frac{2^{\frac{3p_1-4}{2}}l^{p_1-1}(\|u_1\|_2^2+\|\nabla u_0\|_2^2)^{2-p_2}+|\Omega|^{\frac{p_1-2}{2}}}{|\Omega|^{\frac{p_1-2}{2}}},$$

$$\frac{l^{p_2-1}}{C_2}(\varphi(0))^{2-p_2}+1\leq \frac{2^{\frac{3p_2-4}{2}}l^{p_2-1}(\|u_1\|_2^2+\parallel \nabla u_0\parallel_2^2)^{2-p_2}+|\Omega|^{\frac{p_2-2}{2}}}{|\Omega|^{\frac{p_2-2}{2}}}.$$

The above pair inequalities coupling (6.7) give $T\leq T^*\leq \hat{T}$, where $\hat{T}$ is fixed in (6.2).

7 General Comments and Issues

This paper is devoted to studying a model of a nonlinear viscoelastic wave equation with damping and source terms involving variable-exponent nonlinearities (1.3)-(1.5).

1. We prove that the energy grows exponentially, and thus so the L^{p_2} and L^{p_1}-norms. For the case 2≤k(.)<p(.), we reach the exponential growth result in a blow-up in finite time with positive initial energy and get the upper bound for the blow-up time.

2. For the case k(.)=2, we use the concavity method to show a finite time blow-up result and get the upper bound for the blow-up time of the solutions.

3. Furthermore, for the case k(.)≥2, under some conditions on the data, we give a lower bound for the blow-up time when the blow-up occurs.

-The natural question that we can ask is whether the obtained decay rate (3.32) is optimal.

-The second question is the extension of our results to the case of other boundary conditions than (1,4), especially the proof of the lack of exponential stability.

-The last interesting question we note here is proving the stability of (1.3)-(1.5) in the whole space $\mathbb{R}^n$ $(n\geq 1)$ (instead of Ω).

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