Novel Exact Traveling Wave Solutions for Nonlinear Wave Equations with Beta-Derivatives via the sine-Gordon Expansion Method

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Abstract: The main objectives of this research are to use the sine-Gordon expansion method (SGEM) along with the use of appropriate traveling transformations to extract new exact solitary wave solutions of the $(2 + 1)$-dimensional breaking soliton equation and the generalized Hirota-Satsuma coupled Korteweg de Vries (KdV) system equipped with beta partial derivatives. Using the chain rule, we convert the proposed nonlinear problems into nonlinear ordinary differential equations with integer orders. There is then no further demand for any normalization or discretization in the calculation process. The exact explicit solutions to the problems obtained with the SGEM are written in terms of hyperbolic functions. The exact solutions are new and published here for the first time. The effects of varying the fractional order of the beta-derivatives are studied through numerical simulations. 3D, 2D, and contour plots of solutions are shown for a range of values of fractional orders. As parameter values are changed, we can identify a kink-type solution, a bell-shaped solitary wave solution, and an anti-bell shaped soliton solution. All of the solutions have been carefully checked for correctness and could be very important for understanding nonlinear phenomena in beta partial differential equation models for systems involving the interaction of a Riemann wave with a long wave and interactions of two long waves with distinct dispersion relations.

Key-Words: - sine-Gordon expansion method, Exact traveling wave solutions, Beta partial derivatives, Breaking soliton equation, Generalized Hirota-Satsuma coupled KdV

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1 Introduction

Exact solutions are of great importance for describing several nonlinear wave phenomena arising in nature such as the propagation of shallow water waves, [1], nonlinear optics, [2], [3], plasma physics, [4], quantum mechanics, [5], [6], fluid dynamics, [7], hydrodynamics, [8], and acoustics, [9]. Mathematical models for most of the mentioned phenomena can be developed using nonlinear partial differential equations (NLPDEs). Unlike numerical methods for NLPDEs, the powerful analytic methods for obtaining exact solutions which are now available and which use the chain rule do not require any normalization or discretization, [10]. Also, with the fast development of computer science and computerized symbolic computation, the direct search for exact solitary wave solutions of NLPDEs has become practicable and is now attracting much attention from research scholars all over the world. Many techniques have now been developed to obtain exact analytical solutions for NLPDEs equipped with classical, conformable and beta-derivatives, [11], [12], [13], [14], [15], [16]. Useful techniques that have been successfully applied to obtain exact solutions of NLPDEs include the modified Kudryashov method, [17], the $(G'/G, 1/G)$-expansion method, [18], the (G'/G^2)-expansion method, [19], [20], the modified simple equation (MSE) method, [21], the Hirota bilinear approach, [22], the Jacobi elliptic equation method, [23], the $\exp(-\varphi(\xi))$-expansion approach, [24], the Ricatti-Bernoulli sub-ODE method, [25], the Bäcklund transformation, [26], the auxiliary equation method, [27], the trial equation technique, [28], the new extended direct algebraic method, [29], [30], and the Sardar sub-equation method, [31].

Recently, many researchers have given new definitions of derivatives with a fractional order which have been used for NLPDEs instead of the classical partial derivative. These derivatives include the conformable derivative, [32], the Atangana conformable derivative, [33], and the truncated Ω-fractional derivative, [34].

In this article, we concentrate on the use of the sine-Gordon expansion method, [35], to generate exact traveling wave solutions of the $(2+1)$-dimensional breaking soliton equation and the generalized Hirota-Satsuma coupled Korteweg de Vries (KdV) system for beta partial derivatives. The definitions of the two systems are as follows.

1. The $(2+1)$-dimensional breaking soliton equation with beta space-time derivatives is defined by:

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) - 4\left(\frac{\partial^\beta u}{\partial x^\beta}\right)\left(\frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right)\right) -2\left(\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right)\right)\left(\frac{\partial^\beta u}{\partial y^\beta}\right) +\frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right)\right)\right) = 0, \quad (1)$$

where $\frac{\partial^\alpha}{\partial t^\alpha}(\cdot)$, $\frac{\partial^\beta}{\partial x^\beta}(\cdot)$ and $\frac{\partial^\beta}{\partial y^\beta}(\cdot)$ denote the beta partial derivatives with respect to t of order $0 < \alpha \leq 1$, to x of order $0 < \beta \leq 1$ and to y of order $0 < \beta \leq 1$, respectively. If $\alpha = \beta = 1$, then equation (1) reduces to the original breaking soliton equation, [36], initially proposed by [37], [38]. The equation explains the $(2+1)$-dimensional interaction of a Riemann wave propagated along the y-axis with a long wave propagated along the x-axis, [39], [40], [41]. If we set $y = x$ in the original breaking soliton equation and integrate the resulting equation, then we obtain the potential KdV equation. This KdV equation is an important mathematical model for the special waves called solitons on shallow water surfaces. The significant characteristic feature of breaking soliton equations is that the spectral parameter used in the Lax representations possesses so-called breaking behavior, [42]. Details of some existing methods for finding exact solutions of the $(2+1)$-dimensional breaking soliton equation of integer-orders are given in [43], [44], [45].

2. The generalized Hirota-Satsuma coupled Korteweg de Vries (KdV) system with beta time derivative is defined by:

$$\begin{aligned} \frac{\partial^\eta u}{\partial t^\eta} &= \frac{1}{4}u_{xxx} + 3uu_x + 3(-v^2+w)_x, \\ \frac{\partial^\eta v}{\partial t^\eta} &= -\frac{1}{2}v_{xxx} - 3uv_x, \\ \frac{\partial^\eta w}{\partial t^\eta} &= -\frac{1}{2}w_{xxx} - 3uw_x, \end{aligned} \quad (2)$$

where $\frac{\partial^\eta}{\partial t^\eta}(\cdot)$ represents the beta partial derivative with respect to t of order $0 < \eta \leq 1$. If $\eta = 1$, then equation (2) becomes the first-order generalized Hirota-Satsuma coupled KdV system, which was first introduced by Satsuma and Hirota in 1982. This system can be reduced to the Hirota-Satsuma system by using the four-reduction Kadomtsev-Petviashvili (KP) hierarchy and introducing the new dependent variables u, v and w, [46]. The first-order generalized Hirota-Satsuma coupled KdV equation can then be reduced further to the Hirota-Satsuma coupled KdV (cKdV) equation by taking $w = 0$ and scaling the variables, [46], [47]. The integer-order system, [48], [49], [50], [51], for equation (2) has been extensively studied by researchers because the system is used as a model for dispersive long waves in shallow water which appear in many applications of fluid mechanics and related fields. These applications include shallow-water undulations with weakly nonlinear retrieve vigor, ion-acoustic undulations in a plasma and the interaction of neighboring particles of equal mass in a crystal lattice, [51], [52]. Further interesting applications of the generalized Hirota-Satsuma coupled KdV system can be found in [53], [54].

Since the $(2+1)$-dimensional breaking soliton equation and the generalized Hirota-Satsuma coupled KdV system especially play an important role in shallow-water phenomena, we believe that the generalized models in equations (1) and (2) could give some very useful insights into traveling wave behavior in fluids and related fields. To the best of the authors' knowledge, there are no researchers who have obtained exact traveling wave solutions for the NLPDEs with beta partial derivatives in equations (1) and (2) using the sine-Gordon expansion method. Therefore, some novel exact solutions to these two problems will be reported here for the first time.

The paper is arranged as follows. In Section 2, the definition and some useful features of the beta-derivative are presented. Then, in Section 3, the important steps of the sine-Gordon expansion method are briefly explained. In Section 4, we use the sine_Gordon method to derive new exact solutions of equations (1) and (2). Also, in this section, we use numerical simulations to obtain graphs of the solutions for a range of fractional orders and to give physical interpretations of some selected exact solutions. Finally, in Section 5, we give a discussion and conclusions.

2 Beta-Derivative and Its Properties

A significant advantage of some fractional derivatives such as the Caputo fractional derivative, [55], the Riemann Liouville fractional derivative, [55], the conformable derivative, [56], and the beta-derivative, [57], is that, unlike integer-order derivatives, they can describe memory properties of various materials and processes. In this section, we define the beta-derivative, which was initially proposed by [58], and give some of its properties. The beta-derivative can be considered as a natural generalization of the classical derivative as it obeys most of the fundamental properties of the classical derivative.

Definition 2.1 *Let $f : [0,\infty) \to \mathbb{R}$ be a function. Then, the beta-derivative of f of order β, where $0 < \beta \leq 1$, is defined as [57], [58], [59], [60].*

$$D_t^{\beta} f(t) = \lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - f(t)}{\varepsilon}. \quad (3)$$

Some useful properties of the beta-derivative are as follows, [57], [58], [59], [60]. Let $f(t)$, $g(t)$ be β-differentiable functions for all $t > 0$ and $\beta \in (0,1]$. Then

(1) $D_t^{\beta}(\lambda) = 0,\ \forall\lambda \in \mathbb{R}$.

(2) $D_t^{\beta}(af(t) + bg(t)) = aD_t^{\beta}f(t) + bD_t^{\beta}g(t)$, for all values $a, b \in \mathbb{R}$.

(3) $D_t^{\beta}(f(t)g(t)) = f(t)D_t^{\beta}g(t) + g(t)D_t^{\beta}f(t)$.

(4) $D_t^{\beta}\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)D_t^{\beta}f(t) - f(t)D_t^{\beta}g(t)}{(g(t))^2}$, where $g(t) \neq 0$.

(5) If f is differentiable, then $D_t^{\beta}(f(t)) = \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{df(t)}{dt}$.

Theorem 2.1 *Suppose f, $g : (0, \infty) \to \mathbb{R}$ are differentiable and also β-differentiable. Further assume that g is a function defined in the range of f. Then, the beta-derivative of a composite function $f \circ g$ can be expressed as [57], [58], [59], [60].*

$$D_t^{\beta}(f\circ g)(t) = \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta} f'(g(t))g'(t), \quad (4)$$

where the prime symbol $(')$ denotes the classical derivative.

By equation (3), the beta partial derivative of, for example, a function $u = u(x,t)$ with respect to t of order $\beta \in (0,1]$ can be defined by

$$\partial_t^{\beta}u(x,t) = \frac{\partial^{\beta}}{\partial t^{\beta}}u(x,t) = \lim_{\varepsilon\to 0}\frac{u\left(x, t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - u(x,t)}{\varepsilon},\ t>0. \quad (5)$$

A review of recent literature on applications of the beta-derivative in NLPDEs is as follows. In [57] the authors studied magnetic solitons and periodic wave propagation in a Heisenberg ferromagnetic spin chain using the $(2+1)$-dimensional nonlinear Schrödinger equation (NLSE) with beta-derivative. They found that the beta-derivative parameter significantly affected the rogue wave phenomena in which the amplitudes and widths of such waves are enlarged with an increase of β. The results for this system can be very helpful in analyzing the wave dynamics arising in any non-local and non-conservative/conservative physical system. In [61], the authors studied the space-time fractional modified equal width (FMEW) equation with beta-derivative. This equation is related to the regularized long wave (RLW) equation which has solitary wave solutions with both positive and negative amplitudes but the same width. In this paper, new traveling wave solutions for the equation were constructed using the unified method and the effects of varying the fractional order were studied. These new traveling wave solutions were expressed in both polynomial and rational forms. Further applications of the beta-derivative in physical applications such as group velocity dispersion, unidirectional propagation of long waves and transmission in monomode optical fibers can be found in [59], [60], [62].

3 The sine-Gordon Expansion Method

When compared with other existing methods, the sine-Gordon expansion method has been found to be a simple, direct and powerful mathematical tool for obtaining exact solutions of NLPDEs arising in the fields of science, engineering and mathematical physics, [63], [64]. The solutions obtained by the SGEM show many new behavioral aspects such as wave solutions which are expressed in terms of hyperbolic, exponential and complex function structures. In particular, it has been found that solutions of some real physical models are related to hyperbolic functions, [65]. For example, hyperbolic sine and cosine functions occur in the gravitational potential of a cylinder and in the shape of a hanging cable, respectively. The hyperbolic tangent function occurs in some applications involving special relativity and the hyperbolic secant and cotangent functions occur in the profile of a laminar jet and in the Langevin function for magnetic polarization, respectively.

A review of recent literature in which the sine-Gordon expansion method has been used to obtain exact solutions of NLPDEs is as follows. In [66], the authors applied the sine-Gordon expansion method to obtain exact solutions of some conformable time fractional equations in the Regularized Long Wave (RLW)-class. They found some real-valued and complex-valued solutions which were combinations of powers of hyperbolic tangent and hyperbolic secant functions. A recent study also used their exact solutions and numerical simulations to study the ef-fects of changing the fractional time order. In [67], the authors used the sine-Gordon expansion method to solve Kudryashov's equation. They obtained new so-lutions which included singular and bright-dark opti-cal soliton solutions. Also, in [68], the authors used the sine-Gordon expansion method to derive exact soli-ton solutions for the higher dimensional generalized Boussinesq equation and the Klein-Gordon equation. They found breather, rogue, bell-shaped, bright-dark, kink and periodic solitons. In [69], the authors applied the sine-Gordon expansion method to obtain exact optical soliton solutions of the Fokas-Lenells equa-tion. Further papers on the applications of the sine-Gordon expansion method to obtain exact solutions for NLPDEs can be found in [70], [71], [72], [73].

We will now apply the sine-Gordon expansion method to obtain exact solutions of the following sine-Gordon equation with beta partial derivatives.

$$\partial_x^\beta\left(\partial_x^\beta u\right)-\partial_t^\beta\left(\partial_t^\beta u\right)=m^2\sin(u),\ \beta\in(0,1], \tag{6}$$

where $u=u(x,t)$ and m is a nonzero real constant. Substituting the wave variable

$$\begin{aligned} u&=u(x,t)=U(\xi),\\ \xi&=\mu\left(\frac{\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}-\frac{c\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}\right), \end{aligned} \tag{7}$$

into equation (6) and using the chain rule mentioned above, we obtain the following nonlinear ODE

$$U''=\frac{m^2}{\mu^2(1-c^2)}\sin(U), \tag{8}$$

where $U=U(\xi)$, ξ is the amplitude of the traveling wave, μ is a nonzero real constant and c is the velocity of the traveling wave. After applying integration by parts and some trigonometric identities to equation (8) and then simplifying the result, we get

$$\left[\left(\frac{U}{2}\right)'\right]^2=C^2\sin^2\left(\frac{U}{2}\right)+K, \tag{9}$$

where $C^2=\frac{m^2}{\mu^2(1-c^2)}, c\neq\pm1$ and K is a constant of integration. Then, replacing $\omega(\xi)=\frac{U}{2}$ and $K=0$ in equation (9) and simplifying, we obtain

$$\omega'=C\sin(\omega). \tag{10}$$

By choosing $C=1$ for equation (10), we have

$$\omega'=\sin(\omega). \tag{11}$$

Using the method of separation of variables to solve equation (11), we obtain two important relations as follows, [35]:

$$\sin(\omega)=\sin(\omega(\xi))=\frac{2pe^\xi}{p^2e^{2\xi}+1}\Big|_{p=1}=\operatorname{sech}(\xi), \tag{12}$$

and

$$\cos(\omega)=\cos(\omega(\xi))=\frac{p^2e^{2\xi}-1}{p^2e^{2\xi}+1}\Big|_{p=1}=\tanh(\xi), \tag{13}$$

where p is a constant of integration.

We will now give a summary of the sine-Gordon expansion method (SGEM) for NLDEs. In order to apply the sine-Gordon expansion method to nonlinear space-time partial differential equations with beta partial derivatives, we must first transform the original problem into an ordinary differential equation (ODE) in a new variable ξ. Consider the following nonlinear partial differential equation with beta partial derivatives of a dependent variable $u=u(x_1,x_2,...,x_n,t)$ and independent variables $x_1,x_2,...,x_n$ and t:

$$\begin{aligned} F_1\Big(&u,\partial_t^\beta u,\partial_{x_1}^{\beta_1}u,...,\partial_{x_n}^{\beta_n}u,u_{tt},u_{x_1x_1},...,\\ &u_{x_nx_n},\partial_t^\beta\left(\partial_{x_1}^{\beta_1}u\right),...\Big)=0, \end{aligned} \tag{14}$$

where $0<\beta,\beta_1,\beta_2,...,\beta_n\leq 1$. Further, $\partial_v^\gamma u=\frac{\partial^\gamma}{\partial v^\gamma}u$ is a generic term for the beta partial derivative of the dependent variable u with respect to the independent variable v of order $\gamma\in(0,1]$. Finally, the terms of the form u_{tt} and $u_{x_ix_j}$ represent classical integer-order partial derivatives. We also assume that the function F_1 in equation (14) is a polynomial of u and its various partial derivatives.

Then, applying the following fractional complex traveling wave transformation in ξ to (14)

$$\begin{aligned} u(x_1,x_2,...,x_n,t)&=U(\xi),\\ \xi&=\frac{k_1\left(x_1+\frac{1}{\Gamma(\beta_1)}\right)^{\beta_1}}{\beta_1}+\frac{k_2\left(x_2+\frac{1}{\Gamma(\beta_2)}\right)^{\beta_2}}{\beta_2}\\ &+...+\frac{k_n\left(x_n+\frac{1}{\Gamma(\beta_n)}\right)^{\beta_n}}{\beta_n}+\frac{c\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}}{\beta}, \end{aligned} \tag{15}$$

where $k_1,k_2,...,k_n,c$ are nonzero constants which will be found at a later step, and then integrating the resulting equation with respect to ξ as many times as possible, we obtain an ODE in $U=U(\xi)$ as

$$F_2(U,U',U'',U''',...)=0, \tag{16}$$

where F_2 is a polynomial function of U and its various integer-order derivatives and the prime notation ($'$) denotes the ordinary derivative with respect to ξ.

The next major steps of the sine-Gordon expansion method, [35], [68], [74], [75], are as follows.

Step 1: Suppose that the trial solution of equation (16) is of the form

$$U(\xi) = A_0 + \sum_{i=1}^{N} \tanh^{i-1}(\xi)\left(A_i \tanh(\xi) + B_i \operatorname{sech}(\xi)\right), \tag{17}$$

where the coefficients $A_0,\ A_i,\ B_i\ (i = 1,\ 2,\ ...,\ N)$ will be determined later. Then equation (17) can be rewritten in terms of the sine and cosine functions via equations (12) and (13) as:

$$U(\omega(\xi)) = A_0 + \sum_{i=1}^{N} \cos^{i-1}(\omega(\xi))\left(A_i \cos(\omega(\xi)) + B_i \sin(\omega(\xi))\right). \tag{18}$$

Step 2: The positive integer N in equation (17) (or equation (18)) can be found using the homogeneous balance principle, in other words, by substituting equation (17) (or equation (18)) into equation (16) and balancing the nonlinear terms and the highest-order derivatives appearing in the resulting equation. More precisely, if the degree of $U(\xi)$ is $\text{Deg}[U(\xi)] = N$, then the degree of some specific terms can be calculated using the following formulas, [13]:

$$\text{Deg}\left[\frac{d^qU(\xi)}{d\xi^q}\right] = N + q,$$
$$\text{Deg}\left[(U(\xi))^p\left(\frac{d^qU(\xi)}{d\xi^q}\right)^s\right] = Np + s(N + q). \tag{19}$$

Step 3: Substituting solution form (18) with the known value N obtained from Step 2 into the ODE (16) and setting the summation of coefficients of $\sin^r(\omega(\xi))\cos^s(\omega(\xi))$ with the same power to zero, we obtain a system of nonlinear algebraic equations for the unknowns $A_0,\ A_i,\ B_i\ (i = 1, 2, ..., N)$, $k_j\ (j = 1, 2, ..., n)$ and c. With the help of symbolic software packages such as Maple, it is assumed that the resulting algebraic system can be solved for the unknown constants.

Step 4: Substituting the unknown values obtained from Step 3 and the wave transformation ξ in equation (15) into solution (17), we finally derive the exact traveling wave solutions of equation (14).

In the following section, we justify the performance of the method by applying it to the problems (1) and (2).

4 Applications of the Method and Graphical Simulations

In this section, we use the sine-Gordon expansion method to obtain exact traveling wave solutions of the PDEs with beta partial derivatives in equations (1) and (2).

4.1 Exact solutions for the $(2 + 1)$-dimensional breaking soliton equation with beta space-time derivatives

We first convert equation (1) to an ODE using the chain rule (4) and the following transformation

$$u(x, y, t) = U(\xi),$$
$$\xi = \left(\frac{k_1\left(x + \frac{1}{\Gamma(\beta)}\right)^\beta}{\beta} + \frac{k_2\left(y + \frac{1}{\Gamma(\beta)}\right)^\beta}{\beta} - \frac{c\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha}{\alpha}\right), \tag{20}$$

where $k_1,\ k_2$ are c are nonzero constants which will be determined at a later step. The resulting ODE in the variable $U = U(\xi)$ is then

$$k_1^4k_2U^{(4)} - 6k_1^3k_2U'U'' - ck_1^2U'' = 0, \tag{21}$$

where the prime notation $(')$ denotes the ordinary derivative with respect to ξ and $U^{(4)}$ denotes the fourth derivative. Integrating equation (21) with respect to ξ and letting a constant of integration be zero, we eventually get the following ODE

$$k_1^4k_2U''' - 3k_1^3k_2(U')^2 - ck_1^2U' = 0. \tag{22}$$

Utilizing the solution form (18) and balancing the highest-order derivative U''' with the nonlinear term of the highest-order $(U')^2$ via the formulas in equation (19), we get $N = 1$. Then, using equation (17), we can write the solution of equation (22) as

$$U(\xi) = A_0 + A_1 \tanh(\xi) + B_1 \operatorname{sech}(\xi), \tag{23}$$

or equivalently from equation (18), we obtain

$$U(\omega(\xi)) = A_0 + A_1 \cos(\omega(\xi)) + B_1 \sin(\omega(\xi)), \tag{24}$$

where $A_0,\ A_1,\ B_1$ are unknown constants with $A_1^2 + B_1^2 \neq 0$ which will be determined at a later step. Then, by differentiating equation (24), we can obtain the following expressions for U' and U'''.

$$\begin{aligned} U'(\omega(\xi)) &= -A_1 \sin^2(w(\xi)) \\ &\quad + B_1 \sin(w(\xi))\cos(w(\xi)), \\ U'''(\omega(\xi)) &= -4A_1 \sin^2(w(\xi))\cos^2(w(\xi)) \\ &\quad + 2A_1 \sin^4(w(\xi)) \\ &\quad + B_1 \sin(w(\xi))\cos^3(w(\xi)) \\ &\quad - 5B_1 \sin^3(w(\xi))\cos(w(\xi)). \end{aligned} \tag{25}$$

Inserting equation (25) into equation (22), we obtain

$$\begin{aligned}
&3k_1^3k_2A_1^2\sin^2(w(\xi))\cos^2(w(\xi))\\
&-6k_1^4k_2A_1\sin^2(w(\xi))\cos^2(w(\xi))\\
&-3k_1^3k_2B_1^2\sin^2(w(\xi))\cos^2(w(\xi))\\
&-6k_1^3k_2A_1B_1\sin(w(\xi))\cos^3(w(\xi))\\
&+6k_1^4k_2B_1\sin(w(\xi))\cos^3(w(\xi))\\
&-3k_1^3k_2A_1^2\sin^2(w(\xi))+2k_1^4k_2A_1\sin^2(w(\xi))\\
&+6k_1^3k_2A_1B_1\sin(w(\xi))\cos(w(\xi))\\
&-5k_1^4k_2B_1\sin(w(\xi))\cos(w(\xi))\\
&+ck_1^2A_1\sin^2(w(\xi))\\
&-ck_1^2B_1\sin(w(\xi))\cos(w(\xi))=0.
\end{aligned} \tag{26}$$

Next, setting the summation of the coefficients of each power of $\sin^r(\omega(\xi))\cos^s(\omega(\xi))$ to zero, we obtain the following set of nonlinear algebraic equations:

$$\begin{aligned}
\sin(\omega(\xi))\cos(\omega(\xi)) :\ & 6k_1^3k_2A_1B_1-5k_1^4k_2B_1\\
&-ck_1^2B_1=0,\\
\sin(\omega(\xi))\cos^3(\omega(\xi)) :\ & -6k_1^3k_2A_1B_1\\
&+6k_1^4k_2B_1=0,\\
\sin^2(\omega(\xi)) :\ & -3k_1^3k_2A_1^2+2k_1^4k_2A_1\\
&+ck_1^2A_1=0,\\
\sin^2(\omega(\xi))\cos^2(\omega(\xi)) :\ & 3k_1^3k_2A_1^2-6k_1^4k_2A_1\\
&-3k_1^3k_2B_1^2=0.
\end{aligned}$$

By solving the algebraic set of equations with the aid of symbolic computation software such as Maple 17, we get the following two cases of the unknown constants $c, k_1, k_2, A_0, A_1, B_1$ listed in Table 1, where c, k_2, A_0 are arbitrary nonzero constants such that $ck_2 > 0$.

Table 1. Values of the unknown coefficients and parameters for the solutions of equation (1)

Case 1	Case 2
$k_1=\pm\frac{1}{2}\sqrt{\frac{c}{k_2}}, k_2\neq 0$	$k_1=\pm\sqrt{\frac{c}{k_2}}$
$A_1=2k_1$	$A_1=k_1$
$B_1=0$	$B_1=\pm k_1 i$, where $i=\sqrt{-1}$

We therefore get two independent solutions.

Case 1: After substituting the values of the unknown constants in Case 1 of Table 1 and the transformation (20) into equation (23), we obtain one exact traveling wave solution of equation (1) as

$$u_1(x,y,t)=A_0+2k_1\tanh(\xi), \tag{27}$$

where $\xi=k_1\left(\frac{\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}+\frac{k_2\left(y+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha}{\alpha}\right)$.

Case 2: After substituting the values of the unknown constants in Case 2 of Table 1 and the transformation (20) into equation (23), we obtain a second exact traveling wave solution of equation (1) as

$$u_2(x,y,t)=A_0+k_1\tanh(\xi)\pm ik_1\,\mathrm{sech}(\xi), \tag{28}$$

where $\xi=k_1\left(\frac{\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}+\frac{k_2\left(y+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta}-\frac{c\left(t+\frac{1}{\Gamma(\alpha)}\right)^\alpha}{\alpha}\right)$.

Next, with the help of the Maple program, we will plot graphs of the exact solutions of (27) and (28) in which both real-valued and complex-valued solutions consisting of the hyperbolic functions are constructed via the SGEM. In the graphs, we will show how the solutions change as the temporal and spatial fractional- orders α and β of equation (1) are varied. In particular, $\alpha=\beta=1$, $\alpha=\beta=0.8$ and $\alpha=\beta=0.4$ are used for the plots. The solutions (27) and (28) are shown as 3D, 2D, and contour plots according to the values of α and β. All of the 3D solution graphs are plotted on the domain $\{(x,y,t) \mid 0\le x\le 10,\ \ y=1 \text{ and } 0\le t\le 10\}$. All 2D solution graphs, showing relationships between $u(x)$ and x, are plotted on $\{0\le x\le 10,\ \ y=1 \text{ and } t=7\}$. The variables y and t are fixed for all 2D plots since a variation of the fractional-order β of the spatial beta-derivative for x is being studied. In the contour plots, the contours of $u(x,t)$ for $y=1$ are plotted on the $x-t$ plane. The physical meanings of all plotted solutions will also be described. All of the graphs mentioned in this section are shown in the Appendix.

In Fig. 1, graphs of the exact solution $u_1(x,y,t)$ in equation (27) are plotted on the specified domains using the parameter values $c=k_2=1$, $A_0=1$. In particular, Fig. 1 (a)-(c), Fig. 1 (d)-(f) and Fig. 1 (g)-(i) show the 3D, 2D and contour plots for the exact solution (27) evaluated at $\alpha=\beta=1$, $\alpha=\beta=0.8$ and $\alpha=\beta=0.4$, respectively. As can be seen from the 3D plots of Fig. 1, the solution (27) can be characterized as a kink-type solution.

Fig. 2 and Fig. 3 show the plots of the real and imaginary parts of the solution $u_2(x,y,t)$ in equation (28), respectively, when plotted on the domains defined above using the parameter values $c=k_2=1$, $A_0=1$. In particular, Fig. 2 (a)-(c), Fig. 2 (d)-(f) and Fig. 2 (g)-(i) show the 3D, 2D and contour plots for the real part of equation (28), i.e., $\mathrm{Re}(u_2(x,y,t))$ computed at $\alpha=\beta=1$, $\alpha=\beta=0.8$ and $\alpha=\beta=0.4$, respectively. We can see from the 3D plots of Fig. 2 that $\mathrm{Re}(u_2(x,y,t))$ represents a kink-type solution. In a similar manner, the 3D, 2D and contour plots for the imaginary part of equation (28), i.e.,

$\mathrm{Im}(u_2(x,y,t))$ evaluated at $\alpha=\beta=1$, $\alpha=\beta=0.8$ and $\alpha=\beta=0.4$ are plotted in Fig. 3 (a)-(c), Fig. 3 (d)-(f) and Fig. 3 (g)-(i), respectively. By classifying the shape of the 3D graphs in Fig. 3, the imaginary solution $\mathrm{Im}(u_2(x,y,t))$ can be identified as a bell-shaped solitary wave solution. Finally, Fig. 4 (a)-(c), Fig. 4 (d)-(f) and Fig. 4 (g)-(i) show the 3D surface plots for $y=1$, the 2D curves for $y=1$, $t=7$, and the contour plots when $y=1$ for the magnitudes $|u_2(x,y,t)|$ for the fractional orders $\alpha=\beta=1$, $\alpha=\beta=0.8$ and $\alpha=\beta=0.4$, respectively.

To conclude this section, we summarize results reported by previous authors for the $(2+1)$-dimensional breaking soliton equation. In [76], the authors used the generalized exponential rational function (GERF) method to solve the integer-order $(2+1)$-dimensional breaking soliton equation. They obtained exact solutions including hyperbolic, trigonometric, exponential and rational functions and identified the dynamics of some solutions as a singular kink wave and a soliton. In [77], the author solved the classical $(2+1)$-dimensional breaking soliton equation to obtain exact non-traveling wave solutions using the idea of merging the generalized variable separation method with the extended homoclinic test approach. By combining the trigonometric, exponential, hyperbolic functions and three arbitrary functions, Shang identified non-traveling wave solutions as a periodic solitary wave, a cross soliton-like wave, a periodic cross-kink wave and a period two solitary wave. In [78], the authors obtained analytical solutions for the space-time conformable fractional $(2+1)$-dimensional breaking soliton equation using the simplified $\tan(\frac{\phi(\xi)}{2})$-expansion method. They obtained traveling wave solutions in terms of trigonometric, exponential and rational functions with arbitrary parameters. Inserting specific values for the parameters, they identified the physical behaviors of the solutions as periodic and anti-kink wave solutions. They also investigated how the graphical behaviors of the solutions changed as the fractional-orders of the equation were varied.

We have found it quite difficult to directly compare our obtained solutions with those described in the abovementioned references except by comparing them in terms of their mathematical expressions and graphical structures. Using the SGEM, we have obtained results for equation (1), including the hyperbolic tangent and hyperbolic secant functions with the special transformation ξ in equation (20), which are novel.

4.2 Exact solutions for the generalized Hirota-Satsuma coupled KdV system with beta time derivative

Using the traveling wave transformation,

$$u(x,t)=U(\xi),\ v(x,t)=V(\xi),\ w(x,t)=W(\xi),$$
$$\xi=k\left(x-\frac{c\left(t+\frac{1}{\Gamma(\eta)}\right)^{\eta}}{\eta}\right), \tag{29}$$

where k and c are nonzero constants, we can reduce the generalized Hirota-Satsuma coupled KdV system with beta time derivative in (2) to the following system of nonlinear ODEs:

$$-ckU'=\frac{1}{4}k^3U'''+3kUU'+3k(-V^2+W)', \tag{30}$$

$$-ckV'=-\frac{1}{2}k^3V'''-3kUV', \tag{31}$$

$$-ckW'=-\frac{1}{2}k^3W'''-3kUW', \tag{32}$$

where the prime notation $(')$ represents the ordinary derivative with respect to ξ.

Let, [12], [79], [80],

$$U=\alpha V^2+\beta V+\gamma,\ \ W=AV+B, \tag{33}$$

where $\alpha,\ \beta,\ \gamma,\ A$ and B are real constants to be determined later. Substituting equation (33) into equations (31) and (32), and then integrating once, we find that equations (31) and (32) can be converted to the following equation:

$$k^2V''=-2\alpha V^3-3\beta V^2+2(c-3\gamma)V+c_1, \tag{34}$$

where c_1 is a constant of integration. Multiplying equation (34) by V' and then integrating the resulting equation with respect to ξ, we obtain

$$\begin{aligned}k^2(V')^2=&-\alpha V^4-2\beta V^3+2(c-3\gamma)V^2\\&+2c_1V+c_2,\end{aligned} \tag{35}$$

where c_2 is another constant of integration.

Then, differentiating equation (33) with respect to ξ and using equations (34) and (35), we get

$$\begin{aligned}k^2U''&=2\alpha k^2(V')^2+k^2(2\alpha V+\beta)V'',\\&=2\alpha[-\alpha V^4-2\beta V^3+2(c-3\gamma)V^2\\&\quad+2c_1V+c_2]\\&\quad+(2\alpha V+\beta)[-2\alpha V^3-3\beta V^2\\&\quad+2(c-3\gamma)V+c_1].\end{aligned} \tag{36}$$

Integrating equation (30) with respect to ξ once, we obtain

$$\frac{1}{4}k^2U''+\frac{3}{2}U^2+cU+3(-V^2+W)+c_3=0, \tag{37}$$

where c_3 is a constant of integration. Substituting equations (33) and (36) into equation (37), we find that the coefficients of the resulting polynomial must be zero as shown below:

$$\begin{aligned} 3\alpha c - 3\alpha\gamma + \frac{3}{4}\beta^2 - 3 = 0, \\ \frac{1}{2}\left(\alpha c_1 + \beta c + \gamma\beta\right) + A = 0, \\ \frac{1}{4}\left(2\alpha c_2 + \beta c_1\right) + \frac{3}{2}\gamma^2 + c\gamma + 3B + c_3 = 0. \end{aligned} \tag{38}$$

Let

$$\begin{aligned} c_1 &= \frac{1}{2\alpha^2}(\beta^3 + 2c\alpha\beta - 6\alpha\beta\gamma), \\ V(\xi) &= aP(\xi) - \frac{\beta}{2\alpha}, \end{aligned} \tag{39}$$

where $a \neq 0$ is a constant. From (38) and after some algebraic manipulations, we have

$$\begin{aligned} \alpha &= \frac{\beta^2 - 4}{4(\gamma - c)}, \;\; A = \frac{4\beta(c-\gamma)}{\beta^2 - 4}, \; c \neq \gamma, \; \beta \neq \pm 2, \\ B &= \frac{1}{6\left(c-\gamma\right)\left(\beta^2-4\right)^2}\Big(16\, c_3 c\, \beta^2 \\ &\quad - 2\, c_3 c\, \beta^4 - 16\, c_3\gamma\, \beta^2 + 2\, c_3\gamma\, \beta^4 \\ &\quad + 56\, \gamma c^2\, \beta^2 - 48\, c\gamma^2\, \beta^2 - 16\, c_2 + \frac{1}{4}c_2\beta^6 \\ &\quad - 3\, c_2\beta^4 + 12\, c_2\beta^2 - 16\, c\gamma^2 \\ &\quad - 32\, \gamma c^2 - 8\, \beta^2 c^3 + \gamma^3\beta^4 - 2\, c^3\beta^4 \\ &\quad + 32\, c_3\gamma - 32\, c_3 c + 48\, \gamma^3 + c\gamma^2\beta^4\Big). \end{aligned} \tag{40}$$

Then, from equation (34), we have

$$k^2 P'' - \left(2c - 6\gamma + \frac{3\beta^2}{2\alpha}\right)P + 2\alpha a^2 P^3 = 0. \tag{41}$$

Applying the homogeneous balance principle and the formulas in equation (19) to the terms P'' and P^3 of the above equation, we obtain

$$\text{Deg}\left[P''\right] = N + 2 = \text{Deg}\left[P^3\right] = 3N, \tag{42}$$

which leads to $N = 1$. Therefore, the solution form of the ODE (41) using equation (17) is

$$P(\xi) = A_0 + A_1\tanh(\xi) + B_1\,\text{sech}(\xi), \tag{43}$$

or equivalently, by using equation (18), the solution form is

$$P(\omega(\xi)) = A_0 + A_1\cos(\omega(\xi)) + B_1\sin(\omega(\xi)), \tag{44}$$

where the constant coefficients A_0, A_1 and B_1 are determined at a later step, provided that $A_1^2 + B_1^2 \neq 0$.

After performing the appropriate algebraic manipulations for equation (44), we have

$$\begin{aligned} P^3(\omega(\xi)) &= -2A_1\sin^2(\omega(\xi))\cos(\omega(\xi)) \\ &\quad + B_1\sin(\omega(\xi))\cos^2(\omega(\xi)) \\ &\quad - B_1\sin^3(\omega(\xi)), \\ P''(\omega(\xi)) &= A_0^3 + 3A_0^2A_1\cos(\omega(\xi)) \\ &\quad + 3A_0^2B_1\sin(\omega(\xi)) \\ &\quad + 3A_0A_1^2\cos^2(\omega(\xi)) \\ &\quad + A_1^3\cos^3(\omega(\xi)) \\ &\quad + 6A_0A_1B_1\cos(\omega(\xi))\sin(\omega(\xi)) \\ &\quad + 3A_0B_1^2\sin^2(\omega(\xi)) \\ &\quad + 3A_1^2B_1\cos^2(\omega(\xi))\sin(\omega(\xi)) \\ &\quad + 3A_1B_1^2\cos(\omega(\xi))\sin^2(\omega(\xi)) \\ &\quad + B_1^3\sin^3(\omega(\xi)). \end{aligned} \tag{45}$$

Then, substituting equations (44) and (45) into (41), we obtain

$$\begin{aligned} &2k^2B_1\sin(\omega(\xi))\cos^2(\omega(\xi)) + 2B_1^3\alpha a^2\sin(\omega(\xi)) \\ &+ 6a^2\alpha A_0B_1^2 + 2\alpha a^2A_1^3\cos^3(\omega(\xi)) \\ &- \frac{3\beta^2B_1\sin(\omega(\xi))}{2\alpha} - \frac{3\beta^2A_1\cos(\omega(\xi))}{2\alpha} \\ &+ 6\alpha a^2{A_1}^2B_1\sin(\omega(\xi))\cos^2(\omega(\xi)) \\ &- \frac{3\beta^2A_0}{2\alpha} + 12\alpha a^2A_0A_1B_1\sin(\omega(\xi))\cos(\omega(\xi)) \\ &- 2cA_0 + 6\gamma A_0 + 6\alpha a^2A_0A_1^2\cos^2(\omega(\xi)) \\ &+ 6\alpha a^2A_0^2B_1\sin(\omega(\xi)) + 6\alpha a^2A_0^2A_1\cos(\omega(\xi)) \\ &- 6\alpha a^2A_1B_1^2\cos^3(\omega(\xi)) \\ &- 6\alpha a^2A_0B_1^2\cos^2(\omega(\xi)) \\ &- 2\alpha a^2B_1^3\sin(\omega(\xi))\cos^2(\omega(\xi)) \\ &+ 6\alpha a^2A_1B_1^2\cos(\omega(\xi)) \\ &- 2cA_1\cos(\omega(\xi)) + 6\gamma B_1\sin(\omega(\xi)) \\ &+ 6\gamma A_1\cos(\omega(\xi)) - 2k^2A_1\cos(\omega(\xi)) + 2\alpha a^2A_0^3 \\ &- k^2B_1\sin(\omega(\xi)) + 2k^2A_1\cos^3(\omega(\xi)) \\ &- 2cB_1\sin(\omega(\xi)) = 0. \end{aligned} \tag{46}$$

The following algebraic system of equations can then be derived by summing up the coefficients of the trigonometric functions with the same power and equating each sum to zero:

$$\begin{aligned}
\text{constant}: & -2cA_0+2\alpha a^2A_0^3+6\gamma A_0 \\
& -\frac{3\beta^2A_0}{2\alpha}+6a^2\alpha A_0B_1^2=0, \\
\sin(\omega(\xi)): & \, 6\gamma B_1-k^2B_1-2cB_1 \\
& +6\alpha a^2A_0^2B_1 \\
& -\frac{3\beta^2B_1}{2\alpha}+2B_1^3\alpha a^2=0, \\
\cos(\omega(\xi)): & \, 6\alpha a^2A_1B_1^2+6\alpha a^2A_0^2A_1 \\
& +6\gamma A_1-2cA_1-2k^2A_1 \\
& -\frac{3\beta^2A_1}{2\alpha}=0, \\
\sin(\omega(\xi))\cos(\omega(\xi)): & \, 12\alpha a^2A_0A_1B_1=0, \\
\cos^2(\omega(\xi)): & \, 6a^2\alpha A_0A_1^2-6a^2\alpha A_0B_1^2=0, \\
\cos^3(\omega(\xi)): & \, 2\alpha a^2A_1^3-6\alpha a^2A_1B_1^2 \\
& +2k^2A_1=0, \\
\sin(\omega(\xi))\cos^2(\omega(\xi)): & \, 6\alpha\, a^2A_1^2B_1 \\
& -2\alpha a^2B_1^3+2k^2B_1=0.
\end{aligned}$$

Solving the above system with the help of Maple, we obtain the three cases of the unknown coefficients shown in Table 2 in which c, a, β and γ are arbitrary nonzero constants and $\alpha=\frac{\beta^2-4}{4(\gamma-c)}\neq 0$ with $\gamma\neq c$.

Case 1: Substituting the values of the unknown coefficients and parameters in Case 1 of Table 2 and the transformation (29) into equations (43), (39) and (33), respectively, we obtain the following exact solution of equation (2):

$$\begin{aligned}
v_1(x,t)&=\pm\frac{k}{\sqrt{\alpha}}\operatorname{sech}(\xi)-\frac{\beta}{2\alpha}, \\
u_1(x,t)&=\alpha\left(v_1(x,t)\right)^2+\beta v_1(x,t)+\gamma, \\
w_1(x,t)&=Av_1(x,t)+B,
\end{aligned} \tag{47}$$

where $\xi=k\left(x-\frac{c\left(t+\frac{1}{\Gamma(\eta)}\right)^\eta}{\eta}\right)$.

Case 2: Substituting the values of the unknown coefficients and parameters in Case 2 of Table 2 and the transformation (29) into equations (43), (39) and (33), respectively, we get the following exact traveling wave solution of equation (2):

$$\begin{aligned}
v_2(x,t)&=\pm\frac{k}{\sqrt{-\alpha}}\tanh(\xi)-\frac{\beta}{2\alpha}, \\
u_2(x,t)&=\alpha\left(v_2(x,t)\right)^2+\beta v_2(x,t)+\gamma, \\
w_2(x,t)&=Av_2(x,t)+B,
\end{aligned} \tag{48}$$

where $\xi=k\left(x-\frac{c\left(t+\frac{1}{\Gamma(\eta)}\right)^\eta}{\eta}\right)$.

Case 3: Substituting the values of the unknown coefficients and parameters in Case 3 of Table 2 and the transformation (29) into equations (43), (39) and (33), respectively, we obtain the following exact solution of equation (2):

$$\begin{aligned}
v_3(x,t)&=\pm\frac{k}{2\sqrt{-\alpha}}\tanh(\xi)\pm\frac{k}{2\sqrt{\alpha}}\operatorname{sech}(\xi)-\frac{\beta}{2\alpha}, \\
u_3(x,t)&=\alpha\left(v_3(x,t)\right)^2+\beta v_3(x,t)+\gamma, \\
w_3(x,t)&=Av_3(x,t)+B,
\end{aligned} \tag{49}$$

where $\xi=k\left(x-\frac{c\left(t+\frac{1}{\Gamma(\eta)}\right)^\eta}{\eta}\right)$.

Next, with the aid of the Maple software, we will plot graphs of the case 1 solution $v_1(x,t)$ in equation (47) and the case 3 solution $v_3(x,t)$ in equation (49) of system (2). As shown in the Appendix, these graphs will be plotted in Figs. 5-7 for a range of values of the temporal fractional-order η of equation (2), namely $\eta=1$, $\eta=0.8$ and $\eta=0.4$. The solutions $v_1(x,t)$ in equation (47) and $v_3(x,t)$ in equation (49) are plotted as 3D, 2D, and contour plots according to the values of η. All 3D graphs are plotted on the domain $\{(x,t)\mid 0\leq x\leq 10 \text{ and } 0\leq t\leq 10\}$. All 2D graphs which show the relationships between $v(t)$ and t, are plotted on $\{0\leq t\leq 10 \text{ and } x=5\}$. The variable t is varied but $x=5$ is fixed for all of the 2D plots because the effects of changing η in equation (2) on the solutions will be considered. The contour plots show the contours of $v_1(x,t)$ and $v_3(x,t)$ on the $x-t$ plane. As for the breaking soliton equation solutions, the physical meanings of the selected solutions will be discussed.

In Fig. 5, the graphs of the exact solution $v_1(x,t)$ in equation (47) are plotted on the domains described in the previous paragraph using the parameter values $c=1$, $\beta=3$, $\gamma=2$, $a=1$ and $\alpha=\frac{5}{4}$ in equation (40). In particular, the 3D, 2D and contour graphs for the exact solution (47) are plotted for $\eta=1$, $\eta=0.8$ and $\eta=0.4$ in Fig. 5 (a)-(c), Fig. 5 (d)-(f) and Fig. 5 (g)-(i), respectively. From the 3D plots of Fig. 5, the solution of equation (47) can be identified as a bell-shaped solitary wave.

In Fig. 6 and Fig. 7, the graphs of the solutions of the real and imaginary parts of the solution $v_3(x,t)$ in equation (49), respectively, are plotted. The plots are for the domains specified above for parameter values $c=1$, $\beta=1$, $\gamma=2$, $a=1$ and $\alpha=-\frac{3}{4}$ in equation (40). In particular, Fig. 6 (a)-(c), Fig. 6 (d)-(f) and Fig. 6 (g)-(i) show the 3D, 2D and contour plots for the real part of equation (49), i.e., Re$(v_3(x,t))$ evaluated at $\eta=1$, $\eta=0.8$ and $\eta=0.4$, respectively. From the 3D plots of Fig. 6 the solution of Re$(v_3(x,t))$ can be identified as a kink-type solution. The graphs in Fig. 7 (a)-(c), Fig. 7 (d)-(f) and

Table 2. Values of the unknown coefficients and parameters for the solutions of equation (2)

Case 1	Case 2	Case 3
$\Delta = 3\beta^2 + 4\alpha c - 12\alpha\gamma$	$\Delta = 3\beta^2 + 4\alpha c - 12\alpha\gamma$	$\Delta = 3\beta^2 + 4\alpha c - 12\alpha\gamma$
$k = \pm\sqrt{\frac{\Delta}{2\alpha}}$	$k = \pm\frac{1}{2}\sqrt{-\frac{\Delta}{\alpha}}$	$k = \pm\sqrt{-\frac{\Delta}{\alpha}}$
$A_0 = 0$	$A_0 = 0$	$A_0 = 0$
$A_1 = 0$	$A_1 = \pm\frac{k}{a\sqrt{-\alpha}}$	$A_1 = \pm\frac{k}{2a\sqrt{-\alpha}}$
$B_1 = \pm\frac{k}{a\sqrt{\alpha}}$	$B_1 = 0$	$B_1 = \pm\frac{k}{2a\sqrt{\alpha}}$

Fig. 7 (g)-(i) show the 3D, 2D and contour plots for the imaginary part of equation (49), i.e., $\mathrm{Im}(v_3(x,t))$ evaluated at $\eta = 1$, $\eta = 0.8$ and $\eta = 0.4$, respectively. From the 3D graphs in Fig. 7, the physical behavior of $\mathrm{Im}(v_3(x,t))$ can be identified as an anti-bell shaped soliton solution. Finally, Fig. 8 (a)-(c), Fig. 8 (d)-(f) and Fig. 8 (g)-(i) show the 3D, 2D and contour plots of the magnitude $|v_3(x,t)|$ for the fractional orders $\eta = 1$, $\eta = 0.8$ and $\eta = 0.4$, respectively.

To conclude this section, we summarize results reported by previous authors for the generalized Hirota-Satsuma coupled KdV system. In [81], the authors derived exact solutions of the generalized Hirota-Satsuma coupled KdV model with classical derivatives using the generalized Kudryashov method. Some new singular and kink soliton wave solutions, expressed in terms of the exponential function solutions, were obtained. They also plotted some graphs to show the interactions of two long waves with different dispersion relations. In [82], the authors used the auxiliary equation method to obtain exact traveling wave solutions for the generalized Hirota-Satsuma coupled KdV system with the commensurate-order beta time derivative for a range of fractional orders. All of the solutions they obtained were expressed in terms of exponential functions and they characterized the real parts as a kink-type solution. Also, in [83], the authors derived exact solutions for the generalized Hirota-Satsuma coupled KdV system for a commensurate-order conformable derivative using the sub-equation method. They obtained their solutions as hyperbolic tangent and cotangent functions, tangent and cotangent functions and rational functions by using the solutions of the Riccati equation. The effects of variation of the time-fractional order on the 3D solution graphs were also studied.

We again found that only mathematical expressions and graphical representations between our solutions and the solutions described in the mentioned references could be usefully compared. Utilizing the SGEM, we found exact solutions for equation (2) expressed in equations (47)-(49) which contained the hyperbolic tangent and hyperbolic secant functions. These solutions are new due to the special transformation for ξ in equation (29).

5 Discussions and Conclusions

In this paper, the sine-Gordon expansion method has been successfully used to derive exact traveling wave solutions for the $(2+1)$-dimensional breaking soliton equation with beta space-time derivatives in equation (1) and the generalized Hirota-Satsuma coupled KdV system with beta time derivative in equation (2). The exact explicit solutions of the proposed problems derived and plotted in Section 4, are novel and have not been reported in any previous literature. As seen from the results, the sine-Gordon method combined with Maple is a powerful method for obtaining real-valued and complex-valued solutions expressed in terms of sums of hyperbolic tangent and hyperbolic secant functions. In order to check the results, all of the solutions obtained were substituted back into the partial differential equations (1) and (2) using the chain rule of the beta-derivative and the Maple software package. Numerical simulations were carried out for a range of parameter values. From the 3D, 2D and contour plots of selected solutions and their absolute values, it was possible to identify physical behaviors of the solutions which included a kink-type solution, a bell-shaped solitary wave solution and an anti-bell shaped soliton solution. In physical science and engineering, the kink-shaped solitary wave localized traveling structure is maintained by different balances, for instance, nonlinearity is balanced by dissipation or nonlinearity is simultaneously balanced by dispersion and dissipation. Moreover, the bell-shaped solitary wave solution normally occurs as a result of the balance between nonlinearity and dispersion. These two main types of traveling solitary waves are of great significance for various applications of nonlinear physical phenomena described by NLPDEs such as the generalized Benjamin-Bona-Mahony-Burgers (BBMB) equation, [84], and the SIdV equation, [85]. In particular, the variation of the temporal fractional order α and the spatial fractional order β in equation (1) and the temporal fractional order η in equation (2) were found to slightly affect the amplitude and translation of the obtained exact solutions via the transformations (20) and (29). However, as shown in Figs. 1-8, the fractional orders did not change the type of solution structure when compared with those for the integer-order cases. The results shown in this research may be important in diverse fields of nonlinear sciences including physics, applied mathematics and engineering. In conclusion, the SGEM could be a powerful and reliable tool for studying physical applications of a wide range of nonlinear phenomena modeled by NLPDEs with emphasis on problems which can be solved by analytical methods or admit certain special kinds of explicit solutions. Because of the advantages of analytical solutions, many researchers have now developed a large number of different methods for finding these solutions. Finally, one interesting future development for research would be to apply the SGEM to equations (1) and (2) with a truncated M-fractional derivative which is a recent definition of a fractional-order derivative.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Thitthita Iatkliang, Supaporn Kaewta, Nguyen Minh Tuan: Conceptualization, data curation, investigation, methodology, software, visualization, writing-original draft and writing-review and editing. Sekson Sirisubtawee: Conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, supervision, validation, visualization, writing-original draft and writing-review and editing.

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

Thitthita Iatkliang, Supaporn Kaewta,
Nguyen Minh Tuan, Sekson Sirisubtawee

Appendix

Figures described in sections 4.1-4.2.

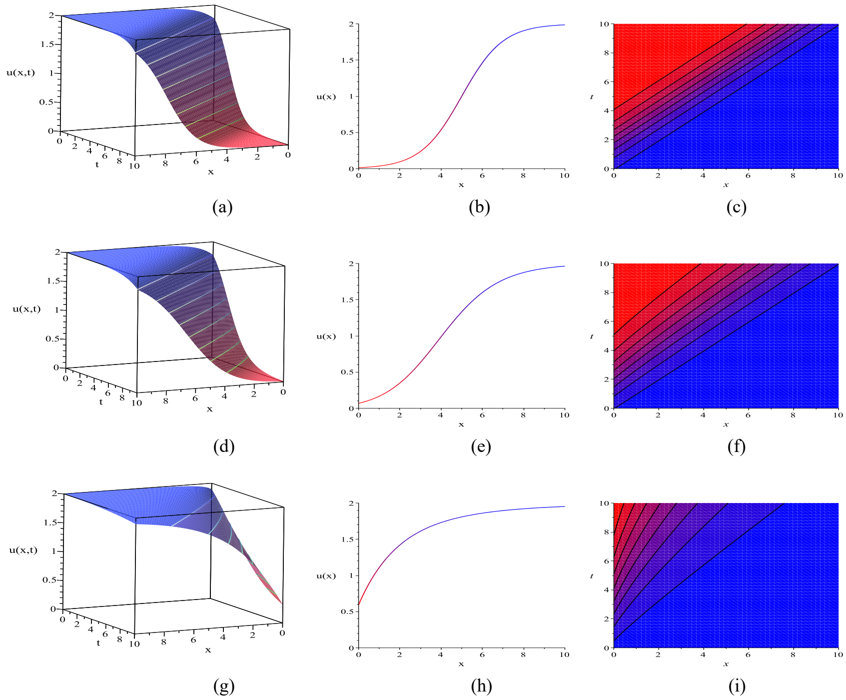

Fig. 1: Plots for solution $u_1(x,y,t)$ in equation (27) constructed using the SGEM: (a)-(c) when $\alpha=\beta=1$; (d)-(f) when $\alpha=\beta=0.8$; (g)-(i) when $\alpha=\beta=0.4$.

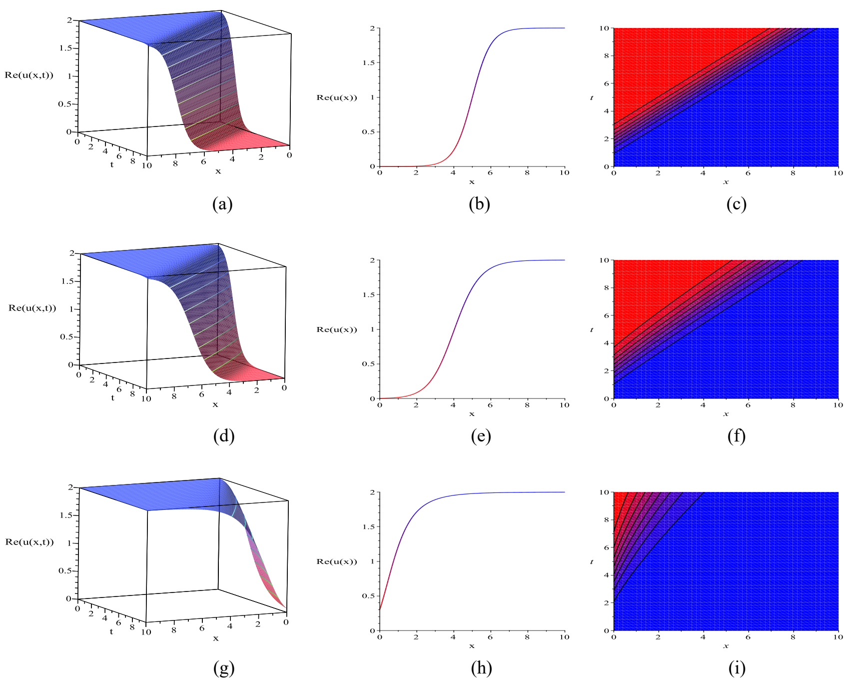

Fig. 2: Plots of real part Re$(u_2(x,y,t))$ of solution $u_2(x,y,t)$ in equation (28) constructed using the SGEM: (a)-(c) when $\alpha=\beta=1$; (d)-(f) when $\alpha=\beta=0.8$; (g)-(i) when $\alpha=\beta=0.4$.

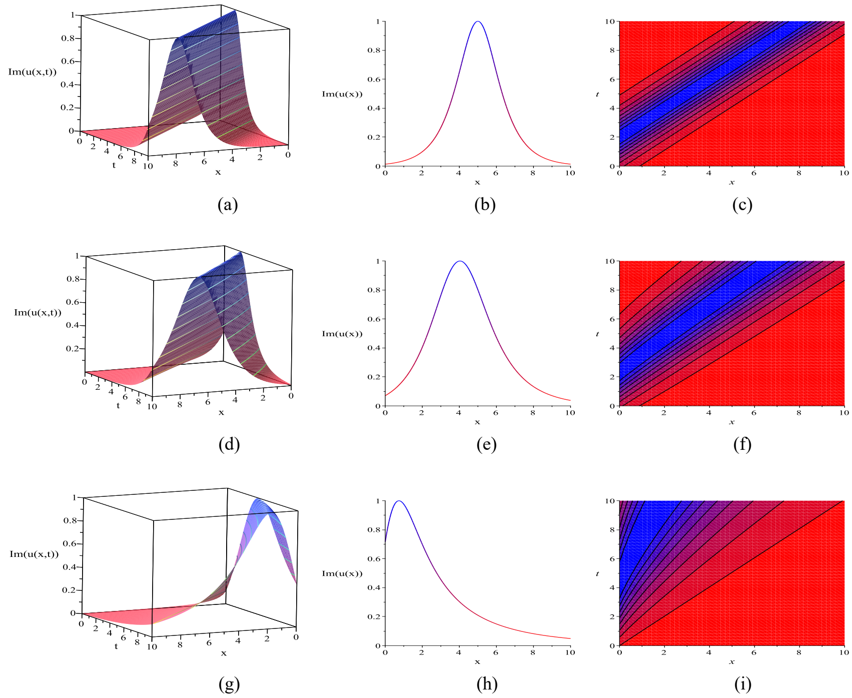

Fig. 3: Plots of imaginary part $\mathrm{Im}(u_2(x, y, t))$ of solution $u_2(x, y, t)$ in (28) constructed using the SGEM: (a)-(c) when $\alpha = \beta = 1$; (d)-(f) when $\alpha = \beta = 0.8$; (g)-(i) when $\alpha = \beta = 0.4$.

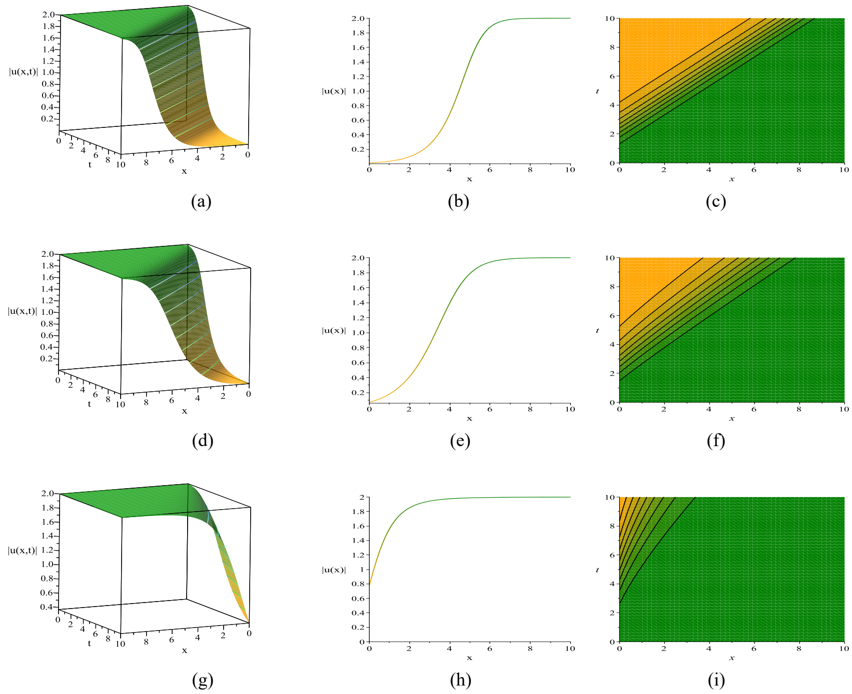

Fig. 4: Plots of magnitude $|u_2(x, y, t))|$ of solution $u_2(x, y, t)$ in equation (28) constructed using the SGEM: (a)-(c) when $\alpha = \beta = 1$; (d)-(f) when $\alpha = \beta = 0.8$; (g)-(i) when $\alpha = \beta = 0.4$.

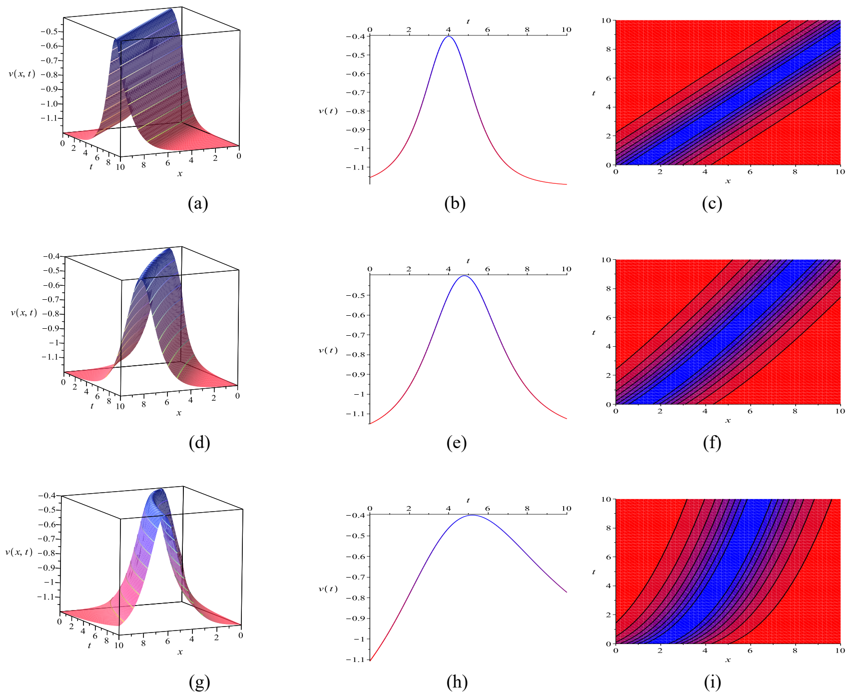

Fig. 5: Plots of solution $v_1(x,t)$ for equation (47) constructed using the SGEM: (a)-(c) when $\eta = 1$; (d)-(f) when $\eta = 0.8$; (g)-(i) when $\eta = 0.4$.

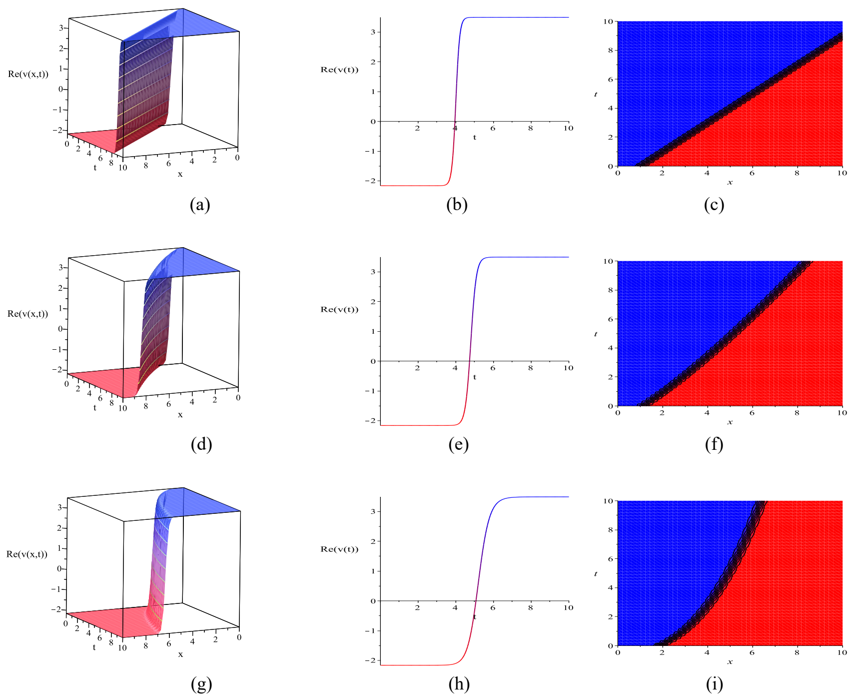

Fig. 6: Plots of real part $\mathrm{Re}(v_3(x,t))$ of solution $v_3(x,t)$ in equation (49) constructed using the SGEM: (a)-(c) when $\eta = 1$; (d)-(f) when $\eta = 0.8$; (g)-(i) when $\eta = 0.4$.

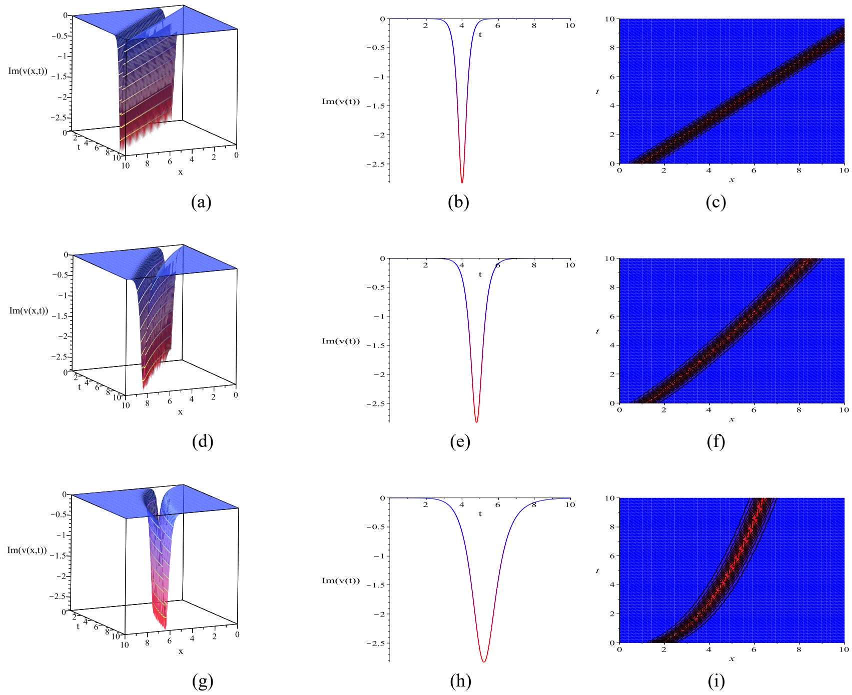

Fig. 7: Plots of imaginary part $\mathrm{Im}(v_3(x,t))$ of solution $v_3(x,t)$ in equation (49) constructed using the SGEM: (a)-(c) when $\eta = 1$; (d)-(f) when $\eta = 0.8$; (g)-(i) when $\eta = 0.4$.

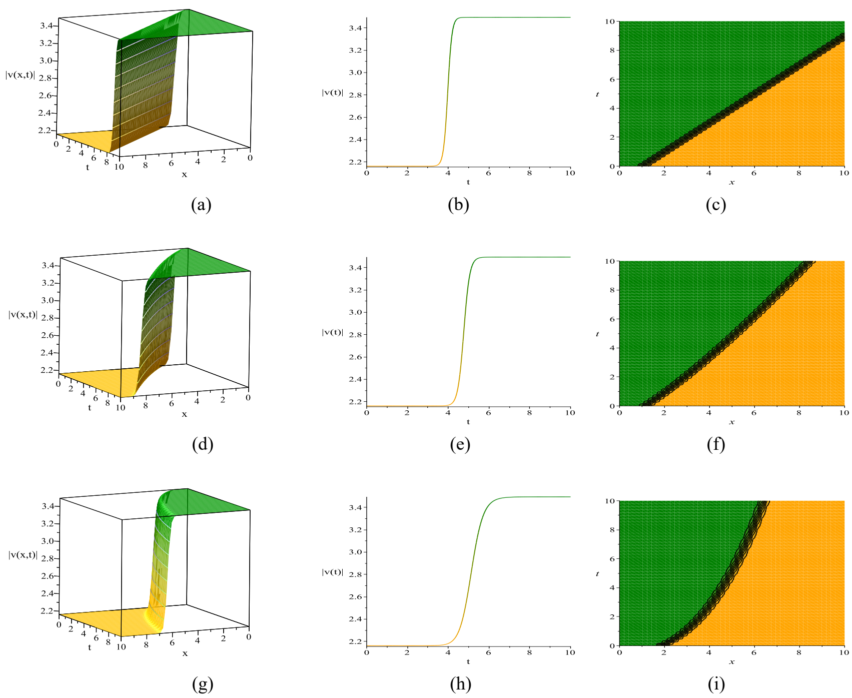

Fig. 8: Plots of magnitude $|v_3(x,t)|$ of solution $v_3(x,t)$ in equation (49) constructed using the SGEM: (a)-(c) when $\eta = 1$; (d)-(f) when $\eta = 0.8$; (g)-(i) when $\eta = 0.4$.