Optimal Statistical Estimation and Dynamic Adaptive Control of Airline Seat Protection Levels for Several Nested Fare Classes under Parametric Uncertainty of Customer Demand Models

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Abstract: - Assigning seats in the same compartment to different fare classes of passengers is a major problem of airline seat allocation. Airlines sell the same seat at different prices according to the time at which the reservation is made and other conditions. Thus the same seat can be sold at different prices. The problem is to find an optimal policy that maximizes total expected revenue. To solve the above problem, this paper presents the novel computational approach to optimization and dynamic adaptive prediction of airline seat protection levels for multiple nested fare classes of single-leg flights under parametric uncertainty. It is assumed that time T (before the flight is scheduled to depart) is divided into h periods, namely a full fare period and h-1 discounted fare periods. The fare structure is given. An airplane has a seat capacity of N. For the sake of simplicity, but without loss of generality, we consider (for illustration) the case of nonstop flight with two fare classes (business and economy). The proposed airline's inventory management policy is based on the use of the proposed computational models. These models emphasize pivotal quantities and ancillary statistics relevant for obtaining statistical predictive limits for anticipated quantities under parametric uncertainty and are applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space. The proposed technique is based on a probability transformation and pivotal quantity averaging. It is conceptually simple and easy to use. Finally, we give illustrative examples, where the proposed analytical methodology is illustrated in terms of the two-parameter exponential distribution. Applications to other log-location-scale distributions could follow directly.

Key-Words: - Airline seat protection levels, statistical optimization, dynamic adaptive control, pivotal quantities, ancillary statistics, unknown (nuisance) parameters, elimination

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1 Introduction

Basically, there have been two static models of airline seat reservation: nested and non-nested. In non-nested model, distinct numbers of seats called buckets are exclusively assigned to each fare class. The sum of these buckets adds up to the total airplane seat capacity. In nested model, each fare class is assigned a booking limit, which is the total number of seats assigned to that fare class (protection level) plus the sum of all seat allocations to its lower fare classes.

Earlier revenue management models considered non-nested seat allocations. However, a major difficulty with non-nested seat allocation is that if the limit for a fare class is reached, a booking request to that class is denied, while a lower fare bucket remains open. In a nested seat allocation, this booking denial does not happen as the inventories are shared among each fare class and its lower classes. The problem of constructing optimal airline seat protection levels for multiple nested fare classes of single-leg flights has been considered in numerous papers.

In [1], the author was the first to propose a solution method of the airline seat allocation problem for a single-leg flight with two fare classes. The idea of his scheme is to equate the marginal revenues in each of the two fare classes. He suggests closing down the low fare class when the certain revenue from selling low fare seat is exceeded by the expected revenue of selling the same seat at the higher fare. That is, low fare booking requests should be accepted as long as

$$c_2 \geq c_1 \Pr(Z_1 > n_1), \tag{1}$$

where c_1 and c_2 are the high and low fare levels respectively, Z_1 denotes the demand for the high fare (or business) class, n_1 is the number of seats to protect for the high fare class and $\Pr(Z_1>n_1)$ is the probability of selling more than n_1 protected seats to high fare class customers. It should be noted that an analytical proof of (1) is not given.

Now we describe how it can be determined protection levels for multiple nested fare classes of single-leg flight when we deal with l=2 nested fare classes. The performance index which can be used to determine the optimal allocation of seats between l=2 dependent (i.e., nested) fare classes, subject to the total airplane seat capacity constraint, is given as follows.

Maximize the total expected revenue for a single-leg flight with l=2 nested fare classes,

$$Q(n_1,n_2) = Q_2(n_2) + E_2\{Q_1(n_1+n_2-Z_2)\}$$
$$= c_2\left[n_2 - \int_0^{n_2} F_2(z_2)dz_2\right]$$
$$+c_1\int_0^{n_2}\left[n_1+n_2-z_2-\int_0^{n_1+n_2-z_2}F_1(z_1)dz_1\right]f_2(z_2)dz_2$$
$$+ c_1\int_{n_2}^{\infty}\left[n_1-\int_0^{n_1}F_1(z_1)dz_1\right]f_2(z_2)dz_2, \quad (2)$$

subject to

$$\sum_{j=1}^{2} n_j = N, \quad n_j \geq 0 \quad \text{for } j \in \{1, 2\}, \quad (3)$$

where

$$Q_j(n_j) = E_j\{c_j \min(n_j, Z_j)\}$$
$$= c_j\left[\int_0^{n_j} z_j f_j(z_j)dz_j + \int_{n_j}^{\infty} n_j f_j(z_j)dz_j\right]$$
$$= c_j\left[z_j F_j(z_j)\Big|_0^{n_j} - \int_0^{n_j} F_j(z_j)dz_j + n_j\left(1-F_j(n_j)\right)\right]$$
$$= c_j\left[n_j - \int_0^{n_j} F_j(z_j)dz_j\right] \quad (4)$$

represents the expected revenue from the jth fare class, c_j is the fare level for the jth fare class, n_j denotes the protection level for the jth fare class, Z_j denotes the customer demand for the jth fare class, $f_j(z_j)$ is the probability density function of Z_j.

Theorem 1. If the performance index is given by (2), (3), then the optimal protection levels have to satisfy the following system of equations:

$$n_1 = \arg\left(c_2 = c_1\overline{F}_1(n_1)\right), \quad n_2 = \max\left(0, N-n_1\right) \quad (5)$$

Proof. A simple application of the Lagrange multipliers technique leads to the optimal solution satisfying

$$\frac{\partial Q(n_1,n_2)}{\partial n_2} = \frac{\partial Q(n_1,n_2)}{\partial n_1}, \quad (6)$$

where

$$\frac{\partial Q(n_1,n_2)}{\partial n_2} = c_2 - c_2F_2(n_2) + c_1n_1f_2(n_2) + c_1\int_0^{n_2} f_2(z_2)dz_2$$
$$-c_1\left(\int_0^{n_1}F_1(z_1)dz_1\right)f_2(n_2) - c_1\int_0^{n_2}F_1(n_1+n_2-z_2)f_2(z_2)dz_2$$
$$-c_1n_1f_2(n_2) + c_1\left(\int_0^{n_1}F_1(z_1)dz_1\right)f_2(n_2), \quad (7)$$

$$\frac{\partial Q(n_1,n_2)}{\partial n_1} = c_1\int_0^{n_2} f_2(z_2)dz_2 - c_1\int_0^{n_2}F_1(n_1+n_2-z_2)f_2(z_2)dz_2$$
$$+ c_1\int_{n_2}^{\infty} f_2(z_2)dz_2 - c_1\int_{n_2}^{\infty}F_1(n_1)f_2(z_2)dz_2. \quad (8)$$

It follows from (6) that

$$c_2\left[1-F_2(n_2)\right] = c_1\overline{F}_2(n_2)\left[1-F_1(n_1)\right] \quad (9)$$

or

$$c_2 = c_1\overline{F}_1(n_1). \quad (10)$$

This ends the proof.

In [2], the authors showed that in the presence of l tariff classes under certain conditions of continuity, the conditions for optimal nested protection levels are reduced to the following set of probabilistic statements:

$$c_2 = c_1 \Pr(Z_1 > n_1),$$
$$c_3 = c_1 \Pr\left(Z_1 > n_1 \cap Z_1 + Z_2 > n_1 + n_2\right),$$
$$\vdots$$
$$c_l = c_1 \Pr\begin{pmatrix} X_1 > n_1 \cap Z_1 + Z_2 > n_1 \\ +n_2 \cap \ldots \cap \sum_{i=1}^{l-1} Z_i > \sum_{i=1}^{l-1} n_i \end{pmatrix}. \quad (11)$$

These statements have an intuitive interpretation, much like Littlewood's rule. To illustrate the method of [2], consider a single-leg flight with l=3 nested fare classes. In [2], the authors show that (for the case of l=3) the conditions for the optimal nested protection levels reduce to the following set of probability statements:

$$c_2 = c_1 \Pr(Z_1 > n_1), \quad (12)$$

$$c_3 = c_1 \Pr\left(Z_1 > n_1 \cap Z_1 + Z_2 > n_1 + n_2\right), \quad (13)$$

where (12) has to be transformed (in terms of probability distributions) to

$$c_2 = c_1\overline{F}(n_1), \quad (14)$$

(13) has to be transformed (in terms of probabilities) to

$$c_3 = c_1 \begin{bmatrix} \Pr(Z_1 > n_1 + n_2) \\ + \Pr\begin{pmatrix} n_1 < Z_1 \le n_1 + n_2 \\ \cap Z_2 > n_1 + n_2 - Z_1 \end{pmatrix} \end{bmatrix}, \quad (15)$$

(15) has to be transformed (in terms of probability distributions) to

$$c_3 = c_1 \begin{bmatrix} \overline{F}_1(n_1 + n_2) \\ + \int_{n_1}^{n_1+n_2} \overline{F}_2\left(\sum_{i=1}^{2} n_i - z_1\right) f_1(z_1) dz_1 \end{bmatrix}. \quad (16)$$

In other words, the method of [2], needs the system of equations (in terms of probabilities),

$$c_2 = c_1 \Pr(Z_1 > n_1),$$

$$c_3 = c_1 \begin{bmatrix} \Pr(Z_1 > n_1 + n_2) + \\ \Pr\begin{pmatrix} n_1 < Z_1 \le n_1 \\ + n_2 \cap Z_2 > n_1 + n_2 - Z_1 \end{pmatrix} \end{bmatrix}, \quad (17)$$

which has to be transformed to the system of equations (in terms of probability distributions),

$$c_2 = c_1 \overline{F}(n_1),$$

$$c_3 = c_1 \begin{bmatrix} \overline{F}_1(n_1 + n_2) \\ + \int_{n_1}^{n_1+n_2} \overline{F}_2\left(\sum_{i=1}^{2} n_i - z_1\right) f_1(z_1) dz_1 \end{bmatrix}. \quad (18)$$

The complex empirical transformations of the system of equations (12), (13) (set of probability statements) to the system of equations (18) (in terms of probability distributions), in order to determine optimal protection levels for *l=3* nested fare classes, show that the method of [2], is not suitable for practical applications if the number of nested fare classes $l \ge 4$.

Unfortunately, we did not find a numerical example in the literature for the case when the number of nested fare classes $l \ge 4$.

2 Optimization of Airline Seat Protection Levels for Nested Fare Classes of Single-Leg Flights

The performance index which can be used to determine the optimal allocation of airline seats between l dependent (i.e., nested) fare classes, subject to N (the total airplane seat capacity), is given as follows.

Maximize the total expected revenue for a single-leg flight with l nested fare classes (say, l=4)

$$\begin{aligned} Q(n_1, n_2, n_3, n_4) &= Q_4(n_4) \\ &+ E_4\left\{ Q_3\left(\sum_{j=3}^{4} n_j - Z_4 \right) \right\} \\ &+ E_4\left\{ E_3\left\{ Q_2\left(\sum_{j=2}^{4} n_j - \sum_{j=3}^{4} Z_j \right) \right\} \right\} \\ &+ E_4\left\{ E_3\left\{ E_2\left\{ Q_1\left(\sum_{j=1}^{4} n_j - \sum_{j=2}^{4} Z_j \right) \right\} \right\} \right\}, \end{aligned} \quad (19)$$

subject to

$$\sum_{j=1}^{4} n_j = N, \quad n_j \ge 0 \quad \text{for } j = 1(1)4, \quad (20)$$

where

$$Q_l(n_l) = E_l\{c_l \min(n_l, Z_l)\} = c_l\left(u_l - \int_0^{u_l} F_l(z_l) dz_l \right) \quad (21)$$

represents the expected revenue from the lth fare class, c_l is the fare level for the lth fare class, ($c_l < c_{l-1} < \ldots < c_1$), n_l denotes the booking limit for the lth fare class, Z_l denotes the customer demand for the lth fare class, $f_l(z_l)$ is the probability density function of Z_l, $F_l(z_l)$ is the cumulative distribution function of Z_l,

$$\begin{aligned} & E_l\left\{ Q_{l-1}\left(\sum_{j=l-1}^{l} n_j - Z_l \right) \right\} \\ &= c_{l-1} \int_0^{n_l} \left(\sum_{j=l-1}^{l} n_j - z_l - \int_0^{\sum_{j=l-1}^{l} n_j - z_l} F_{l-1}(z_{l-1}) dz_{l-1} \right) f_l(z_l) dz_l \\ &+ c_{l-1} \int_{n_l}^{\infty} \left(n_{l-1} - \int_0^{n_{l-1}} F_{l-1}(z_{l-1}) dz_{l-1} \right) f_l(z_l) dz_l \end{aligned} \quad (22)$$

represents the expected revenue from the (l–1)th fare class, n_{l-1} denotes the protection level for the (l–1)th fare class, and so on.

Theorem 2. The optimal solution for the above performance index (19) is given as follows:

$$c_2 = c_1 \overline{F}_1(n_1))$$

$$\left(\text{from } \frac{\partial Q(n_1, n_2)}{\partial n_2} = \frac{\partial Q(n_1, n_2,)}{\partial n_1} \right), \quad (23)$$

$$c_3 = c_2 \overline{F}_2(n_2) + c_1 \int_0^{n_2} \overline{F}_1\left(\sum_{j=1}^{2} n_j - z_2 \right) f_2(z_2) dz_2 \text{ or}$$

$$c_3 = c_1\left(\overline{F}_1(n_1 + n_2) + \int_{n_1}^{n_1+n_2} \overline{F}_2\left(\sum_{j=1}^{2} n_j - z_1 \right) f_1(z_1) dz_1 \right)$$

$$\left(\text{from } \frac{\partial Q(n_1,n_2,n_3)}{\partial n_3} = \frac{\partial Q(n_1,n_2,n_3)}{\partial n_2}\right), \quad (24)$$

$$c_4 = c_1\left(\begin{array}{c} \overline{F}_1\left(\sum_{j=1}^{3} n_j\right) \\ +\int_{n_1}^{n_1+n_2} \overline{F}_2\left(\sum_{j=1}^{3} n_j - z_1\right) f_1(z_1)\,dz_1 \\ +\int_{n_1+n_2}^{n_1+n_2+n_3} \overline{F}_3\left(\sum_{j=1}^{3} n_j - z_1\right) f_1(z_1)\,dz_1 \end{array}\right)$$

$$\left(\text{from } \frac{\partial Q(n_1,n_2,n_3,n_4)}{\partial n_4} = \frac{\partial Q(n_1,n_2,n_3,n_4)}{\partial n_3}\right). \quad (25)$$

Proof. The proof follows using the technique of Lagrange multipliers. Here it is omitted and will appear elsewhere.

For example, consider again a single-leg flight with l=3 nested fare classes. It follows immediately from (23) and (24) that the optimal protection levels have to satisfy the following system of two equations:

$$c_2 = c_1\overline{F}_1(n_1),$$

$$c_3 = c_2\overline{F}_2(n_2) + c_1\int_0^{n_2} \overline{F}_1\left(\sum_{j=1}^{2} n_j - z_2\right) f_2(z_2)dz_2. \quad (26)$$

Theorem 3. It can be shown that the system of two equations (26) can be transformed to (18).

Proof.
It follows from (26) that

$$\int_0^{n_2} \overline{F}_1\left(\sum_{i=1}^{2} n_i - z_2\right) f_2(z_2)dz_2 = \overline{F}_1\left(\sum_{i=1}^{2} n_i - z_2\right) F(z_2)\Bigg|_0^{n_2}$$

$$-\int_0^{n_2} F_2(z_2)\overline{F}_1'\left(\sum_{i=1}^{2} n_i - z_2\right)dz_2 = \overline{F}_1(n_1)F_2(n_2)$$

$$-\int_0^{n_2} (1-\overline{F}_2(z_2))\overline{F}_1'\left(\sum_{i=1}^{2} n_i - z_2\right)dz_2 = \overline{F}_1(n_1)\left(1-\overline{F}_2(n_2)\right)$$

$$-\int_0^{n_2} \overline{F}_1'\left(\sum_{i=1}^{2} n_i - z_2\right)dz_2 + \int_0^{n_2} \overline{F}_2(z_2)\overline{F}_1'\left(\sum_{i=1}^{2} n_i - z_2\right)dz_2$$

$$= \overline{F}_1(n_1) - \overline{F}_1(n_1)\overline{F}_2(n_2) - \overline{F}_1\left(\sum_{i=1}^{2} n_i - z_2\right)\Bigg|_0^{n_2}$$

$$+\int_{n_1+n_2}^{n_1} \overline{F}_2\left(\sum_{i=1}^{2} n_i - z_1\right)\overline{F}_1'(z_1)\,dz_1$$

$$= \overline{F}_1(n_1) - \overline{F}_1(n_1)\overline{F}_2(n_2) - \overline{F}_1(n_1) + \overline{F}_1(n_1+n_2)$$

$$+\int_{n_1}^{n_1+n_2} \overline{F}_2\left(\sum_{i=1}^{2} n_i - z_1\right) f_1(z_1)\,dz_1$$

$$= -\overline{F}_1(n_1)\overline{F}_2(n_2) + \overline{F}_1(n_1+n_2)$$

$$+\int_{n_1}^{n_1+n_2} \overline{F}_2\left(\sum_{i=1}^{2} n_i - z_1\right) f_1(z_1)\,dz_1, \quad (27)$$

where

$$\sum_{i=1}^{2} n_i - z_2 = z_1, \quad (28)$$

$$z_2 = \sum_{i=1}^{2} n_i - z_1. \quad (29)$$

Substitution (27) into (26), we have

$$c_3 = c_2\overline{F}_2(n_2) + c_1\int_0^{n_2} \overline{F}_1\left(\sum_{j=1}^{2} n_j - z_2\right) f_2(z_2)dz_2$$

$$= c_1\overline{F}_1(n_1)\overline{F}_2(n_2) - c_1\overline{F}_1(n_1)\overline{F}_2(n_2) + c_1\overline{F}_1(n_1+n_2)$$

$$+ c_1\int_{n_1}^{n_1+n_2} \overline{F}_2\left(\sum_{i=1}^{2} n_i - z_1\right) f_1(z_1)\,dz_1$$

$$= c_1\left(\overline{F}_1(n_1+n_2) + \int_{n_1}^{n_1+n_2} \overline{F}_2\left(\sum_{i=1}^{2} n_i - z_1\right) f_1(z_1)\,dz_1\right), \quad (30)$$

where

$$c_2 = c_1\overline{F}(n_1). \quad (31)$$

Thus, using two different analytical approaches, the same result (18) was obtained. This indicates the correctness of the used analytical approaches and completes the proof.

For example, it follows immediately from (23) and (24) that the optimal protection levels n_1 and n_2 can be determined as

$$n_1 = \arg\min_{n_1}\left[\overline{F}_1(n_1) - \frac{c_2}{c_1}\right]^2,$$

$$= \arg\min_{n_1}\left[F_1(n_1) - \left(1 - \frac{c_2}{c_1}\right)\right]^2, \quad (32)$$

$$n_2 = \arg\min_{n_2}\left[\left(\begin{array}{c} \overline{F}_1(n_1+n_2) \\ +\int_{n_1}^{n_1+n_2} \overline{F}_2\left(\sum_{j=1}^{2} n_j - z_1\right) \\ \times f_1(z_1)\,dz_1 \end{array}\right) - \frac{c_3}{c_1}\right]^2. \quad (33)$$

For example, in the case of a single-leg flight with l=3 nested fare classes, the performance index is given as follows.

Maximize the total expected revenue for a single-leg flight with l=3 nested fare classes,

$$Q(n_1,n_2,n_3)=c_3\left[n_3-\int_0^{n_3}F_3(z_3)dz_3\right]$$
$$+c_2\int_0^{n_3}\left[n_2+n_3-z_3-\int_0^{n_2+n_3-z_3}F_2(z_2)dz_2\right]f_3(z_3)dz_3$$
$$+c_2\int_{n_3}^{\infty}\left[n_2-\int_0^{n_2}F_2(z_2)dz_2\right]f_3(z_3)dz_3$$
$$+c_1\int_0^{n_3}\left[\int_0^{n_2+n_3-z_3}\left[n_1+n_2+n_3-z_3-z_2\right.\right.$$
$$\left.\left.-\int_0^{n_1+n_2+n_3-z_3-z_2}F_1(z_1)dz_1\right]f_2(z_2)dz_2\right]f_3(z_3)dz_3$$
$$+c_1\int_0^{n_3}\left[\int_{n_2+n_3-z_3}^{\infty}\left[n_1-\int_0^{n_1}F_1(z_1)dz_1\right]f_2(z_2)dz_2\right]f_3(z_3)dz_3$$
$$+c_1\int_{n_3}^{\infty}\left[\int_0^{n_2}\left[n_1+n_2-z_2-\int_0^{n_1+n_2-z_2}F_1(z_1)dz_1\right]f_2(z_2)dz_2\right]$$
$$\times f_3(z_3)dz_3$$
$$+c_1\int_{n_3}^{\infty}\left[\int_{n_2}^{\infty}\left[n_1-\int_0^{n_1}F_1(z_1)dz_1\right]f_2(z_2)dz_2\right]f_3(z_3)dz_3, \quad (34)$$

subject to

$$\sum_{j=1}^{3}n_j=N,\quad n_j\ge 0\quad \text{for}\, j\in\{1,2,3\}. \quad (35)$$

A simple application of the Lagrange multipliers technique leads to the optimal solution satisfying (35) and

$$\left(\begin{array}{c} c_2=c_1\overline{F}_1(n_1) \\ \text{from }\dfrac{\partial Q(n_1,n_2,n_3)}{\partial n_2}=\dfrac{\partial Q(n_1,n_2,n_3)}{\partial n_1}\end{array}\right),$$

$$\left(\begin{array}{c} c_3=c_2\overline{F}_2(n_2)+c_1\displaystyle\int_0^{n_2}\overline{F}_1(n_1+n_2-z_2)f_2(z_2)dz_2 \\ \text{from }\dfrac{\partial Q(n_1,n_2,n_3)}{\partial n_3}=\dfrac{\partial Q(n_1,n_2,n_3)}{\partial n_2}\end{array}\right). \quad (36)$$

3 Exponential Distribution

Let $\mathbf{U}=(U_1\le ...\le U_n)$ be the n ordered observations (order statistics) in a sample of size n from the two-parameter exponential distribution with the probability density function (pdf)

$$f_\omega(u)=\frac{1}{\rho}\exp\left(-\frac{u-\upsilon}{\rho}\right),\quad \rho>0,\ u\ge\upsilon, \quad (37)$$

and the cumulative distribution function (cdf)

$$F_\omega(u)=1-\exp\left(-\frac{u-\upsilon}{\rho}\right),\quad \overline{F}_\omega(u)=1-F_\omega(u), \quad (38)$$

where $\omega=(\upsilon,\rho)$, υ is the shift parameter and ρ is the scale parameter. It is assumed that these parameters are unknown. In Type II censoring, which is of primary interest here, the number of survivors is fixed and U_r is a random variable. In this case, the likelihood function is given by

$$L(\upsilon,\rho)=\prod_{j=1}^{n}f_\omega(u_j)$$
$$=\frac{1}{\rho^n}\exp\left(-\left[\sum_{j=1}^{n}\left(u_j-\upsilon\right)\right]\Big/\rho\right)$$
$$=\frac{1}{\rho^n}\exp\left(-\left[\sum_{j=1}^{n}\left(u_i-u_1+u_1-\upsilon\right)\right]\Big/\rho\right)$$
$$=\frac{1}{\rho^{n-1}}\exp\left(-\left[\sum_{j=1}^{n}(u_i-u_1)\right]\Big/\rho\right)$$
$$\times\frac{1}{\rho}\exp\left(-\frac{n(u_1-\upsilon)}{\rho}\right)$$
$$=\frac{1}{\rho^{n-1}}\exp\left(-\frac{s_n}{\rho}\right)\times\frac{1}{\rho}\exp\left(-\frac{n(s_1-\upsilon)}{\rho}\right), \quad (39)$$

where

$$\mathbf{S}=\left(S_1=U_1,\ S_n=\sum_{j=1}^{n}(U_j-U_1)\right) \quad (40)$$

is the complete sufficient statistic for ω. The probability density function of $\mathbf{S}$=(S_1, S_n) is given by

$$f_\omega(s_1,s_n)$$
$$=\frac{\dfrac{1}{\rho^{n-1}}\exp\left(-\dfrac{s_n}{\rho}\right)\times\dfrac{1}{\rho}\exp\left(-\dfrac{n(s_1-\upsilon)}{\rho}\right)}{\dfrac{1}{s_n^{n-2}}\displaystyle\int_0^{\infty}\frac{s_n^{n-2}}{\rho^{n-1}}\exp\left(-\frac{s_n}{\rho}\right)ds_n\times\frac{1}{n}\int_0^{\infty}\frac{n}{\rho}\exp\left(-\frac{n(s_1-\upsilon)}{\rho}\right)ds_1}$$
$$=\frac{\dfrac{1}{\rho^{n-1}}\exp\left(-\dfrac{s_n}{\rho}\right)\times\dfrac{1}{\rho}\exp\left(-\dfrac{n(s_1-\upsilon)}{\rho}\right)}{\dfrac{\Gamma(n-1)}{s_n^{n-2}}\times\dfrac{1}{n}}$$
$$=\frac{1}{\Gamma(n-1)\rho^{n-1}}s_n^{n-2}\exp\left(-\frac{s_n}{\rho}\right)\times\frac{n}{\rho}\exp\left(-\frac{n(s_1-\upsilon)}{\rho}\right)$$
$$=f_\rho\left(s_n\right)f_\omega\left(s_1\right), \quad (41)$$

where

$$f_\omega(s_1)=\frac{n}{\rho}\exp\left(-\frac{n(s_1-\upsilon)}{\rho}\right),\quad s_1\ge\upsilon, \tag{42}$$

$$f_\rho(s_n)=\frac{1}{\Gamma(n-1)\rho^{n-1}}s_n^{n-2}\exp\left(-\frac{s_n}{\rho}\right),\quad s_n\ge 0. \tag{43}$$

$$V_1=\frac{S_1-\upsilon}{\rho} \tag{44}$$

is the pivotal quantity, the probability density function of which is given by

$$f_1(v_1)=n\exp(-nv_1),\quad v_1\ge 0, \tag{45}$$

$$V_n=\frac{S_n}{\rho} \tag{46}$$

is the pivotal quantity, the probability density function of which is given by

$$f_n(v_n)=\frac{1}{\Gamma(n-1)}v_n^{n-2}\exp(-v_n),\quad v_n\ge 0. \tag{47}$$

3.1 Pivot-Based Elimination of Unknown (Nuisance) Parameters from the Two-Parameter Exponential Distribution

Let us suppose that U is a future observation from the same distribution (38), independent of $\mathbf{U}=(U_1\le \ldots \le U_n)$. Then a statistical estimate of (38) can be determined as follows.

Step 1. Invariant embedding of S_1 in (38) to isolate the unknown parameter υ from the problem through V_1 (44),

$$F_\omega(u)=1-\exp\left(-\frac{u-\upsilon}{\rho}\right)$$

$$=1-\exp\left(-\frac{z-s_1+s_1-\delta}{\vartheta}\right)$$

$$=1-\exp\left(-\frac{u-s_1}{\rho}\right)\exp(-v_1),\quad u\ge s_1, \tag{48}$$

Step 2. Averaging (48) over the probability distribution of the pivotal quantity V_1 to eliminate unknown parameter υ from the problem. It follows from (48) and (46) that the pivot-based estimate of the cumulative distribution function (38) (obtained through the pivot-based method) is given by

$$F_{s_1,\rho}(u)=\int_0^\infty F_\omega(u)f_1(v_1)dv_1$$

$$=\int_0^\infty\left[1-\exp\left(-\frac{u-s_1}{\rho}\right)\exp(-v_1)\right]n\exp(-nv_1)dv_1$$

$$=1-\exp\left(-\frac{u-s_1}{\rho}\right)\int_0^\infty n\exp(-v_1[n+1])dv_1$$

$$=1-\exp\left(-\frac{u-s_1}{\rho}\right)\frac{n}{n+1},\quad u\in(s_1,\infty). \tag{49}$$

Since

$$\frac{dF_{s_1,\rho}(u)}{du}=\frac{n}{n+1}\frac{1}{\rho}\exp\left(-\frac{u-s_1}{\rho}\right) \tag{50}$$

and

$$\frac{\frac{n}{n+1}\frac{1}{\rho}\exp\left(-\frac{u-s_1}{\rho}\right)}{\int_{s_1}^\infty\frac{n}{n+1}\frac{1}{\rho}\exp\left(-\frac{u-s_1}{\rho}\right)du}=\frac{1}{\rho}\exp\left(-\frac{u-s_1}{\rho}\right), \tag{51}$$

it follows from (51) that the probability density function (pdf) of U is given by

$$f_{s_1,\rho}(u)=\frac{1}{\rho}\exp\left(-\frac{u-s_1}{\rho}\right),\quad u\ge s_1, \tag{52}$$

with the cumulative distribution function

$$F_{s_1,\rho}(u)=1-\exp\left(-\frac{u-s_1}{\rho}\right). \tag{53}$$

Step 3. Invariant embedding of S_n in (53) to isolate the unknown parameter υ from the problem through V_n (46),

$$F_{s_1,\rho}(u)=1-\exp\left(-\frac{u-s_1}{\rho}\right)$$

$$=1-\exp\left(-\frac{u-s_1}{s_n}\frac{s_n}{\rho}\right)$$

$$=1-\exp\left(-\frac{u-s_1}{s_n}v_n\right),\quad u\ge s_1. \tag{54}$$

Step 4. Averaging (54) over the probability distribution of the pivotal quantity V_n to eliminate unknown parameter ρ from the problem. It follows from (54) and (47) that the pivot-based estimate of the cumulative distribution function (38) (obtained through the pivot-based method) is given by

$$\int_0^\infty F_{s_1,\rho}(u)f_n(v_n)dv_n=\int_0^\infty\left[1-\exp\left(-\frac{u-s_1}{s_n}v_n\right)\right]$$

$$\times\frac{1}{\Gamma(n-1)}v_n^{n-2}\exp(-v_n)dv_n$$

$$=1-\left(1+\frac{u-s_1}{s_n}\right)^{-(n-1)}=F_{\mathrm{s}}(u). \tag{55}$$

where

$$\overline{F}_s(u)=1-F_s(u)=\left(1+\frac{u-s_1}{s_n}\right)^{-(n-1)}. \quad (56)$$

The pivot-based estimate of the probability density function (37) is given by

$$f_s(u)=\frac{dF_s(u)}{du}=\frac{n-1}{s_n}\left(1+\frac{u-s_1}{s_n}\right)^{-n}, \quad u\geq s_1. \ (57)$$

It follows from (55) that the cumulative distribution function of the ancillary statistic

$$X=\frac{U-S_1}{S_n} \quad (58)$$

is given by

$$F(x)=1-\frac{1}{\left(1+x\right)^{n-1}}. \quad (59)$$

The probability density function of the ancillary statistic (58) is given by

$$f(x)=\frac{dF(x)}{dx}=\frac{n-1}{(1+x)^n}, \quad \mathrm{x}\geq 0. \quad (60)$$

3.2 Constructing Shortest Length or Equal Tails Confidence Intervals for Future Observations from the Two-Parameter Exponential Distribution under Parametric Uncertainty

Using (58) and (59), it can be obtained a 100(1−α)% confidence interval for U from

$$\Pr\left(x_1\leq X\leq x_2\right)=\Pr\left(x_1\leq\frac{U-S_1}{S_n}\leq x_2\right)$$

$$=\Pr\left(x_1S_r+S_1\leq U\leq x_2S_n+S_1\right)=1-\alpha. \quad (61)$$

by suitably choosing the decision variables x_1 and x_2. Hence, the statistical confidence interval for U is given by

$$\left[x_1s_n+s_1, x_2s_n+s_1\right]. \quad (62)$$

The length of the statistical confidence interval for U is given by

$$L(x_1,x_2\mid s_n)=\left(x_2s_n-x_1s_n\right)=\left(x_2-x_1\right)s_n. \quad (63)$$

In order to find the shortest length confidence interval $L(x_1,x_2\,|\,s_n)$, we should find a pair of decision variables x_1 and x_2 such that $L(x_1,x_2\,|\,s_n)$ is minimum.

It follows from (60) and (63) that

$$\int_{x_1}^{x_2}f(x)dx=\int_0^{x_2}f(x)dx-\int_0^{x_1}f(x)dx$$

$$=F(x_2)-F(x_1)=\left(1-\alpha+p\right)-p=1-\alpha, \quad (64)$$

where p $(0\leq p\leq\alpha)$ is a decision variable,

$$\int_0^{x_2}f(x)dx=F(x_2)=\left(1-\alpha+p\right) \quad (65)$$

and

$$\int_0^{x_1}f(x)dx=F(x_1)=p. \quad (66)$$

Then x_2 represents the $\left(1-\alpha+p\right)$ - quantile, which is given by

$$x_2=q_{1-\alpha+p}=\left(\frac{1}{\alpha-p}\right)^{1/(n-1)}-1, \quad (67)$$

x_1 represents the p - quantile, which is given by

$$x_1=q_p=\left(\frac{1}{1-p}\right)^{1/(n-1)}-1. \quad (68)$$

The shortest length confidence interval for U can be found as follows:

Minimize

$$L^2(x_1,x_2\mid s_n)=\left[\left(x_2-x_1\right)s_n\right]^2=\left[\left(q_{1-\alpha+p}-q_p\right)s_n\right]^2$$

$$=\left[\left(\frac{1}{\alpha-p}\right)^{1/(n-1)}-\left(\frac{1}{1-p}\right)^{1/(n-1)}\right]^2 s_n^2. \quad (69)$$

subject to

$$0\leq p\leq\alpha, \quad (70)$$

The optimal numerical solution minimizing $L(x_1, x_2\mid s_n)$ can be obtained using the standard computer software "Solver" of Excel 2016. If, for example, n = 4, α = 0.05, then the optimal numerical solution is given by

$$p=0 \quad (71)$$

with the 100(1−α)% shortest-length confidence interval

$$L(x_1,x_2\mid s_n)=1.1s_n. \quad (72)$$

The 100(1−α)% equal tails confidence interval is given by

$$L(x_1,x_2\mid s_n;p=\alpha/2)=1.5s_n \quad (73)$$

with

$$p=0.025. \quad (74)$$

Relative efficiency. The relative efficiency of $L(x_1,x_2\,|\,s_n;p=\alpha/2)$ as compared with $L(x_1,x_2|\ s_n)$ is given by

$$\text{rel.eff.}_L\left\{\,L(x_1,x_2\mid s_n;p=\alpha/2),L(x_1,x_2\mid s_n)\,\right\}$$

$$=\frac{L(x_1,x_2\mid s_n)}{L(x_1,x_2\mid s_n;p=\alpha/2)}=\frac{1.1s_n}{1.5s_n}=0.7. \quad (75)$$

4 Generalized Pivotal Quantities to Construct Statistical Predictive Limits for Order Statistics in the New Sample

Theorem 4. Suppose we are interested in a new random sample of m ordered observations $U_1\le\ldots\le U_m$ from a known distribution with a probability density function (pdf) $f_\omega(u)$, cumulative distribution function (cdf) $F_\omega(u)$, where ω is the parameter (in general, vector). Then for constructing one-sided predictive limits (for the rth order statistic U_r, $r\in\{1, 2, \ldots, m\}$) with confidence level $1-\alpha$ can be used the following generalized pivotal quantities.

***Generalized Pivotal Quantity GPQ*1:**

$$GPQ1=F_\omega(u_r)$$

$$=z\sim f_{r,m-r+1}(z)=\frac{z^{r-1}(1-z)^{(m-r+1)-1}}{\mathrm{B}(r,m-r+1)},$$

$$0<z<1, \quad (76)$$

where $f_{r,m-r+1}(z)$ is the probability density function (pdf) of the beta distribution $(Beta(r,m-r+1))$ with the shape parameters r and m-r+1,

Proof. It follows from (76) that

$$\frac{d}{du_r}\int_0^{GPQ1} f_{r,m-r+1}(z)dz=\frac{d}{du_r}P_\omega(U_r\le u_r\mid m) \quad (77)$$

with

$$\int_0^{GPQ1} f_{r,m-r+1}(z)dz=P_\omega(U_r\le u_r\mid m), \quad (78)$$

where

$$P_\omega(U_r\le u_r\mid m)$$

$$=\sum_{j=r}^{m}\binom{m}{j}[F_\omega(u_r)]^j\ [1-F_\omega(u_l)]^{m-j}. \quad (79)$$

This ends the proof.

***Generalized Pivotal Quantity GPQ*2:**

$$GPQ2=1-F_\omega(u_r)$$

$$=z\sim f_{m-r+1,r}(z)=\frac{z^{(m-r+1)-1}(1-z)^{r-1}}{\mathrm{B}(m-r+1,r)},$$

$$0<z<1, \quad (80)$$

where $f_{m-r+1,r}(z)$ is the probability density function (pdf) of the beta distribution $(Beta(m-r+1,r))$ with shape parameters $m-r$+1 and r.

Proof. It follows from (80) that

$$\frac{d}{du_r}\int_{GPQ2}^{1} f_{m-r+1,r}(z)dz=\frac{d}{du_r}P_\omega(U_r\le u_r\mid m) \quad (81)$$

with

$$\int_{GPQ2}^{1} f_{m-r+1,r}(z)dz=P_\omega(U_r\le u_r\mid m). \quad (82)$$

This ends the proof.

***Generalized Pivotal Quantity GPQ*3:**

$$GPQ3=\frac{m-r+1}{r}\frac{F_\omega(u_r)}{1-F_\omega(u_r)}$$

$$=z\sim\varphi_{r,m-r+1}(z)=\frac{\dfrac{\frac{r}{m-r+1}}{\mathrm{B}(r,m-r+1)}\left[\frac{r}{m-r+1}z\right]^{r-1}}{\left[1+\frac{r}{m-r+1}z\right]^{m+1}},$$

$$z\in(0,\infty), \quad (83)$$

where $\varphi_{r,m-r+1}(z)$ is the probability density function (pdf) of the F distribution $(F(r,m-r+1))$ with parameters r and $m-r$+1, which are positive integers known as the degrees of freedom for the numerator and the degrees of freedom for the denominator.

Proof. It follows from (83) that

$$\frac{d}{du_r}\int_0^{GPQ3}\varphi_{r,m-r+1}(z)dz=\frac{d}{du_r}P_\omega(U_r\le u_r\mid m) \quad (84)$$

with

$$\int_0^{GPQ3}\varphi_{r,m-r+1}(z)dz=P_\omega(U_r\le u_r\mid m). \quad (85)$$

This ends the proof.

***Generalized Pivotal Quantity GPQ*4:**

$$GPQ4=\frac{r}{m-r+1}\frac{1-F_\omega(u_r)}{F_\omega(u_r)}$$

$$=z\sim\varphi_{m-r+1,r}(z)=\frac{\dfrac{\frac{m-r+1}{r}}{\mathrm{B}(m-r+1,r)}\left[\frac{m-r+1}{r}z\right]^{m-r}}{\left[1+\frac{m-r+1}{r}z\right]^{m+1}},$$

$$z\in(0,\infty), \quad (86)$$

where $\varphi_{m-r+1,r}(z)$ is the probability density function (pdf) of the F distribution $(F(m-r+1,r)$ with parameters $m-r$ +1 and r, which are positive integers known as the degrees of freedom for the numerator and the degrees of freedom for the denominator.

Proof. It follows from (86) that

$$\frac{d}{du_r}\int_{GPQ4}^{\infty}\varphi_{m-r+1,r}(z)dz=\frac{d}{du_r}P_\omega(U_r\le u_r\,|\,m) \quad (87)$$

with

$$\int_{GPQ4}^{\infty}\varphi_{m-r+1,r}(z)dz=P_\omega(U_r\le u_r\,|\,m). \quad (88)$$

This ends the proof.

5 Generalized Pivotal Quantities to Construct Statistical Predictive Limits for Order Statistics in the Same Sample

Theorem 5. Suppose we observe some random sample of m ordered observations $U_1\le\ldots\le U_m$ from a known distribution with a probability density function (pdf) $f_\omega(u)$, cumulative distribution function (cdf) $F_\omega(u)$, where ω is the parameter (in general, vector). The order statistic U_r is known. Then, for constructing one-sided predictive limits (for the kth order statistic U_k, $k\in\{r+1, \ldots, m\}$) with confidence level $1-\alpha$, the following generalized pivotal quantities can be used.

***Generalized Pivotal Quantity GPQ*5:**

$$GPQ5=1-\frac{\bar{F}_\omega(u_r)}{\bar{F}_\omega(u_k)}$$

$$=z\sim\varphi_{r-k,m-r+1}(z)=\frac{z^{r-k-1}(1-z)^{(m-r+1)-1}}{\mathrm{B}\left(r-k,m-r+1\right)},$$

$$0<z<1, \quad (89)$$

where $\varphi_{r-k,m-r+1}(z)$ is the probability density function of the beta distribution $(Beta(r-k,m-r+1))$ with shape parameters $r-k$ and $m-r+1$.

Proof. It follows from (89) that

$$\frac{d}{du_r}\int_{0}^{GPQ5}\varphi_{r-k,m-r+1}(z)dz$$

$$=\frac{d}{du_r}P_\omega(U_r\le u_r\,|\,U_k=u_k;m) \quad (90)$$

with

$$\int_{0}^{GPQ5}\varphi_{r-k,m-r+1}(z)dz=P_\omega(U_r\le u_r\,|\,U_k=u_k;m), \quad (91)$$

where

$$P_\omega(U_r\le u_r\,|\,U_k=u_k;m)$$

$$=\sum_{j=r-k}^{m-k}\binom{m-k}{j}\left[1-\frac{\bar{F}_\omega(u_r)}{\bar{F}_\omega(u_k)}\right]^j\left[\frac{\bar{F}_\omega(z_r)}{\bar{F}_\omega(z_k)}\right]^{m-k-j}. \quad (92)$$

This ends the proof.

***Generalized Pivotal Quantity GPQ*6:**

$$GPQ6=\frac{\bar{F}_\omega(u_r)}{\bar{F}_\omega(u_k)}$$

$$=z\sim\varphi_{m-r+1,r-k}(z)=\frac{z^{(m-r+1)-1}(1-z)^{r-k-1}}{\mathrm{B}\left(m-r+1,r-k\right)},$$

$$0<z<1. \quad (93)$$

where $\varphi_{m-r+1,r-k}(z)$ is the probability density function (pdf) of the beta distribution $(Beta(m-r+1,r-k))$ with shape parameters $m-r+1$ and $r-k$.

Proof. It follows from (93) that

$$\frac{d}{du_r}\int_{GPQ6}^{1}\varphi_{m-r+1,r-k}(z)dz$$

$$=\frac{d}{du_r}P_\omega(U_r\le u_r\,|\,U_k=u_k;m) \quad (94)$$

with

$$\int_{GPQ6}^{1}\varphi_{m-r+1,r-k}(z)dz,=P_\omega(U_r\le u_r\,|\,U_k=u_k;m) \quad (95)$$

This ends the proof.

***Generalized Pivotal Quantity GPQ*7:**

$$GPQ7=\frac{m-r+1}{r-k}\left(1-\frac{\bar{F}_\omega(u_r)}{\bar{F}_\omega(u_k)}\right)\Bigg/\frac{\bar{F}_\omega(u_r)}{\bar{F}_\omega(u_k)}$$

$$=z\sim\varphi_{r-k,m-r+1}(z)$$

$$=\frac{\dfrac{r-k}{m-r+1}}{\mathrm{B}(r-k,m-r+1)}\frac{\left[\dfrac{r-k}{m-r+1}z\right]^{r-k-1}}{\left[1+\dfrac{r-k}{m-r+1}z\right]^{m-k+1}},$$

$$z\in(0,\infty), \quad (96)$$

where $\varphi_{r-k,m-r+1}(z)$ is the probability density function (pdf) of the F distribution $(F(r-k,m-r+1))$ with parameters $r-k$ and $m-r+1$, which are positive integers known as the degrees of freedom for the numerator and the degrees of freedom for the denominator.

Proof. It follows from (96) that

$$\frac{d}{du_r}\int_{0}^{GPQ7}\varphi_{r-k,m-r+1}(z)dz$$

$$= \frac{d}{du_r} P_\omega(U_r \le u_r \mid U_k = u_k; m) \tag{97}$$

with

$$\int_0^{GPQ7} \varphi_{r-k,m-r+1}(z)dz = P_\omega(U_r \le u_r \mid U_k = u_k; m). \tag{98}$$

This ends the proof.

***Generalized Pivotal Quantity GPQ*8:**

$$GPQ8 = \frac{r-k}{m-r+1} \frac{\bar{F}_\omega(u_r)}{\bar{F}_\omega(u_k)} \bigg/ \left(1 - \frac{\bar{F}_\omega(u_r)}{\bar{F}_\omega(u_k)}\right)$$

$$= z \sim \varphi_{m-r+1,r-k}(z)$$

$$= \frac{\dfrac{m-r+1}{r-k}}{\mathrm{B}(m-r+1, r-k)} \frac{\left[\dfrac{m-r+1}{r-k} z\right]^{m-r+1}}{\left[1 + \dfrac{m-r+1}{r-k} z\right]^{m-k+1}},$$

$$z \in (0, \infty), \tag{99}$$

where $\varphi_{m-r+1,r-k}(z)$ is the probability density function (pdf) of the F distribution $(F(m-r+1, r-k))$ with parameters $m-r+1$ and $r-k$, which are positive integers known as the degrees of freedom for the numerator and the degrees of freedom for the denominator,

Proof. It follows from (99) that

$$\frac{d}{du_r} \int_{GPQ8}^{\infty} \varphi_{m-r+1,r-k}(z)dz$$

$$= \frac{d}{du_r} P_\omega(U_r \le u_r \mid U_k = u_k; m) \tag{100}$$

with

$$\int_{GPQ8}^{\infty} \varphi_{m-r+1,r-k}(z)dz = P_\omega(U_r \le u_r \mid U_k = u_k; m). \tag{101}$$

This ends the proof.

6 Illustrative Example

For the sake of simplicity but without loss of generality, consider the problem of optimal allocation of seats between two dependent (i.e., nested) fare classes. The performance index which can be used to determine the optimal allocation of seats between two dependent (i.e., nested) fare classes, subject to the total airplane seat capacity constraint, is given as follows.

Maximize the total expected revenue for a single-leg flight with two nested fare classes (business and economy),

$$Q(n_1, n_2) = Q_2(n_2)$$

$$+ E_2\{Q_1(n_1 + n_2 - \min(n_2, Z_2))\}, \tag{102}$$

where

$$Q_2(n_2) = E_2\{c_2 \min(n_2, Z_2)\}$$

$$= c_2\left(n_2 - \int_0^{n_2} F_2(z_2)dz_2\right), \tag{103}$$

$$E_2\{Q_1(n_1 + n_2 - \min(n_2, Z_2))\}$$

$$= c_1 \int_0^{n_2} \left(n_1 + n_2 - z_2 - \int_0^{n_1+n_2-z_2} F_1(z_1)dz_1\right) f_2(z_2)dz_2$$

$$+ c_1 \int_{n_2}^{\infty} \left(n_1 - \int_0^{n_1} F_1(z_1)dz_1\right) f_2(z_2)dz_2. \tag{104}$$

subject to

$$n_1 + n_2 = N, \quad n_j \ge 0 \quad \text{for } j = 1, 2, \tag{105}$$

where c_1 and c_2 are the high and low fare levels respectively ($c_1 > c_2$), n_j denotes the booking limit for the jth fare class, Z_j denotes the customer demand for the jth fare class, $f_j(z_j)$ is the probability density function of Z_j, N is the total capacity of the cabin to be shared among the two fare classes. A simple application of the Lagrange multipliers technique leads to the optimal solution satisfying

$$n_1 = \arg\left[c_2 = c_1 \Pr(Z_1 > n_1)\right] = \arg\left[\frac{c_2}{c_1} = \bar{F}_1(n_1)\right]$$

$$= \arg\min_{n_1} \left[F_1(n_1) - \frac{c_1 - c_2}{c_1}\right]^2,$$

$$n_2 = \min(0, N - n_1), \tag{106}$$

where n_1 denotes the optimal protection level for the high fare class, and n_2 denotes the optimal booking limit for the low fare class. Thus, (106) suggests closing down the low fare class when the certain revenue from selling low fare seats is exceeded by the expected revenue of selling the same seat at the higher fare. It should be remarked that there is no protection level for the low fare (or economy) class; n_2 is the booking limit, or the number of seats available, for the low fare class; the low fare class is open as long as the number of bookings in this class remains less than this limit. Thus, (n_1+n_2) is the booking limit or number of seats available for the high fare class at the time. The high fare class is open as long as the number of bookings in this and low classes remains less than this limit.

6.1 Model of Optimal Statistical Estimation of Airline Seat Protection Levels for Nested Fare Classes under Parametric Uncertainty

The model is given as follows:

Step 1. It follows from (56) that

$$\overline{F}_1(n_1) = \overline{F}_{\mathrm{s}}(n_1) = 1 - F_{\mathrm{s}}(n_1) = \left(1 + \frac{n_1 - s_1}{s_n}\right)^{-(n-1)}. \quad (107)$$

Step 2. It follows from (106) and (107) that

$$n_1 = \arg\left[\frac{c_2}{c_1} = \overline{F}_1\left(n_1\right)\right] = \arg\left[\frac{c_2}{c_1} = \left(1 + \frac{n_1 - s_1}{s_n}\right)^{-(n-1)}\right]$$

$$= s_1 + s_n\left[\left(\frac{c_1}{c_2}\right)^{1/n} - 1\right]. \quad (108)$$

Step 3.

$$n_2 = \min\left[0, N - \left(s_1 + s_n\left[\left(c_1/c_2\right)^{1/n} - 1\right]\right)\right]. \quad (109)$$

The proposed policies of the dynamic adaptive airline seat inventory control are based on the use of order statistics of cumulative customer demand, which have such properties as bivariate dependence and conditional predictability. Dynamic adaptation of the airline seat reservation system to airline customer demand is carried out via the bivariate dependence of order statistics of cumulative customer demand. Dynamic anticipatory adaptive optimization of the airline seat allocation includes dynamic anticipatory adaptive nested optimization of protection levels over time T. It is carried out via the conditional predictability of order statistics. The airline seat reservation system makes online decisions as to whether to accept or reject any customer request using established decision rules based on order statistics of the current cumulative customer demand. The computer simulation results are promising.

6.2 Model of Dynamic Adaptive Control of Airline Seat Protection Levels for Nested Fare Classes under Parametric Uncertainty

For example, consider a single-leg flight with two fare classes (business and economy) for a single departure date with predefined reading dates at which the dynamic policy is to be updated, i.e., the booking period before departure is divided into h reading periods: (τ_0=0, τ_1], (τ_1, τ_2], …, (τ_{h-1}, τ_h] determined by the h reading dates: τ_1, τ_2, …, τ_h. These reading dates are indexed in increasing order: $0<\tau_1<\tau_2< \cdots <\tau_h$, where ($\tau_{h-1}$, τ_h] denotes the reading period immediately preceding a departure, and τ_h is at departure. Typically, the reading periods that are closer to departure cover much shorter periods of time than those further from departure. For example, the reading period immediately preceding departure may cover 1 day whereas the reading period (1 month) from departure may cover 1 week.

Let us suppose that the cumulative customer demand for the high (business) fare class at the kth reading date (time τ_k, $1\le k\le h$) is U_k representing the kth order statistic from the underlying distribution with the probability density function $f_\omega(u)$ and cumulative distribution function $F_\omega(u)$, where ω is a parameter (in general, vector). This parameter is assumed to be unknown, but there is a sample of order statistics $U_1 \le ... \le U_h$ (statistical estimates of cumulative customer demands for the high (business) fare class of past flights).

Also, suppose that the cumulative customer demands for the high and low fare classes are stochastically independent. Each booking of a seat of the high fare class in the reading period (τ_{k-1}, τ_k] generates revenue of c_1. Each booking of a seat of the low fare class in the reading period (τ_{k-1}, τ_k] generates revenue of c_2, where $c_1 > c_2$ for all $k\in\{1, …, h)$. Then the model of optimal statistical estimation of airline seat protection levels for nested fare classes under parametric uncertainty includes the following steps:

Step 1. Let's assume that U_k , $\forall k \in \{1, ..., h\}$, represents the kth order statistic from the two-parameter exponential distribution (38), where the parameter ω is unknown. It follows from (91) that

$$\int_0^{GPQ5} \varphi_{r-k,h-r+1}(z)dz$$

$$= P_\omega(U_r \le u_r \mid U_k = u_k; h) = \alpha, \quad (110)$$

where

$$GPQ5 = 1 - \frac{\overline{F}_\omega(u_r)}{\overline{F}_\omega(u_k)} = q_{r-k,h-r+1;\alpha} \text{ (α-quantile)}, \quad (111)$$

$$\overline{F}_\omega(u) = \exp\left(-\frac{u-\upsilon}{\rho}\right). \quad (112)$$

It follows from (111) and (112) that

$$u_r = u_k + \rho\ln\left(\frac{1}{1-q_{r-k,h-r+1;\alpha}}\right). \quad (113)$$

Step 2. Assuming r=h, ρ=ρ_k and α=α_k, it follows from (113) that

$$u_h = u_k + \rho_k \ln\left(\frac{1}{1-q_{h-k,1;\alpha_k}}\right). \quad (114)$$

Step 3. For each k=1, …, h-1 we define ρ_k and α_k such that

$$u_h = u_k + \rho_k \ln\left(\frac{1}{1-q_{h-k,1;\alpha_k}}\right)$$

$$= n_1 = \arg\left[\frac{c_2}{c_1} = \overline{F}_1(n_1)\right] = s_1 + s_n\left[\left(\frac{c_1}{c_2}\right)^{1/n} - 1\right] \quad (115)$$

It should be noted that equation (115) is used to determine the exact fragment estimate

$$\rho_k \ln\left(\frac{1}{1-q_{h-k,1;\alpha_k}}\right),$$

$$\forall k \in \{1, ..., h-1\}, \quad (116)$$

based on accurate statistical information obtained from the process of selling and reserving air tickets for the high (business) fare class and past single-leg flights.

Step 4. For a new flight with new cumulative customer demand values of U_k^{new} for each k=1, …, h-1, the exact fragment estimate (116) can be used for dynamic adaptive control of airline seat protection level for the high (business) fare class under parametric uncertainty of customer demand models during the process of selling and reserving air tickets for a new future flight as follows:

$$N_1^{new} = U_k^{new} + \rho_k \ln\left(\frac{1}{1-q_{h-k,1;\alpha_k}}\right),$$

$$\forall k \in \{1, ..., h-1\}, \quad (117)$$

where N_1^{new} is the dynamic adaptive airline seat protection level for the high (business) fare class under parametric uncertainty of customer demand models during the process of selling and reserving air tickets for a new future flight.

7 Conclusion

New rigorous formulations of the problems of statistical optimization and dynamic adaptive control of airline seat protection levels for several nested fare classes under parametric uncertainty of consumer demand models are presented. Several results useful for practical application have been obtained. Illustrative examples are given.

The new intelligent analytical technique proposed in this paper represents the conceptually simple, efficient, and useful method for constructing optimal airline seat protection levels for multiple nested fare classes of single-leg flights with any practical number l (≥2) of nested fare classes. The technique yields the optimal allocation of airline seats between l dependent (i.e., nested) fare classes, subject to N (the total airplane seat capacity), that takes into account not only the past observations but also the future observation program and the associated statistics. The optimum procedure is to consider the situation as a dual optimization of allocation problem where information and action are interrelated.

The technique used in this article is based on a probability transformation and pivotal quantity averaging, [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]. It is conceptually simple and easy to use.

The methodology presented in this article can be useful for solving problems of the optimal allocation of resources in physics and engineering.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Nicholas Nechval carried out the proof of theorems, which are the basis of analytical methodology.
Gundars Berzins, Konstantin Nechval were responsible for illustrative examples.

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Conflict of Interest

The authors have no conflict of interest to declare.