A Special Note on the Logistic Functions with Complex Parameters and Some of Related Implications

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Abstract: - By this special note, certain necessary information pertaining to the logistic function together with some of its special forms (with real parameters) will be firstly introduced, and some results consisting of several differential inequalities associated with various versions of the complex logistic functions will be then determined. In addition, a number of special implications concerning those results will be also indicated.

Key-Words: - The complex plane, analytic functions, power series, the logistic function, the sigmoid function, the modified sigmoid function

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1 Introduction and Preliminary Information

As it is well known, in written literature, especially, in mathematical science, there are many special functions. These functions, which possess a wide range of applications, appear as functions with both real independent variables and complex independent variables. One of those functions is the logistic function, or, more specifically, the sigmoid function. In this special research, some necessary-basic information about those special functions (with real independent parameters) will be firstly introduced, and for some of the possible results of our main research, which will consist of various theoretical information, some of the numerous complex forms of those indicated functions will be then concentrated.

We can now begin to present (or evoke) various necessary information dealing with this special investigation.

For those, we begin by introducing the logistic function with a real variable (or parameter) (or the sigmoid function with a real variable (or parameter)). We denote this special function (with real variable) by the following-equivalent notations:

$$\mathrm{L}(x) \equiv \mathrm{L}_{\lambda,\ell}(x_0;\ x)$$

and also define by the form given by

$$\mathrm{L}_{\lambda,\ell}(x_0;\ x) = \frac{\ell}{1+\mathrm{e}^{-\lambda(x-x_0)}}\,, \qquad (1)$$

where, of course, the associated parameters x_0, ℓ ($\ell \neq 0$) and λ ($\lambda \neq 0$) belong to the set $\mathbb{R}$ consisting of the real numbers, which are usually expressed by the midpoint, the maximum value, and the logistic growth rate (or steepness) of the related function $\mathrm{L}(x)$, respectively. Notably, in terms of these specific technical terms, we should emphasize here that values greater than zero of the parameters ℓ and λ are significant values that can be reasonable for various applications of the logistic function with real parameters.

As certain theoretical-historical information, the important function just above was introduced in a series of three basic papers by Pierre Francois Verhulst between the years 1838 and 1847, who devised it as a classical model of population growth by adjusting the exponential growth model, under the guidance of Adolphe Quetelet. For each of their details, respectively, one may refer to the primary works given by, [33], [34], [35] in the references.

Most particularly, in the literature, it also appeared as a more specialized form of the logistic function $\mathrm{L}(x)$, which is generally denoted by the notation given by

$$s(x) := \mathrm{L}_{1,1}(0;\ x)$$

and also called the standard logistic function (or the sigmoid function), being of the form given by

$$s(x) = \frac{1}{1 + e^{-x}}, \tag{2}$$

where the main variable x will belong to any interval, determined by the relevant field of related study, in the well-known set $\mathbb{R}$.

Specially, as a great number of interesting implications of various special functions (with real (or complex) parameters), those two special functions (just above) has an important part in numerous applications in a range of scientific fields including biology, ecology, biomathematics, demography, chemistry, sociology, political science, geoscience, economics, mathematical psychology, linguistics, probability, statistics, mathematics, artificial neural networks and so on. For some different applications of them, as an example, one may concentrate on the relevant results in the earlier papers given by, [3], [4], [6], [19], [20], [24], [25], [27], [29], [31], [32].

For both necessary information and the scope of this investigation, there is a need to evoke the well-known Taylor-Maclaurin series expansions of the following-elementary functions with real variable x:

$$u(z) \coloneqq e^x \text{ and } v(z) \coloneqq \frac{1}{1+x}$$

given by

$$u(z) = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{s!}x^s + \cdots \tag{3}$$

and

$$v(z) = 1 - x + x^2 + \cdots + (-1)^s x^s + \cdots, \tag{4}$$

respectively.

By means of the well-known series expansions presented in (3) and (4), the following equivalent-extensive relations:

$$\begin{aligned} \mathrm{L}(x) \equiv\ & \mathrm{L}_{\lambda,\ell}(x_0;\ x) \\ &= \frac{\ell}{1+e^{-\lambda(x-x_0)}} \\ &\equiv \ell \textstyle\sum_{n=0}^{\infty}\left(-e^{-\lambda(x-x_0)}\right)^n & (5) \\ &\equiv \frac{\ell}{1+e^{\lambda x_0}e^{-\lambda x}} \\ &\equiv \frac{\ell}{1+e^{\lambda x_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\lambda x)^n} & (6) \\ &\equiv \textstyle\sum_{m=0}^{\infty}\left(-e^{\lambda x_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\lambda x)^n\right)^m & (7) \end{aligned}$$

can be easily constituted, where

$$\ell \in \mathbb{R}^* \coloneqq \mathbb{R} - \{0\}, \lambda \in \mathbb{R}^*, \ x_0 \in \mathbb{R}^* \tag{8}$$

As has been informed above, we have mentioned some application areas of both the function $\mathrm{L}(x)$ and the function $s(x)$, and we have also offered numerous basic references for those.

However, since the basic purpose of this special research will relate to certain complex forms of those functions indicated above, some extra information and also references about both certain functions and special functions with complex variables will also be needed. For more information about both the relevant special functions, introduced by (1) and (2), and, especially, possible complex forms of those functions, the main works given by the references in, [2], [5], [7], [9], [13], [15], [16], [21], [22], [25], [28], [38], [41] can be recommended to concerned readers as main works (or scientific researchers). Furthermore, under some extra conditions associated with the mentioned parameters similar to (8) and for several relations in relation to various special functions (with complex parameters), a number of extra researches, given by, [1], [10], [12], [18], [22], [30], [37], [40], can be also checked as different type investigations for interested researchers.

2 Related Lemma and Main Results

In the previous section, we presented various basic information and numerous special results about the logistic function and its special forms with real variables (or parameters). In this section, the conditions in which some special cases of the conditions specified in (8) will be determined, by considering all possible-complex variable structures in relation to those relevant special functions, which are similar forms between (1)-(7), several new (special) results regarding those complex functions can be determined. Here, only two of those new results will be also constituted.

As it is well known, by making use of the important assertions presented by [14], the following-useful lemma was proven in the paper given by, [23], which will be needed for stating and also proving each one of our main results. At the same time, for extra information, it can be also focused on the earlier works given by, [8], [17], [26].

Lemma 1. Let $\mathbb{N} \coloneqq \{1,2,3,\ldots\}$ and also let

$$q(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (\forall s \in \mathbb{N}\,; c_s \in \mathbb{C}) \tag{9}$$

be an analytic function in the open unit disk:

$$\mathbb{U} \coloneqq \{\omega : \omega \in \mathbb{C} \text{ and } |\omega| < 1\},$$

and suppose that there exists a point ω_0 belonging to $\mathbb{U}$ such that

$$\Re e(q(z)) > 0 \text{ for } |z| \le |\omega_0| \tag{10}$$

and
$$\Re e\big(q(z)\big) = 0. \tag{11}$$
Then, the inequality:
$$\omega_0 q'(\omega_0) \leq -\frac{1}{2}(1 + |q(\omega_0)|^2) \tag{12}$$
is satisfied.

In light of the information indicated right after the second part and the equivalent forms of the functions consisting of real parameters between (5) and (7), if we consider the logistic function (with complex variable z), which has the same notation in the mentioned form:
$$\mathrm{L}(z) \equiv \mathrm{L}_{\lambda,\ell}(z_0;\ z)\,, \tag{13}$$
various new results relating to both these new-special functions consisting of complex mentioned parameters and their special forms can be also constituted, where
$$z_0 \in \mathbb{C}\ ,\ 0 < \lambda \leq \frac{\pi}{2} \quad \text{and} \quad \ell \in \mathbb{R}^*. \tag{14}$$

Since each one of the complex functions like the forms as in (13), which consists of any one of the same forms between (5)-(7), is an important function with the same properties as in the form in Lemma 1, the mentioned lemma will be interesting for stating and then proving each one of main results of this paper.

Let us now present some of our main results and then prove them by considering Lemma 1. As the first main result, by considering the form with complex parameters of the expression given in (6), the following theorem, Theorem 1, can be then constituted.

Theorem 1. Under the conditions specified by the admissible values of the parameters emphasized in (14), for the equivalent complex functions presented by (13), if
$$\Re e\left(\frac{ze^{-\lambda z_0}\frac{d}{dz}\left(\frac{\ell}{\mathrm{L}(z)}\right)}{1+e^{-\lambda z_0}\left(\frac{\ell}{\mathrm{L}(z)}-1\right)}\right) > -\frac{1+\left|e^{-\lambda z_0}\left(\frac{\ell}{\mathrm{L}(z)}-1\right)\right|^2}{2\left|1+e^{-\lambda z_0}\left(\frac{\ell}{\mathrm{L}(z)}-1\right)\right|^2} \tag{15}$$
holds, then
$$\Re e\left\{e^{-\lambda z_0}\left(\frac{\ell}{\mathrm{L}(z)} - 1\right)\right\} > 0 \tag{16}$$
holds, where $z \in \mathbb{U}$.

Proof. For the related proof, we want to use lemma 1. Therefore, first of all, of course, we need to describe a complex function like $q(z)$ that is as in the relevant lemma and also has the series-expansion form in (9). For it, under the mentioned conditions in (14), let us consider the implicit form consisting of both the function $q(z)$ situated as in Lemma and the complex logistic function indicated in (13), which is
$$\mathrm{L}(z) = \frac{\ell}{1+e^{\lambda z_0} q(z)} \quad (z \in \mathbb{U}). \tag{17}$$
Then, in consideration of the relation having the same form presented by (1), for a suitable-analytic function, the function $q(z)$ can be taken as the necessary form:
$$q(z) = e^{-\lambda z},$$
where
$$0 < \lambda \leq \frac{\pi}{2} \ \text{ and } \ z \in \mathbb{U}\,,$$
or, equivalently, when the series expansion possessing the same form given in (6) is considered here, it also has the complex series expansion given by
$$q(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\lambda z)^n\,,$$
where
$$0 < \lambda \leq \frac{\pi}{2} \ \text{ and } \ z \in \mathbb{U}.$$
As a result of a simple focus, clear, $q(z)$ both is an analytic function in disc U and satisfies the condition $q(0) = 1$. Namely, it is one of the main conditions of Lemma 1.

We next suppose that there exists a point $\omega_0 \in \mathbb{U}$, which satisfies the conditions given by (10) and (11) of the lemma. By means of the implicit function defined in (17), we then get that
$$\begin{aligned}
&\Re e\left(\left.\frac{ze^{-\lambda z_0}\frac{d}{dz}\left(\frac{\ell}{\mathrm{L}(z)}-1\right)}{1+e^{-\lambda z_0}\left(\frac{\ell}{\mathrm{L}(z)}-1\right)}\right|_{z:=\omega_0}\right) \\
&= \Re e\left(\left.\frac{zq'(z)}{1+q(z)}\right|_{z:=\omega_0}\right) \\
&= \Re e\left(\frac{\omega_0 q'(\omega_0)}{1+q(\omega_0)}\right) \\
&= \Re e\left(\frac{\omega_0 q'(\omega_0)}{|1+q(\omega_0)|^2}\overline{1+q(\omega_0)}\right) \\
&= \frac{\omega_0 q'(\omega_0)}{|1+q(\omega_0)|^2}\Re e\left(\overline{1+q(\omega_0)}\right) \\
&= \frac{\omega_0 q'(\omega_0)}{|1+q(\omega_0)|^2}\left[1+\Re e\left(\overline{q(\omega_0)}\right)\right],
\end{aligned} \tag{18}$$
and, by using the mentioned conditions in (10) and (11) of Lemma 1 for the result determined by (18), we then arrive at
$$\Re e\left(\left.\frac{ze^{-\lambda z_0}\frac{d}{dz}\left(\frac{\ell}{\mathrm{L}(z)}\right)}{1+e^{-\lambda z_0}\left(\frac{\ell}{\mathrm{L}(z)}-1\right)}\right|_{z:=\omega_0}\right)$$

$$= \frac{\omega_0 q'(\omega_0)}{|1+q(\omega_0)|^2}\left[1+\Re e\left(\overline{q(\omega_0)}\right)\right]$$
$$= \frac{\omega_0 q'(\omega_0)}{|1+q(\omega_0)|^2}$$
$$\leq -\frac{1+|q(\omega_0)|^2}{2|1+q(\omega_0)|^2}$$

or, equivalently, with help of the implicit function formed by (17), we next derive that

$$\Re e\left(\left.\frac{ze^{-\lambda z_0}\frac{d}{dz}\left(\frac{\ell}{\mathbf{L}(z)}\right)}{1+e^{-\lambda z_0}\left(\frac{\ell}{\mathbf{L}(z)}-1\right)}\right|_{z:=\omega_0}\right)$$
$$\leq -\frac{1+|q(\omega_0)|^2}{2|1+q(\omega_0)|^2}$$
$$= -\frac{1+\left|e^{-\lambda z_0}\left(\frac{\ell}{\mathbf{L}(z_0)}-1\right)\right|^2}{2\left|1+e^{-\lambda z_0}\left(\frac{\ell}{\mathbf{L}(z_0)}-1\right)\right|^2},$$

which also is inconsistent with the hypothesis of Theorem 1 in equation (15). Therefore, in accordance with

$$z=\rho e^{i\Theta}\ (0\leq\rho<1)\ ,\ 0\leq\Theta<2\pi$$

and

$$0<\lambda\leq\frac{\pi}{2},$$

one can easily arrive at

$$\Re e\big(q(z)\big)=\Re e\big(e^{-\lambda z}\big)$$
$$=\Re e\left(e^{-\rho\lambda e^{i\Theta}}\right)$$
$$=e^{-\rho\lambda e^{i\cos\Theta}}\cos(\rho\lambda\sin\Theta)$$
$$>0,$$

which immediately requires the inequality given by (16). So, this also ends the desired proof.

As the second main result, by considering the form with complex parameters of the expression given in (7), the following theorem, Theorem 2, can be also composed.

Theorem 2. Let $z\in\mathbb{U}$ and $\tau\in\boldsymbol{Z}:=\mathbb{Z}^+\cup\mathbb{Z}^-$. Under the conditions specified by the admissible values of the parameters established by (14), for the equivalent complex functions presented by (13), if any one of the cases constituted by the conditions:

$$\Re e\left\{\frac{z\mathbf{L}'(z)}{\mathbf{L}(z)}\left(\frac{\mathbf{L}(z)}{\ell}\right)^{\tau}\right\}$$
$$\begin{Bmatrix} > -\frac{1}{2\tau}\left(1+\left|\frac{\mathbf{L}(z)}{\ell}\right|^{2\tau}\right) \ if\ \tau\in\mathbb{Z}^+ \\ < -\frac{1}{2\tau}\left(1+\left|\frac{\mathbf{L}(z)}{\ell}\right|^{2\tau}\right)\ if\ \tau\in\mathbb{Z}^- \end{Bmatrix} \quad (19)$$

holds, then

$$\Re e\left\{\left(\frac{\mathbf{L}(z)}{\ell}\right)^{\tau}\right\}>0 \quad (20)$$

holds, where the value of the complex power stated in (19) (or in (20)) is considered as its principal value, and, of course, the mentioned notations: $\mathbb{Z}^+$ and $\mathbb{Z}^-$ denote the set of positive integers and the set of negative integers, respectively.

Proof. First of all, let $z\in\mathbb{U}$ and $\tau\in\boldsymbol{Z}$, and also let the mentioned conditions satisfy those complex parameters given by (14). Then, for its pending proof, in view of the relation in (7) and by considering an analytic function $q(z)$ being of the form given by

$$\left(\frac{\mathbf{L}(z)}{\ell}\right)^{\tau}=\left(\sum_{m=0}^{\infty}\left(-e^{\lambda z_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\lambda z)^n\right)^{\tau}\right)$$
$$=$$
$$q(z), \quad (21)$$

Lemma 1 can be considered again as it was used in the proof of Theorem 1. Because this defined function with complex variable, namely, $q(z)$ satisfies the hypotheses given in Lemma 1. It also follows from (21) that

$$z\frac{d}{dz}\left\{\left(\frac{\mathbf{L}(z)}{\ell}\right)^{\tau}\right\}=\frac{z\mathbf{L}'(z)\,[\mathbf{L}(z)]^{\tau-1}}{\ell^{\tau}}$$
$$=zq'(z)\quad(z\in\mathbb{U}). \quad (22)$$

From here on, the similar ways, which have been followed in the proof of Theorem 1, are then considered for the equation obtained in (22) and if Lemma 1 is also used there, all necessary steps of the desired proof can be easily constructed. Consequently, its proof is omitted here and its detail is also brought to the attention of interested researchers.

3 Conclusion and Recommendations

In this last section, we want to put emphasis on certain special information. In this present research, it is clearly seen that the fundamental theorem of our research is directly related to the theory of complex functions. For the emphasized information, one can center on the references in, [2], [8], [14], [25], [26], [36], [37].

As the first implication, especially, we have focused on only two of any number of the possible theories relating to the logistic function with complex parameter by considering the basic relations regarding the logistic function with real parameter set out in the equivalent-equations presented by (1), (5), (6) and (7). These presented theories are only two important results and many new theories can be also created by making use of the main-highlighted relations, which will be two interesting examples for interested researchers.

As the next implications, all right, various special (or remodified) results of the logistic function can be also determined. For those, of course, various type sigmoid functions with the complex variable, which are some special cases of the special function with real parameters introduced by (2), can be also concentrated as certain interesting implications. Those modified functions with complex variables, are frequently encountered in the mathematical literature and include various different results in many different fields. For some of them, especially, see also the papers in, [5], [6], [10], [11], [12], [13], [18], [28], as certain special examples. At the same time, for other possible implications relating to those complex functions, it is enough to select the values of the mentioned parameters contained in the definition in (13), of course, under the conditions accentuated in (14).

For extra information relating to various possible results of those special functions with real (or complex) variable, the research works presented in, [2], [8], [12], [15], [20], [23], [25], [26], [34] can also be as some main works for related researchers. These extensive results are only two important implications of our main results and they also associate with Theorem 1. In addition, of course, to reveal other extra new special implications concerning our main results, it will be enough to consider the appropriate parameters used in their respective theorems. Nevertheless, as a special implication of our investigation concerning the sigmoid function with complex variables, we want to present it as a proposition.

When taking account of the special relationship between the logistic function (with real parameters) in (1) and the sigmoid function (with real parameters) in (2), of course, naturally, there is a matter of the following important relationships between the logistic function and certain exponential type functions, which are the special functions with complex variable given by the following forms:

$$s(z) = \frac{1}{1+e^{-z}} \equiv \mathrm{L}_{1,1}(0;z) \tag{23}$$

and

$$ms(x) = \frac{2}{1+e^{-z}} \equiv \mathrm{L}_{1,2}(0;z) \tag{24}$$

where $z \in \mathbb{U}$. We remark that the function $ms(x)$ is known as the modified Sigmoid function in the literature.

As additional extra information, it is clear that when considering the complex exponential function:

$$k(z) = e^{-z} \quad (z = x + iy),$$

the following function:

$$\begin{aligned}\Re e(e^{-z}) &= \Re e\left(e^{-x-iy}\right)\\ &= e^{-x}\cos(-y)\\ &\equiv e^{-x}\cos(y)\end{aligned}$$

depends upon the sign of the trigonometric function with real variable given by the form:

$$g(y) = Cos(y).$$

Naturally, in terms of the corresponding complex exponential function $k(z)$, which will play an important role both in the complex logistic function and also in its special forms, the real parameter x and the imaginary parameter y change in the open interval (-1,1) when the complex parameter z changes in the open set $\mathbb{U}$.
Indeed, since

$$x, y \in (-1,1) \subset \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \Rightarrow [e^{-x} > 0 \text{ and } \cos(y) > 0] \tag{25}$$

for all $z = x + iy \in \mathbb{U}$, it is obvious that

$$\Re e(e^{-z}) = e^{-x}\cos(y) > 0. \tag{26}$$

In special, we should point out here that the complex functions:

$$s(z) \quad \text{and} \quad ms(x)$$

are also known as, respectively, the (complex) sigmoid function and the modified (complex) sigmoid function in the literature. For both functions above, we think that the earlier results, cited in, [12], [18], [19], [28], [30], [37], [39], [40], in the references, are interesting papers containing various analytical-geometric results of functions with complex variables. So, for those special functions presented in (23) and (24), by leaving the details of the results to the interested readers, we just want to present a new special implication regarding the complex function given in (23).

For only one of the indicated implications of Theorem 1, in consideration of the information in (7) (or (6)) along with (25) and (26), by means of Theorem 1 and also by choosing the values of the parameters λ as $\lambda := 1$, the value of the parameter ℓ as $\ell := 1$ and the value of the parameter z_0 as $z_0 := 0$ there, namely, by using the special function with complex variable composed as in (23), which is the complex-sigmoid function being also of the series expansion given by

$$\frac{1}{1+e^{-z}} \equiv \frac{e^z}{1+e^z} = \frac{1}{2} + \frac{1}{4}z - \frac{1}{48}z^3 + \frac{1}{480}z^5 + \cdots,$$

the following-special proposition can be easily created as only one of a great number of the possible implications of Theorem 1.

Proposition 1. Let $z \in \mathbb{U}$. Then, the assertion:

$$\Re e\left(\frac{z}{1+e^z}\right) > -\frac{1+e^{2\Re e(z)}}{2|1+e^z|} \Rightarrow \Re e\left(\frac{1}{e^z}\right) > 0$$

holds true.

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