Freezing Sets Invariant-based Characteristics of Digital Images

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Abstract: - Due to the widespread use of digital images of real-world objects as mathematical models, this research examines the freezing sets invariant-base properties of digital images. In contrast to earlier studies that only covered a discrete or limited collection of points, fixed points of digitally continuous functions are approved to deal with a variety of characteristics of digital images.

Key-Words: - Digital image, Freezing sets, Boundary, Irreducible

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1 Introduction

Mathematical models commonly use illustrations of the world's objects. A digital representation of the notion of a continuous function, which was drawn from topology, is usually useful for the analysis of digital images. However, the digital picture is frequently a distinct, limited collection of points. As a result, methods other than topology-based methods for digital picture analysis are usually needed. In this work, we examine a number of digital picture features that are connected to the fixed points of digitally continuous functions.

These characteristics include discrete measurements that do not naturally correspond to the characteristics of $\mathbb{R}^n$ subsets. (U, κ) is a digital image where for some integer n, $U \subset \mathbb{Z}^n$ and κ is an adjacency on U which is considered to be finite, [6]. If U is a vertex set and κ is an edge set, then the pair (U, κ) is a graph. Adjacency is a measure of how "closedness" two points are to one another in $\mathbb{Z}^n$. When these conditions (finiteness of X) and (closedness of adjacency points) are satisfied, the digital image may be viewed as a model of a white-and-black "real world" image, where white points in the background are declared by elements of $\mathbb{Z}^n - \{U\}$ and the black points in the foreground by members of U, [1].

$\alpha\beta$ indicates that α and β are $\kappa -$adjacent and $\alpha \leftrightarrows \beta$ are $\kappa -$adjacent or equal.

If z is an integer such that $1 \leq z \leq n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n\}) \neq (\beta_1, \beta_2, \ldots, \beta_n) = \beta$, then $\alpha \leftrightarrow_{c_z} \beta$ iff

(i) For at most indices i, $|\alpha_i - \beta_i|$.

(ii) For all indices j, $|\alpha_i - \beta_i|$ implies $\alpha_j = \beta_j$.

The number of adjacent points is frequently used to indicate the $c_z -$adjacencies.

Examples:

(i) The $2 -$adjacency in $\mathbb{Z}$ is $c_1 -$adjacency.

(ii) The $4 -$adjacency is $c_1 -$adjacency is and the $8 -$adjacency in $\mathbb{Z}^2$ is $c_2 -$adjacency.

(iii) The $8 -$adjacency is $c_1 -$adjacency, $18 -$adjacency is $c_2 -$adjacency, and $26 -$adjacency in $\mathbb{Z}^3$ is $c_3 -$adjacency.

If (U, κ) and (V, λ) are two digital images, then $NP(\kappa, \lambda)$ denotes the strong product adjacency or normal adjacency, [2], on $U \times V$ iff

$\forall \alpha_0, \alpha_1 \in U$ and $\beta_0, \beta_1 \in V$ where $p_0 = (\alpha_0, \beta_0) \neq p_1 = (\alpha_1, \beta_1)$, $p_0 \leftrightarrow_{NP(\kappa,\lambda)} p_1$ if one of the following conditions is valid:

(i) $\alpha_0 \leftrightarrow_\kappa \alpha_1$ and $\beta_0 = \beta_1$.

(ii)$\beta_0 \leftrightarrow_\kappa \beta_1$ and $\alpha_0 = \alpha_1$.

(iii) $\alpha_0 \leftrightarrow_\kappa \alpha_1$ and $\beta_0 \leftrightarrow_\kappa \beta_1$.

Typically if z and υ are two natural numbers such that $1 \leq z \leq \upsilon$, (U_i,κ_i) $\forall$ $1 \leq i \leq \upsilon$ and $U = \prod_{i=1}^{\upsilon} U_i$, then the adjacency $NP_z(\kappa_1, \kappa_2, \ldots, \kappa_\upsilon)$, [3], is defined as: For some α_i and $\alpha_{i'}$ in U_i, if $p = (\alpha_1, \alpha_2, \ldots, \alpha_\upsilon) \neq q = (\alpha_1{}', \alpha_2{}', \ldots, \alpha_\upsilon{}')$, then: $p \leftrightarrow_{NP_z(\kappa_1,\kappa_2,\ldots,\kappa_\upsilon)}$q if for at least 1 and at most z indices i, $x_i \leftrightarrow_{\kappa_i} x_i{}'$ and $\forall$ j indices, $\alpha_j = \alpha_j{}'$. In this paper, "digital images" is referred to as D.I.

2 (κ, λ) −Digitally Continuous Function

Definition 1.1:

i. [4], If (U, κ) and (V, λ) are two D.I, then $f: U \to V$ is a (κ, λ) −digitally continuous function, if $(U, \kappa) = (V, \lambda)$, then f is (κ, κ) −continuous.

ii. The path from α to β is the set $\{\alpha_i\}^m{}_{i=0}$ such that $\alpha_0 = \alpha$, $\alpha_m = \beta$ and $\alpha_i \rightleftarrows \alpha_{i+1}$ $\forall$ $i = 1,2,\ldots, m-1$ $\forall$ $\alpha, \beta \in U$.
Now, if $\alpha_i \neq \alpha_j$ $\forall$ $i \neq j$, then the length of the path is m.

iii. The path from α to β is a $(2, \kappa) - P$, where $P: [0, m]_{\mathbb{Z}} \to U$ is a continuous function $\forall$ $m \in \mathbb{Z}$ and $P(0) = \alpha$ and $P(m) = \beta$.

Theorem 1.2: [5], If (U, κ) and (V, λ) are two D.Is', then:

i. $f: U \to V$ is a (κ, λ) −digitally continuous function iff $\forall$ $\alpha, \beta \in U$, if $\alpha \leftrightarrow_\kappa \beta$, then $f(\alpha) \rightleftarrows_\lambda f(\beta)$.

ii. If (Z, γ) is a D.I and $g: (V, \lambda) \to (Z, \gamma)$ is (λ, γ) −continuous, then $g \circ f: (U, \kappa) \to (Z, \gamma)$ is (κ, γ) −continuous.

Definition 1.3:

i. [1], [6], Let (U, κ) and (V, λ) be two D.I and, $f, g: (U, \kappa) \to (V, \lambda)$ are two (κ, λ) −continuous functions and $h: [U \times, m]_{\mathbb{Z}} \to V$ is defined as $h(\alpha, 0) = f(\alpha)$ and $h(\alpha, m) = g(\alpha)$ $\forall$ $m \in \mathbb{Z}$ and $\alpha \in U$.

ii. A function h is a digital (κ, λ) −homotopy, and f, g are (κ, λ) digitally homotopic in V (denoted by $f \sim g$.

iii. If $h(\alpha, t) = \alpha$ $\forall$ $t \in [0, m]_{\mathbb{Z}}$, then h holds α fixed.

iv. [1], If A is a subset of U and $r: U \to A$ is a κ −continuous function, then r is a retraction. If $r(a) = a$ $\forall$ $a \in A$, then A is a retract.

v. If $i: A \to U$ is an inclusion function, and $i \circ r \sim_\kappa id_U$, then A is a κ −deformation retract of U.

iv. The function $f: (U, \kappa) \to (V, \lambda)$ is an isomorphisim (homeomorphisim) if f is a bijective continuous function and f^{-1} is continuous.

v. If (U, κ) is a digital image, then $C(U, \kappa) = \{f: U \to U: f$ is continuous$\}$.

vi. If $f(\alpha) = \alpha$ $\forall$ $\alpha \in U$ and $f \in C(U, \kappa)$, then α is a fixed point.

vii. $Fix(f)$ is the set of all fixed points of U.

Theorem 1.4: [3], If (U_i,κ_i) and (V_i, λ_i) are D.I $\forall$ $1 \leq i \leq \nu$, f_i: (U_i,κ_i) $\to$$(V_i, \lambda_i)$ and f:$\prod_{i=1}^{\nu} U_i$ $\to$$\prod_{i=1}^{\nu} V_i$ given by $f(\alpha^1, \alpha^2, \ldots, \alpha_\nu) = \left(f^1(\alpha^1), f^2(\alpha^2), \ldots, f_\nu(\alpha_\nu)\right)$ which is $(NP_\nu(\kappa_1, \kappa_2, \ldots, \kappa_\nu), NP_\nu(\lambda_1, \lambda_2, \ldots, \lambda_\nu))$ − continuous iff f_i is (κ_i,λ_i) −ψontinuous $\forall$ $\alpha_i \in U_i$.

Definition 1.5, [1]:

i. A continuous function $f: (U, \kappa) \to (V, \lambda)$ is rigid if there is no continuous map homotopic to f except itself.

ii. U is rigid if $id: (U, \kappa) \to (U, \kappa)$ is rigid.

iii. [7], If a finite image U is homotopy equivalent to an image with fewer points, it is said to be reducible. Otherwise, U is irreducible.

iv. [1], If (U, κ) is irreducible, then for some point $\alpha \in U$ $\exists f \in C(U, \kappa)$ such that id$\simeq_\kappa f$ and $\alpha \notin f$, α is a reduction point.

Remark 1.6: [7], A finite image (U,κ) is reducible if $id:(U,\kappa)\to(U,\kappa)$ is homotopic to a non-surjective function.

Definition 1.7: [8], For the D.I (U,κ) and $f\in C(U,\kappa)$:

i. The set $S(f)=\{no.Fix(h): h\sim_\kappa f\}$ is the homotopy fixed point spectrum of the function .

ii. The set $S(f,\alpha_0)=\{no.Fix(h): h\sim_\kappa f$ holding α_0 fixed$\}$ is the pointed homotopy fixed point spectrum of the function f for some $\alpha_0\in Fix(f)$.

iii. The set $F(U,\kappa)=\{no.Fix(f): f\in C(U,\kappa)\}$ is the fixed point spectrum of (U,κ).

iv. The set $F(U,\kappa,\alpha_0)=\{no.Fix(f): f\in C(U,\kappa), \alpha_0\in Fix(f)\}$ is the pointed fixed point spectrum of (U,κ, α_0).

Theorem 1.8: i. [8], If V is a retract of the D.I (A,κ), then $F(A)\subset F(U)$.

ii. If (A,κ,α_0) is a retract of (U,κ,α_0), then $F(A,\kappa,\alpha_0)\subset F(U,\kappa,\alpha_0)$

3 Freezing Sets

Definition 2.1: [1], If (U,κ) is a D.I, then A is a freezing subset for U if $A\subset Fix(g)\ \Rightarrow g=id_U$ for some $g\in C(U,\kappa)$

Theorem 2.2: If (U,κ) is a D.I and A is a freezing subset for U, then:

i. id_A has a unique extension of id_U to a member of $C(U,\kappa)$.

ii. If $h;(U,\kappa)\to(V,\lambda)$ is an isomorphisim, $g:(U,\kappa)\to(V,\lambda)$ is continuous and $h|_A=g|_A$ then $g=h$.

iii. A continuous function $f:(A,\kappa)\to(V,\lambda)$ has one extension to an isomorphism $F:(U,\kappa)\to(V,\lambda)$.

Lemma 2.3: Freezing sets are topological invariants.

Theorem 2.4: If (U,κ) is a D.I and V is a freezing subset for U and $f:(U,\kappa)\to(V,\lambda)$ is an isomorphism, then $f(A)$ is a freezing set for (V,λ).

Proof: Suppose that $f\in C(U,\kappa)$ and $f|_{F(A)}=id_V||_{F(A)}$. Now, $f\circ F(A)=f|_{F(A)}\circ F|_{F(A)}=id_V|_{F(A)}\circ F|_{F(A)}=F|_{F(A)}$ and by theorem 3.2,

$f\circ F=F$, then

$$f=(f\circ F)\circ F^{-1}=F\circ F^{-1}=id_V$$

Thus $F(A)$ is a freezing set for V.

Theorem 2.5: If $(U,C_Z)\subset\mathbb{Z}^n$ is a D.I for $z\in[1,n]$, $f\in C(U,c_z\})$, $\alpha,\alpha'\in U:\alpha\leftrightarrow_{c_z}\alpha'$ and $p_i(f(\alpha))\le p_i(\alpha)\le p_i(\alpha')$, then $p_i\ (f(\alpha))\le p_i(\alpha')$.

Proof: If $p_i\ (f(\alpha))\le p_i(\alpha)\le p_i(\alpha')$ and $p_i(\alpha)=m$ then $p_i(q')=m-1$.

Hence $p_i\ (f(\alpha))>m$, but $f\in C(U,c_Z)$, so $f(\alpha)\leftrightarrow_{c_z}\ f(\alpha')$.

Therefore, $p_i\ (f(\alpha))\le p_i(\alpha)\le p_i(\alpha')$.

Theorem 2.6:

i. If (U,κ) is a D.I and $A\subset U$ is a retract of U, then (A,κ) has no freezing sets for (U,κ).

ii. If (U,κ) is a reducible digital image and A is a freezing subset for U, then if $a\in U$ is a reduction point of U, $\alpha\in A$.

Proof: i. If $i:(A,\kappa)\to(U,\kappa)$ is an inclusion function, then $r:(U,\kappa)\to(A,\kappa)$ is a retraction and $f=i\circ r$ is (κ,κ) −continuous.

Now, $f|_A=id_A$, but f is not the identity function.

ii. If $a\in U$ is a reduction point of U,

then $\exists r: U\to U-\{a\}$ where $U-\{a\}$ has no freezing sets for (U,κ).

4 Boundaries of Freezing Sets

Definition 3.1:

i. [1], If (U,κ) is a D.I and $A\subset U$ is a freezing set for U, then A is minimal if no proper subset of A is a freezing set for U.

ii. [9], If $U\subset\mathbb{Z}^n$, then the boundary of U is $Bd(U)=\{\alpha:\alpha\leftrightarrow_{c_1}\beta$ for some $\beta\in\mathbb{Z}^n-U\}$.

iii. [1], The interior of U is $int(U) = U - Bd(U)$.

Theorem 3.2: Let $U \subset \mathbb{Z}^n$ be finite, A is a subset of U, $f \in C(U, c_z) \forall z \in [1, n]$. If $Bd(A) \subset Fix(f)$ and $Bd(A)$ is a freezing set for (U, c_z), then $A \subset Fix(f)$.

Proof: Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \notin Fix(f)$ but α is an interior point of A.

Now, $\exists j \in [1, n]$ such that $p_j(f(\alpha)) \neq \alpha_j$, and because of the finiteness, there exists a path $P = \{(\alpha_1, \alpha_2, \dots, \alpha_{j-1}, a_i, \alpha_{j+1}, \dots, \alpha_n\} \ \forall \ i \in [1, m]$.

For $i = 1$ and m, the path belongs to $Bd(A)$, and for $i = 2, \dots, m-1$, the path belongs to $int(A)$.

By theorem 3.5, the path does not belong to $Fix(f)$ for $i = 1$ and m which contradicts the assumption.

Thus, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \notin Fix(f)$.

Theorem 3.3: [10], If $\prod_{j=1}^{n} [0, m_j]_{\mathbb{Z}} \subset \mathbb{Z}^n$ such that $m_j > 1 \ \forall \ j$, then $Bd(U)$ is a minimal freezing set for (U, c_n).

Proof: Let $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in Bd(U) - A$ for some proper set A of $Bd(U)$.

For some index j, we have $\beta_j \in \{0, m_j\}$. If β_j=0, then for the function $f: U \to U$ given by

$f(\alpha) = \alpha \ \forall \ \alpha \neq \beta$ and $f(\beta) = f(\beta_1, \dots, \beta_{j-1}, \beta_{j+1}, \dots, \beta_n)$ we have $f \in C(U, c_n)\})$ where $f|_A = id_A$ and $f \neq id_U$.

Now, if $\beta_j = m_j$, then $f(\alpha) = \alpha \ \forall \ \alpha \neq \beta$ and $f(\beta) = (\beta_1, \dots, \beta_{j-1}, m_j - 1, \beta_{j+1}, \dots, \beta_n)$ where $f \in C(U, c_n)$ where $f|_A = id_A$ and $f \neq id_U$.

Theorem 3.4: If (U_i, κ_i) is a set of D.I $\forall i \in [1, v]_{\mathbb{Z}}$, U= $\prod_{i=1}^{v} U_i$ and a subset A of U is a freezing set for $\left(U, NP_v(\kappa^1, \kappa^2, \dots, \kappa_v)\right)$, then for the projection function $p_j: \prod_{i=1}^{v} U_i \to U_j$ given by $p(\alpha_1, \alpha_2, \dots, \alpha_v) = \alpha_j$ we have $p_i(A)$ is a freezing set for $(U_i, \kappa_i) \forall \ i \in [1, v]_{\mathbb{Z}}$.

Proof: Suppose $f_i \in C(U_i, \kappa_i)$ and $g: U \to U$ is defined as

$g(\alpha_1, \alpha_2, \dots, \alpha_v) = (f_1(\alpha_1), f_2(\alpha_2), \dots, f_v(\alpha_v))$, then $g \in C(U, NP_v(\kappa_1, \kappa_2, \dots, \kappa_v))$.

Now, $f_i(\alpha_i)$= $\alpha_i \ \forall \ i \in p_i(A)$, but A is a freezing set for U, hence $g = id_U$,

Therefore, f_i=id_{U_i}.

5 Conclusion

Freezing sets are topological invariants. So, If (U, κ) is a D.I, V is a freezing subset for U and $f: (U, \kappa) \to (V, \lambda)$ is an isomorphism, then $f(A)$ is a freezing set for (V, λ). $i: (A, \kappa) \to (U, \kappa)$ is an inclusion function, then $r: (U, \kappa) \to (A, \kappa)$ is a retraction and $f = i \circ r$ is (κ, κ) −continuous. Moreover, if $U \subset \mathbb{Z}^n$ is finite, A is a subset of U, $f \in C(U, c_z) \forall z \in [1, n]$. If $Bd(A) \subset Fix(f)$ and $Bd(A)$ is a freezing set for (U, c_z), then $A \subset Fix(f)$.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

-Eman Almuhur is in charge of both conceptualizing the research challenge and overseeing the effort.

-Eman A. AbuHijleh and Ghada Alafifi are in charge of doing the formal analysis and composing the paper's initial draft.

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