Banach Function Space Property of A New Type of Grand Lorentz Spaces

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Abstract: - The concept of Lorentz space has been generalized to the grand Lorentz space. In this paper, a new generalization of Lorentz spaces is defined as $L_{\theta}^{p,q)}\left(X,\Sigma,\mu\right)$ with $\theta\geq 0$ by using the maximal function of a measurable function. Besides an explicit proof of being Banach function space with a rearrangement invariant norm seemed to be missing in the literature. Therefore, the authors provided such proof which can serve as a reference for the next studies and literature and as a fundamental reference for subsequent results.

Key-Words: - Grand Lorentz space, Banach function space, Iwaniec-Sbordone space, Distribution function

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1 Introduction and Preliminaries

Iwaniec and Sbordone generalized the concept of Lebesgue spaces and introduced a new space of measurable, almost everywhere equal integrable function classes, which they called grand Lebesgue spaces. Now, let X be a locally compact Hausdorff space and suppose that $\left(X,\Sigma,\mu\right)$ is a finite measure space. According to [10], grand Lebesgue spaces are the collection of equivalence classes of functions obtained according to almost everywhere relation of all $\mu-$measurable functions defined on $\left(X,\Sigma,\mu\right)$ and denoted by $L^{p)}$ for $1<p<\infty$. For any $v\in L^{p)}$, the functional

$$\left\|v\right\|_{p)}=\sup_{0<\varepsilon<p-1}\varepsilon^{\frac{1}{p-\varepsilon}}\left(\int_X\left|v(x)\right|^{p-\varepsilon}d\mu\right)^{\frac{1}{p-\varepsilon}}$$

defines a norm on $L^{p)}$ and makes them Banach function spaces with rearrangement invariant norm. Also $L^{p}\subset L^{p)}\subset L^{p-\varepsilon}$ if $0<\varepsilon\leq p-1$. New results on grand Lebesgue spaces can be observed in current studies, [3], [6], [7], [9], [11], [12], [15]. Presented in terms of the Jacobian integrability problem, these works have proven useful in various applications of partial differential equations and variational problems, where they are used in the study of maximum functions, extrapolation theory, etc. The harmonic analysis of these spaces, and the related small Lebesgue spaces, has been intensively developed in recent years and continues to attract the attention of researchers due to various applications.

There have been several generalizations of the grand Lebesgue spaces in recent years. One such generalization denoted by $L^{p),\theta}\left(X\right)$ is defined as the set all functions belong to $\bigcap_{0<\varepsilon<p-1}L^{p-\varepsilon}$ in [8]. According to [1], [8], $L^{p),\theta}\left(X\right)$ is a rearrangement-invariant space, i.e. Banach function space generated with the rearrangement-invariant norm

$$\left\|v\right\|_{p),\theta}=\sup_{0<\varepsilon<p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\left(\int_X\left|v(x)\right|^{p-\varepsilon}d\mu\right)^{\frac{1}{p-\varepsilon}}$$
$$=\sup_{0<\varepsilon<p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\left\|v\right\|_{p-\varepsilon}$$

for all $v\in L^{p),\theta}\left(X\right)$ where $1<p<\infty$ and $\theta\geq 0$. $L^{p),\theta}\left(X\right)$ reduces to classical Lebesgue spaces $L^{p}\left(X\right)$ when $\theta=0$ and reduces to grand Lebesgue spaces $L^{p)}(X)$ when $\theta = 1$, [3], [8]. Also, we have $L^p(X) \subset L^{p),\theta}(X) \subset L^{p-\varepsilon}(X)$ for $0 < \varepsilon \le p-1$ and $L^p(X) \subset L^{p),\theta_1}(X) \subset L^{p),\theta_2}(X)$ for $0 \le \theta_1 < \theta_2$, [1], [8]. It is important to remember that the subspaces of simple functions S and the subspace of test functions $C_0^\infty(X)$ is not dense in $L^{p),\theta}(X)$. If we call the closure of $C_0^\infty(X)$ in $L^{p),\theta}(X)$ as $E^{p),\theta}(X)$, then

$$E^{p),\theta}(X) = \left\{ v \in L^{p),\theta}(X) : \lim_{\varepsilon \to 0^+} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|v\|_{p-\varepsilon} = 0 \right\}$$

and $L^{p),\theta_1}(X) \subset E^{p),\theta_2}(X)$ for $0 \le \theta_1 < \theta_2$, [8]. The Marcinkiewicz class, denoted by $weak-L^p(X)$ or $L^{p,\infty}(X)$, consists of all measurable functions $f : X \to \mathbb{C}$ such that $\sup_{\lambda>0} \lambda^p D_f(\lambda) < \infty$ where the distribution function of f is

$$D_f(\lambda) = \mu\left\{ x \in X : |f(x)| > \lambda \right\}, \quad \lambda \ge 0 .$$

Then $L^{p,\infty}(X) \subset L^{p),\theta}(X)$. It is commonly known that in the sense of [2], the grand and small Lebesgue spaces are Banach Function Spaces. An explicit proof of these facts seemed to be missing in the literature. In [1], the author provided such a proof, which can serve as a reference for the next studies and literature and as a fundamental reference for subsequent results. The proofs in [1], show that the grand Lebesgue spaces content all the axioms which are necessary to be Banach Function Spaces, including the Fatou property. These results are important because they establish the grand Lebesgue spaces as a well-behaved class of function spaces that can be used to study various problems in analysis and partial differential equations. Overall, the results in [1], fill an important gap in the literature and provide a solid foundation for further research in this area.

Throughout this paper $X = (X, \Sigma, \mu)$ will show a $\sigma-$finite measure space, and the collection of all extended scalar-valued (real or complex) $\mu-$measurable functions on X will be shown by M. Also, M_0 will stand for the class of functions in M which are finite-valued $\mu-$a.e. The function χ_U will be employed for the characteristic function of any subset U.

The distribution function of a complex-valued, measurable function f defined on the measure space X is

$$\lambda_f(y) = \mu\left\{ x \in X : |f(x)| > y \right\}, \; y \ge 0 .$$

The nonnegative rearrangement function $f^*(\cdot)$ of f is given by

$$\begin{aligned} f^*(t) &= \inf\left\{ y > 0 : \lambda_f(y) \le t \right\} \\ &= \sup\left\{ y > 0 : \lambda_f(y) > t \right\}, \; t \ge 0 \end{aligned}$$

where we assume that $\inf \varnothing = \infty$ and $\sup \varnothing = 0$. Likewise, the (average) maximal function $f^{**}(\cdot)$ of f which is defined on $(0, \infty)$ is given by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds.$$

Note that $\lambda_f(\cdot)$, $f^*(\cdot)$ and $f^{**}(\cdot)$ are right continuous and non-increasing functions.

The generalization of ordinary Lebesgue spaces are Lorentz spaces $L(p,q)$ which are the collection of all classes of the functions f such that $\|f\|_{p,q}^* < \infty$, where

$$\|f\|_{p,q}^* = \begin{cases} \left(\dfrac{q}{p} \displaystyle\int_0^\infty t^{\frac{q}{p}-1} \left[f^*(t) \right]^q dt \right)^{\frac{1}{q}}, 0 < p, q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), \qquad 0 < p \le \infty, \; q = \infty. \end{cases} \tag{1}$$

In general, however, $\|\cdot\|_{p,q}^*$ is not a norm if not $1 \le q \le p < \infty$ or $p = q = \infty$ since the Minkowski inequality may fail. But by replacing $f^*(\cdot)$ with $f^{**}(\cdot)$ in (1), we get that $L(p,q)$ is a normed space, with the functional $\|\cdot\|_{p,q}$ defined by

$$\|f\|_{p,q} = \begin{cases} \left(\dfrac{q}{p} \displaystyle\int_0^\infty t^{\frac{q}{p}-1} \left[f^{**}(t) \right]^q dt \right)^{\frac{1}{q}}, 0 < p, q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t), \qquad 0 < p \le \infty, \; q = \infty. \end{cases} \tag{2}$$

This functional defined in (2) is sub-additive and equivalent to (1) if $1 < p < \infty$ and $1 \le q \le \infty$, that is

$$\| f\|_{p,q}^{*} \le \| f\|_{p,q} \le \frac{p}{p-1}\| f\|_{p,q}^{*}.$$

Here the (left) first inequality is coming from the fact that $f^{*}(\cdot)\le f^{**}(\cdot)$ and the second (right) is an immediate consequence of Hardy inequality. For detailed knowledge of Lorentz spaces, we can refer to [3], [4], [5], [13], [14], and references therein.

2 Main Results

In [14], the grand Lorentz space $L^{p,q)}$ is defined as the collection of all the complex-valued, measurable functions which are defined on $(0,1)$ such that the quasi-norm $\| f\|_{p,q)}^{*}<\infty$ where

$$\| f\|_{p,q)}^{*} = \begin{cases} \sup\limits_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left[f^{*}(t)\right]^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}, 1\le p,q<\infty \\ \sup\limits_{0<t<1} t^{\frac{1}{p}} f^{**}(t), \quad 1\le p<\infty, q=\infty \end{cases} \tag{3}$$

for any $f\in L_{\theta}^{p,q)}$.

Using the maximal function $f^{**}(\cdot)$, instead of the nonnegative rearrangement $f^{*}(\cdot)$ used in the definition of grand Lorentz space defined in [14], we generalized the grand Lorentz spaces as follows.

Definition 1 The grand Lorentz space $L_{\theta}^{p,q)}$ is classes of all the complex-valued, measurable functions which are defined on the measure space $(0,\mu(X))$ such that $\| f\|_{p,q)}^{\theta}<\infty$ where

$$\| f\|_{p,q)}^{\theta} = \begin{cases} \sup\limits_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^{\theta}\int_0^{\mu(X)} t^{\frac{q}{p}-1}\left[f^{**}(t)\right]^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}, 1< p,q<\infty \\ \sup\limits_{0<t<\mu(X)} t^{\frac{1}{p}} f^{**}(t), \quad 1< p<\infty, q=\infty \end{cases} \tag{4}$$

for any $f\in L_{\theta}^{p,q)}$. In particular, if $1<p<\infty$ and $1\le q\le\infty$; $p=q=1$ or $p=q=\infty$, then the normed space $\left(L_{\theta}^{p,q)},\|\cdot\|_{p,q)}^{\theta}\right)$ is a Banach space.

Proposition 1 For any $f\in L_{\theta}^{p,q)}$, the inequality

$$\| f\|_{p,q)}^{\theta,*} \le \| f\|_{p,q)}^{\theta} \le \frac{p}{p-q}\| f\|_{p,q)}^{\theta,*}$$

exists, i.e. the quasi-norm $\| f\|_{p,q)}^{\theta,*}$ and $\| f\|_{p,q)}^{\theta}$ are equivalent.

Proof. Since $f^{*}(\cdot)\le f^{**}(\cdot)$, for any $f\in L_{\theta}^{p,q)}$, we have

$$\begin{aligned}\| f\|_{p,q)}^{\theta,*} &= \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^{\theta}\int_0^{\mu(X)} t^{\frac{q}{p}-1}\left[f^{*}(t)\right]^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}} \\ &\le \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^{\theta}\int_0^{\mu(X)} t^{\frac{q}{p}-1}\left[f^{**}(t)\right]^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}} \\ &= \| f\|_{p,q)}^{\theta}.\end{aligned}$$

On the other side, if one uses, [3], then

$$\begin{aligned}\| f\|_{p,q)}^{\theta} &= \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^{\theta}\int_0^{\mu(X)} t^{\frac{q}{p}-1}\left[f^{**}(t)\right]^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}} \\ &= \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^{\theta}\int_0^{\mu(X)} t^{\frac{q}{p}-1}\left[\frac{1}{t}\int_0^t f^{*}(s)ds\right]^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}} \\ &= \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^{\theta}\int_0^{\mu(X)} t^{\frac{q}{p}+\varepsilon-q-1}\left[\int_0^t f^{*}(s)ds\right]^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}} \\ &\le \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^{\theta}\left(\frac{pq-p\varepsilon}{pq-q-p\varepsilon}\right)^{q-\varepsilon}\right. \\ &\qquad \left.\int_0^{\mu(X)}\left(sf^{*}(s)\right)^{q-\varepsilon}s^{\frac{q}{p}+\varepsilon-q-1}ds\right)^{\frac{1}{q-\varepsilon}} \\ &= \sup_{0<\varepsilon<q-1}\left(\frac{p(q-\varepsilon)}{pq-q-p\varepsilon}\right) \\ &\qquad \left(\frac{q}{p}\varepsilon^{\theta}\int_0^{\mu(X)}\left(f^{*}(s)\right)^{q-\varepsilon}s^{\frac{q}{p}-1}ds\right)^{\frac{1}{q-\varepsilon}} \\ &= \frac{p}{p-q}\| f\|_{p,q)}^{\theta,*}\end{aligned}$$

can be found.

Definition 2 Let X be a measure space and M^{+} be the cone of M. A mapping $\rho:\mathrm{M}^{+}\to[0,\infty]$ is named as Banach function norm if, for all u,v,u_n, $(n = 1,2,3,...)$, in M^{+}, for all constants $\lambda\ge 0$, and for all $\mu-$measurable subsets U of X, the following properties hold:

(P1) $\rho(u)\ge 0$

(P2) $\rho(u)=0 \Leftrightarrow u=0$ *a.e.* in X

(P3) $\rho(\lambda u)=\lambda\rho(u)$
(P4) $\rho(u+v)\le\rho(u)+\rho(v)$
(P5) if $|v|\le|u|$ *a.e.* in X, then $\rho(v)\le\rho(u)$
(P6) if $0\le u_n\uparrow u$ in X, then $\rho(u_n)\uparrow\rho(u)$
(P7) if $\mu(U)<\infty$, then $\rho(\chi_U)<\infty$
(P8) if $\mu(U)<\infty$, then $\int_U |v|\,d\mu\le C_U\rho(v)$

for some constant C_U depending on U and ρ but independent of v, [2].

Lemma 1 If $x,y>0$, $r>1$ and $\alpha\in(0,1)$ are real numbers, then

$$(x+y)^r\le\alpha^{1-r}x^r+(1-\alpha)^{1-r}y^r.$$

The equality holds if and only if $\frac{x}{\alpha}=\frac{y}{1-\alpha}$.

Proof. If $r>1$, then $\varphi(x)=x^r$ is strictly convex. Therefore $(\alpha a+(1-\alpha)b)^r\le\alpha a^r+(1-\alpha)b^r$. Setting $x=\alpha a$ and $y=(1-\alpha)b$, we get the result immediately.

Theorem 1 If $1<p<\infty$ and $1\le q\le\infty$; $p=q=1$ or $p=q=\infty$, then the normed space $\left(L_\theta^{p,q)},\|\cdot\|_{p,q)}^\theta\right)$ is a rearrangement-invariant Banach function space.

Proof. The first three (P1-P3) properties of being Banach function norm come from the identical properties true for Lorentz spaces.

Proof of P4.
For any $f,g\in L_\theta^{p,q)}$, we have

$$\|f+g\|_{p,q)}^\theta=\sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^\theta\int_0^{\mu(X)}t^{\frac{q}{p}-1}\left[(f+g)^{**}(t)\right]^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}$$

$$\le\sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^\theta\int_0^{\mu(X)}t^{\frac{q}{p}-1}\left[f^{**}(t)+g^{**}(t)\right]^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}$$

by the property of maximal function, [3]. If we use Lemma 1, then we get

$$\|f+g\|_{p,q)}^\theta\le$$

$$\le\sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^\theta\int_0^{\mu(X)}t^{\frac{q}{p}-1}\left[f^{**}(t)+g^{**}(t)\right]^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}$$

$$\le\sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^\theta\int_0^{\mu(X)}t^{\frac{q}{p}-1}\left[\alpha^{1-q+\varepsilon}\left(f^{**}(t)\right)^{q-\varepsilon}+(1-\alpha)^{1-q+\varepsilon}\left(g^{**}(t)\right)^{q-\varepsilon}\right]dt\right)^{\frac{1}{q-\varepsilon}}$$

because $q-\varepsilon>1$. Finally with $\frac{1}{q-\varepsilon}<1$,

$$\|f+g\|_{p,q)}^\theta$$

$$\le\sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^\theta\int_0^{\mu(X)}t^{\frac{q}{p}-1}\left[\alpha^{1-q+\varepsilon}\left(f^{**}(t)\right)^{q-\varepsilon}+(1-\alpha)^{1-q+\varepsilon}\left(g^{**}(t)\right)^{q-\varepsilon}\right]dt\right)^{\frac{1}{q-\varepsilon}}$$

$$=\sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon^\theta\int_0^{\mu(X)}t^{\frac{q}{p}-1}\alpha^{1-q+\varepsilon}\left(f^{**}(t)\right)^{q-\varepsilon}dt+\frac{q}{p}\varepsilon^\theta\int_0^{\mu(X)}t^{\frac{q}{p}-1}(1-\alpha)^{1-q+\varepsilon}\left(g^{**}(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}$$

$$\le\sup_{0<\varepsilon<q-1}\left[\left(\frac{q}{p}\varepsilon^\theta\int_0^{\mu(X)}t^{\frac{q}{p}-1}\alpha^{1-q+\varepsilon}\left(f^{**}(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}+\left(\frac{q}{p}\varepsilon^\theta\int_0^{\mu(X)}t^{\frac{q}{p}-1}(1-\alpha)^{1-q+\varepsilon}\left(g^{**}(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}\right]$$

$$=\sup_{0<\varepsilon<q-1}\left[\alpha^{\frac{1-q+\varepsilon}{q-\varepsilon}}\left(\frac{q}{p}\varepsilon^\theta\int_0^{\mu(X)}t^{\frac{q}{p}-1}\left(f^{**}(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}+(1-\alpha)^{\frac{1-q+\varepsilon}{q-\varepsilon}}\left(\frac{q}{p}\varepsilon^\theta\int_0^{\mu(X)}t^{\frac{q}{p}-1}\left(g^{**}(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}\right]$$

can be written. If we take the supremum of $\alpha^{\frac{1-q+\varepsilon}{q-\varepsilon}}$, $(1-\alpha)^{\frac{1-q+\varepsilon}{q-\varepsilon}}$ over $0<\varepsilon<q-1$, then we get that

$$\|f+g\|_{p,q)}^\theta\le\|f\|_{p,q)}^\theta+\|g\|_{p,q)}^\theta.$$

Proof of P5.
Let $|g|\le|f|$ *a.e.* in X. By [3], it is known that $g^*(t)\le f^*(t)$ if $|g|\le|f|$. Then

$$g^{**}(t)=\frac{1}{t}\int_0^t g^*(s)ds\le\frac{1}{t}\int_0^t f^*(s)ds=f^{**}(t)$$

and so the result comes from the definition of the norm in (4).

Proof of P6.
Let $0\le f_n\uparrow f$ in X. Then $f_n^*(t)\le f^*(t)$ for $n=1,2,3,\ldots$ by [3]. At the same time, if $|u(x)|\le\liminf_{n\to\infty}|u_n(x)|$ for all $x\in X$ $\mu-$a.e., then

$D_u(\lambda) \leq \liminf_{n\to\infty} D_{u_n}(\lambda)$ for any $\lambda \geq 0$, where $(u_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions in M and so $f_n^*(t) \to f^*(t)$ for $t \geq 0$. Since $f_n^{**}(t) \leq f^{**}(t)$, we get $f_n^{**}(t) \to f^{**}(t)$ for all $t > 0$ by Monotone Convergence Theorem. Therefore

$$\|f_n\|_{p,q)}^{\theta} \to \sup_n \|f_n\|_{p,q)}^{\theta}$$

$$= \sup_n \sup_{0<\varepsilon<q-1} \left(\frac{q}{p} \varepsilon^{\theta} \int_0^{\mu(X)} t^{\frac{q}{p}-1} \left(f_n^{**}(t) \right)^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}}$$

$$= \sup_{0<\varepsilon<q-1} \sup_n \left(\frac{q}{p} \varepsilon^{\theta} \int_0^{\mu(X)} t^{\frac{q}{p}-1} \left[f_n^{**}(t) \right]^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}}$$

$$= \sup_{0<\varepsilon<q-1} \left(\frac{q}{p} \varepsilon^{\theta} \int_0^{\mu(X)} t^{\frac{q}{p}-1} \left(f^{**}(t) \right)^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}}$$

$$= \|f\|_{p,q)}^{\theta}.$$

Proof of P7.
Let $E \in \Sigma$ be a measurable subset of X with $\mu(E) < \infty$. Then

$$\chi_E^*(t) = \chi_{[0,\mu(E))}(t), \; \chi_E^{**}(t) = \min\left\{1, \frac{\mu(E)}{t}\right\}$$

and so

$$\|\chi_E\|_{p,q)}^{\theta} = \sup_{0<\varepsilon<q-1} \left(\frac{q}{p} \varepsilon^{\theta} \int_0^{\mu(X)} t^{\frac{q}{p}-1} \left[\chi_E^{**}(t) \right]^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}}$$

$$= \sup_{0<\varepsilon<q-1} \left(\frac{q}{p} \varepsilon^{\theta} \int_0^{\mu(E)} t^{\frac{q}{p}-1} \left[\chi_E^{**}(t) \right]^{q-\varepsilon} dt + \frac{q}{p} \varepsilon^{\theta} \int_{\mu(E)}^{\mu(X)} t^{\frac{q}{p}-1} \left[\chi_E^{**}(t) \right]^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}}$$

$$= \sup_{0<\varepsilon<q-1} \left(\frac{q}{p} \varepsilon^{\theta} \int_0^{\mu(E)} t^{\frac{q}{p}-1} dt + \frac{q}{p} \varepsilon^{\theta} \int_{\mu(E)}^{\mu(X)} t^{\frac{q}{p}-1} \left(\frac{\mu(E)}{t} \right)^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}}$$

$$= \sup_{0<\varepsilon<q-1} \left(\frac{q}{p} \varepsilon^{\theta} \frac{p}{q} \mu(E)^{\frac{q}{p}} + \frac{q}{p} \varepsilon^{\theta} \int_{\mu(E)}^{\mu(X)} t^{\frac{q}{p}-1} \left(\frac{\mu(E)}{t} \right)^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}}$$

$$= (q-1)^{\theta} \mu(E)^{\frac{q}{p}} + \sup_{0<\varepsilon<q-1} \left(\frac{q}{p} \varepsilon^{\theta} \mu(E)^{q-\varepsilon} \int_{\mu(E)}^{\mu(X)} t^{\frac{q}{p}-1-q+\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}}$$

$$= (q-1)^{\theta} \mu(E)^{\frac{q}{p}} + \sup_{0<\varepsilon<q-1} \left(\frac{q}{p} \varepsilon^{\theta} \mu(E)^{q-\varepsilon} \frac{p}{q-p-qp+p\varepsilon} \left[\mu(X)^{\frac{q}{p}-q+\varepsilon} - \mu(E)^{\frac{q}{p}-q+\varepsilon} \right] \right)^{\frac{1}{q-\varepsilon}}$$

$$= (q-1)^{\theta} \mu(E)^{\frac{q}{p}} + (q-1)^{\theta} \mu(E) \frac{1}{1-p} \left(\mu(X)^{\frac{q}{p}-1} - \mu(E)^{\frac{q}{p}-1} \right)$$

can be found if $q < \infty$. Otherwise

$$\|\chi_E\|_{p,\infty)}^{\theta} = \sup_{0<t<\mu(X)} t^{\frac{1}{p}} \chi_E^{**}(t) = \sup_{0<t<\mu(E)} t^{\frac{1}{p}} + \sup_{\mu(E)<t<\mu(X)} t^{\frac{1}{p}-1} \mu(E)$$

$$= 2\mu(E)^{\frac{1}{p}}$$

when $q = \infty$. As a result, $\|\chi_E\|_{p,q)}^{\theta}$ is finite.

Proof of P8.
Let $E \in \Sigma$ be a measurable subset of X with $\mu(E) < \infty$. By using Hardy-Littlewood inequality, one can write that

$$\int_E |f| d\mu = \int_X |f(t)| \chi_E(t) d\mu(t) \leq \int_0^{\infty} f^*(t) \chi_E^*(t) dt$$

$$= \int_0^{\infty} f^*(t) \chi_{[0,\mu(E))}(t) dt = \int_0^{\mu(E)} f^*(t) dt$$

$$\leq \int_0^{\mu(E)} f^{**}(t) dt.$$

Now fix $0 < \varepsilon < q-1$. If one uses Hölder's inequality in the preceding inequality, then

$$
\begin{aligned}
\int_E |f| d\mu &\leq \int_0^{\mu(E)} f^{**}(t)dt = \int_0^{\mu(E)} t^{\frac{q-p}{p(q-\varepsilon)}} f^{**}(t) t^{\frac{p-q}{p(q-\varepsilon)}} dt \\
&\leq \left(\int_0^{\mu(E)} t^{\frac{q-p}{p}} \left(f^{**}(t) \right)^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}} \\
&\qquad \left(\int_0^{\mu(E)} t^{\frac{p-q}{p(q-\varepsilon)} \frac{q-\varepsilon}{q-\varepsilon-1}} dt \right)^{\frac{q-\varepsilon-1}{q-\varepsilon}} \\
&\leq \left(\frac{p}{q\varepsilon^{\theta}} \frac{q\varepsilon^{\theta}}{p} \int_0^{\mu(X)} t^{\frac{q-p}{p}} \left(f^{**}(t) \right)^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}} \\
&\qquad \left(\int_0^{\mu(E)} t^{\frac{p-q}{p(q-\varepsilon-1)}} dt \right)^{\frac{q-\varepsilon-1}{q-\varepsilon}} \\
&\leq \frac{p(q-1)^{-\theta}}{q} \|f\|_{p,q)}^{\theta} \\
&\qquad \left(\frac{pq-p\varepsilon-p}{pq-p\varepsilon-q} \mu(E)^{\frac{pq-p\varepsilon-q}{pq-p\varepsilon-p}} \right)^{\frac{q-\varepsilon-1}{q-\varepsilon}} \\
&\leq \frac{p(q-1)^{-\theta}}{q} \|f\|_{p,q)}^{\theta} \\
&\qquad \sup_{0<\varepsilon<q-1} \left[\left(\frac{pq-p\varepsilon-p}{pq-p\varepsilon-q} \right)^{\frac{q-\varepsilon-1}{q-\varepsilon}} \mu(E)^{\frac{pq-p\varepsilon-q}{pq-p\varepsilon}} \right] \\
&= \frac{p(q-1)^{-\theta}}{q} \|f\|_{p,q)}^{\theta} \mu(E)^{\frac{p-1}{p}}
\end{aligned}
$$

is found.

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