Binormal Measures

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Abstract: Our starting point is the measure $\varepsilon_X - \alpha_X \rho_X^{\omega_1} + \beta_X \rho_X^{\omega_2}$, where $\rho_X^{\omega_i}$ is the harmonic measure relative to $x \epsilon \omega_1 \subset \overline{\omega}_1 \subset \omega_2$ and ω_i are concentric balls of $\mathbb{R}^n$; α_X, β_X are functions depending on x and the radii of ω_i ($i = 1,2$). Generalizing the above measure, we introduce and study the binormal measures as well as their relation to biharmonic functions.

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1 Introduction

The characteristic mean value property of harmonic (respectively parabolic) functions involves the measures $\lambda = \varepsilon_X - \rho_X^{\omega}$, where ε_X is the Dirac measure at $x \epsilon \omega$ and ρ_X^{ω} is the harmonic (respectively parabolic) measure relative to ω and $x \epsilon \omega$, supported by the sphere $\partial\omega$ (respectively by the level surface $\partial\omega$ of the heat kernel). The adjoint potential of these measures is equal to zero on $\complement\overline{\omega}$ (the complement of $\overline{\omega}$), or equivalently, their swept measures satisfy $\lambda^{\complement\overline{\omega}} = 0$.

In 1944, G. Choquet and J. Deny generalized the measure $\varepsilon_X - \rho_X^{\omega}$, and introduced the normal distribution. Moreover, they proved some characteristic properties of solutions of the equations $\Delta u = 0$, and $\Delta^p u = 0$ in $\mathbb{R}^n$. Next, in 1967, de La Pradelle following an idea of [8], extended the notion of normal measure to the setting of Brelot's theory, [3]. Finally, in 1971, E. Smyrnelis, using the extended notion of normal measure, proved several characteristic properties of normal measures and harmonic functions in Brelot spaces, applicable to solutions of $Lu = 0$, where L is a second-order linear elliptic operator in $\mathbb{R}^n$.

On the other hand, biharmonic functions (that is, solutions of $\Delta^2 u = 0$) satisfy a mean value property which involves the measures $\varepsilon_X - \alpha_X \rho_X^{\omega_1} + \beta_X \rho_X^{\omega_2}$, where α_X, β_X are functions of $x \epsilon \omega_1 \subset \overline{\omega}_1 \subset \omega_2$ and of the radii R_1, R_2 of the concentric spheres $\partial\omega_1$, $\partial\omega_2$ (cf., [17]). The scope of this article is to generalize this property and study some related issues, for the solutions of the equation $(L_2 L_1)h = 0$, where L_i $(i = 1,2)$ is a second-order linear elliptic differential operator. The idea is to work in a biharmonic elliptic space, and use special general measures, applicable to the above equation, in particular; note that to this biharmonic elliptic space, we associate a 1-harmonic and a 2-harmonic space that in the applications correspond respectively to the solutions of the equations $L_1 h = 0$, and $L_2 u = 0$.

To this end, we first introduce in Section 2, the *binormal pair* of measures $\Theta = (\lambda, \mu)$ supported by the compact set K, as the pair such that the swept measures on $\complement K$ of $\Lambda := (\lambda, 0)$ and $M := (0, \mu)$ vanish. Since $\Theta = \Lambda + M$, it follows that $\Theta^{\complement K} = \Lambda^{\complement K} + M^{\complement K}$ or $(\lambda, \mu)^{\complement K} = (\lambda, 0)^{\complement K} + (0, \mu)^{\complement K}$ (cf., [15]).

The pair $(\lambda, 0)$ is called a *pure biharmonic pair* if $(\lambda, 0)^{\complement K} = (0,0)$ or equivalently if the pure adjoint potential pair vanishes on $\complement K$.

The pair $(0, \mu)$ is called 2-*normal* if $(0, \mu)^{\complement K} = (0,0)$ or equivalently if the 2-adjoint potential vanishes on $\complement K$ (see, [14]).

Several examples of the aforementioned pairs of measures are given in Section 3.

In Section 4, we prove the characteristic mean value properties of biharmonic pairs in relation to biharmonic pairs of measures.

Section 5 is devoted to the study of the properties of binormal pairs of measures. Furthermore, we show that the linear combinations of the pairs $(\varepsilon_X - {\mu_X}^{CK}, {\nu_X}^{CK})$ are dense for the vague topology, in the space of the pure binormal pairs of measures, where $({\mu_X}^{CK}, {\nu_X}^{CK}) = (\varepsilon_X, 0)^{CK}$. Analogous results hold for the measures $\varepsilon_X - {\mu_X}^{CK}$ (respectively $\varepsilon_X - {\lambda_X}^{CK}$) in the space of 1-normal (respectively 2-normal) measures, where ${\mu_X}^{CK}$ is the swept nonnegative measure of ε_X in the 1-harmonic space (respectively ${\lambda_X}^{CK}$ is the swept nonnegative measure of ε_X in the 2-harmonic space). Finally, we examine the relation between binormal and normal measures.

Note. In this work, we use the term `measure' for `signed measure'.

2 Reminders, Definitions, and Preliminary Results

Let us first point out there are equivalent views of potential theory. We refer for instance to [1], [11].

In this paper, our setting is a general biharmonic space, as the space of solutions of the system $L_1u_1 = -u_2$, $L_2u_2 = 0$, where L_i $(i = 1,2)$ is a second-order linear elliptic or parabolic differential operator (cf., [15]). From this space, one can construct using Green's pairs, the associated adjoint space corresponding to the system $L_2^*h_2 = -h_1$, $L_1^*h_1 = 0$, which is in duality with the initial space (cf., [20]). In this context, the potential theory of the harmonic case can be extended, and appropriate tools are provided to study boundary value problems. We also point out in [21], [22], two different approaches to the study of the biharmonic boundary value problem.

In what follows, we briefly present the main facts about biharmonic spaces. These spaces have been inspired by the classical biharmonic equation $\Delta^2 u = \Delta(\Delta u) = 0$, and we point out that the polyharmonic case can be studied with the same approach. For more details, we refer to [12].

We consider a locally compact, connected space Ω with a countable basis. We denote by $\mathcal{U}$ (respectively $\mathcal{U}_c$) the set of all nonempty open sets (respectively the set of all nonempty relatively compact open sets) in Ω.

Let $\mathcal{H}$ be a map that associates to each $U \in \mathcal{U}$ a linear subspace of $C(U) \times C(U)$ which is composed of compatible pairs (u_1, u_2) in the sense that if $u_1 = 0$ on an open set, then u_2 also vanishes there. The pairs of $\mathcal{H}(U)$ are called *biharmonic* on U.

On the other hand, a set $\omega \in \mathcal{U}_c$ with $\partial\omega \neq \varnothing$ is called *$\mathcal{H}$-regular* if the following conditions hold:

- The Riquier boundary value problem has only one solution $\left(H_1^{\omega,f}, H_2^{\omega,f}\right)$ associated to the pair $f = (f_1, f_2) \in C(\partial\omega) \times C(\partial\omega)$.
- The inequalities $f_j \geq 0$ $(j = 1,2)$ imply that $H_1^{\omega,f} \geq 0$, while the inequality $f_2 \geq 0$ implies that $H_2^{\omega,f} \geq 0$. Hence, for every $x \in \omega$, there exists a unique system $(\lambda_x^\omega, \mu_x^\omega, \nu_x^\omega)$ of Radon nonnegative measures on $\partial\omega$, such that $H_1^{\omega,f}(x) = \int f_1\, d\mu_x^\omega + \int f_2\, d\nu_x^\omega$, while $H_2^{\omega,f}(x) = \int f_2\, d\lambda_x^\omega$.

Next, we recall that a pair of functions (v_1, v_2) defined on $U \in \mathcal{U}$, is called *hyperharmonic* if

- $v_j: U \to (-\infty, +\infty]$,
- v_j is lower semi-continuous,
- and the inequalities $v_1(x) \geq \int v_1\, d\mu_x^\omega + \int v_2\, d\nu_x^\omega$, as well as $v_2(x) \geq \int v_2\, d\lambda_x^\omega$, hold for every regular set $\omega \subset \overline{\omega} \subset U$, and every $x \in \omega$.

Let us also mention that if the function v_1 is finite on a dense subset of U, then the hyperharmonic pair (v_1, v_2) is called *superharmonic* on U. Finally, a nonnegative superharmonic pair $p = (p_1, p_2)$ will be called *potential pair* (on U), if $(h_1, h_2) = (0,0)$ is the only biharmonic pair satisfying $0 \leq h_j \leq p_j$ $(j = 1,2)$.

The space $(\Omega, \mathcal{H})$ with the axioms I, II, III, and IV introduced in [15], is called *biharmonic*. A biharmonic space is called *elliptic* if, for every $x \in \Omega$ and every regular set $\omega \ni x$, we have $\operatorname{supp}(\lambda_x^\omega) = \operatorname{supp}(\mu_x^\omega) = \operatorname{supp}(\nu_x^\omega) = \partial\omega$; it will be called *strong* if there exists a strictly positive potential pair on Ω. In a biharmonic space, we associate the underlying harmonic spaces $(\Omega, \mathcal{H}_1)$ and $(\Omega, \mathcal{H}_2)$, which correspond respectively to the solutions of the equations $L_1u_1 = 0$, and $L_2u_2 = 0$ in the classical case. We use respectively the prefixes 1 or 2, to refer to the harmonic spaces defined previously.

We shall say that the hyperharmonic (respectively superharmonic/potential) pair (v_1, v_2) is *pure*, if given a nonnegative 2-hyperharmonic function v_2 on U, v_1 is the smallest nonnegative function such that (v_1, v_2) is a nonnegative hyperharmonic (respectively superharmonic/potential) pair on U. The j-harmonic (respectively biharmonic) support of a j-hyperharmonic function (respectively hyperharmonic pair) is defined as the smallest closed set such that the function (respectively the pair) is j-harmonic (respectively biharmonic) in its complement ($j = 1,2$). We call Green's pair, a pure potential pair with punctual biharmonic support. We also recall that if φ is a numerical function on an open set U, the function $\hat{\varphi}$ is defined as follows:

$$\hat{\varphi}(x) = \lim_{\substack{y \to x \\ y \in U}} \inf \varphi(y).$$

In [20], we define and study the adjoint biharmonic spaces corresponding to the adjoint equation $(L_2 L_1)^* h = 0$, that is, to the system:

$$L_2^* h_2 = -h_1,\ L_1^* h_1 = 0.$$

The asterisk symbol is used in the sequel to refer to adjoint spaces.

Our setting will be a strong biharmonic elliptic connected space. We assume the proportionality of i-Green's potentials and i-adjoint Green's potentials, and also the existence of a topological basis of completely determining domains for the associated i-harmonic spaces ($i = 1,2$). For the notions and notations not explained in this work, we refer to [15], [9].

Definition 1. Let λ, μ be Radon measures supported by a compact set $K \subset \Omega$, and let $\lambda = \lambda_1 - \lambda_2, \mu = \mu_1 - \mu_2$, with $\lambda_j \geq 0, \mu_j \geq 0$, ($j = 1,2$).

- The pair (λ, μ) is called *binormal* for K if $(\lambda, 0)^{\complement K} = (0,0)$ and $(0, \mu)^{\complement K} = (0,0)$.
- The pair $(\lambda, 0)$ is called *pure binormal* for K if $(\lambda, 0)^{\complement K} = (0,0)$.
- The pair $(0, \mu)$ is called 2-*normal* for K if $(0, \mu)^{\complement K} = (0,0)$.

Let us consider the open subset $\omega \subset \Omega$, the points $x, y \in \Omega$, the Green's pair (w_y, p_y^2) of biharmonic support $\{y\}$ and the adjoint Green's pair (w_x^*, p_x^{1*}) of support $\{x\}$ (cf., [19], [20]). We denote by $(W_y^\omega, P_y^{2,\omega})$ the swept pair on ω of the former pair, and by $(W_x^{*,\omega}, P_x^{1*,\omega})$ the swept pair on ω of the latter pair. We also consider the adjoint pure potential pair $p_\nu^* = (w_\nu^*, p_\nu^{1*})$ with associated nonnegative measure ν, where

- $w_\nu^*(u) = \int w_x^*(u) d\nu(x)$,
- $p_\nu^{1*}(u) = \int p_x^{1*}(u) d\nu(x)$,

and $(W_\nu^{*,\omega}, P_\nu^{1*,\omega})$ the swept pair corresponding to the open set ω.

Lemma 2. We assert that

$$W_\nu^{*,\omega}(y) = \int W_x^{*,\omega}(y) d\nu(x) = \int W_y^\omega(x) d\nu(x).$$

Proof. If $(\alpha_y^\omega, \beta_y^\omega)$ is the adjoint swept pair of $(\epsilon_y, 0)$ on ω, then it holds:

$$\begin{aligned} W_\nu^{*,\omega}(y) &= \int w_\nu^*(u) d\alpha_y^\omega(u) \\ &\quad + \int p_\nu^{1*}(u) d\beta_y^\omega(u) \\ &= \int \left(\int w_x^*(u) d\nu(x)\right) d\alpha_y^\omega(u) \\ &\quad + \int \left(\int p_x^{1*}(u) d\nu(x)\right) d\beta_y^\omega(u) \\ &= \int \left(\int w_x^*(u) d\alpha_y^\omega(u)\right) d\nu(x) \\ &\quad + \int \left(\int p_x^{1*}(u) d\beta_y^\omega(u)\right) d\nu(x). \end{aligned}$$

On the other hand, since we have $W_x^{*,\omega}(y) = \int w_x^*(u) d\alpha_y^\omega(u) + \int p_x^{1*}(u) d\beta_y^\omega(u)$, using, [20], Lemma 4, and a remark after the proof of [20], Proposition 4.2, we obtain $W_y^\omega(x) = W_x^{*,\omega}(y)$, which completes the proof.

Theorem 3. Let $(\lambda, 0)$ be a pair of measures supported by the compact set K. Then, the following properties are equivalent:

(i) $(\lambda, 0)$ is pure binormal relative to K.
(ii) The adjoint pure potential pair $(w_\lambda^*, p_\lambda^{1*})$ vanishes on $\complement K$.

Proof. First, we notice that as the pair $(w_\lambda^*, p_\lambda^{1*})$ is adjoint biharmonic on $\complement K$, and therefore compatible, if $w_{\lambda_1}^* = w_{\lambda_2}^*$ holds on $\complement K$, then $p_{\lambda_1}^{1*} = p_{\lambda_2}^{1*}$ also holds on $\complement K$. In other words, if $(\lambda, 0)$ is pure binormal, then λ is 1-normal.

$(ii) \Rightarrow (i)$. The equality $w_{\lambda_1}^* = w_{\lambda_2}^*$ on $\complement K$ implies that the respective reduced functions satisfy $W_{\lambda_1}^{*\complement K} =$

$W^{*\complement K}_{\lambda_2}$ in Ω, and it follows from Lemma 2 that $\int W^{\complement K}_y(x)d\lambda_1(x)=\int W^{\complement K}_y(x)d\lambda_2(x)$. In view of [15], Theorem 7.11, we have

$$\int W^{\complement K}_y(x)d\lambda_i(x) = \int w_y(x)dB^{\complement K}_{i,1}(x) + \int p^2_y(x)dB^{\complement K}_{i,2}(x),$$

where $\Lambda := (\lambda, 0)$, $B_i := (\lambda_i, 0)$ and $\Lambda^{\complement K} = (\Lambda^{\complement K}_1, \Lambda^{\complement K}_2)$, $B^{\complement K}_i = (B^{\complement K}_{i,1}, B^{\complement K}_{i,2})$, are the respective swept pairs on $\complement K$; therefore $\Lambda^{\complement K}_i = B^{\complement K}_{1,i} - B^{\complement K}_{2,i}$, $(i = 1,2)$. Finally, using [15], Theorem 7.1, (cf. also, [14]), we deduce that $B^{\complement K}_{1,1} = B^{\complement K}_{2,1}$, and $\int p^2_y(x)dB^{\complement K}_{1,2}(x) = \int p^2_y(x)dB^{\complement K}_{2,2}(x)$ or equivalently $P^{*2}_{B^{\complement K}_{1,2}} = P^{*2}_{B^{\complement K}_{2,2}}$ in Ω; it follows that $B^{\complement K}_{1,2} = B^{\complement K}_{2,2}$, hence $\Lambda^{\complement K}_1 = \Lambda^{\complement K}_2 = 0$.

$(i) \Rightarrow (ii)$. The previous arguments can be reversed to prove the converse implication.

Remark 4. The case of the pair $(0, \mu)$ with $(0, \mu)^{\complement K} = (0,0)$ was studied in [14], and it was established that $(0, \mu)^{\complement K} = (0,0) \Leftrightarrow P^{2*}_{\mu} = 0$ on $\complement K$.

Corollary 5. We suppose that $\mathcal{H}_1 = \mathcal{H}_2$ (that is, $L_1 = L_2$ in the classical case). Let $(\lambda, 0)$ be a pure binormal pair for the compact set K. Then, λ is 1- and 2-normal, while (λ, λ) is binormal for K.

Proof. It follows from Theorem 3 that $w^*_{\lambda_1} = w^*_{\lambda_2}$; as the pair $(w^*_{\lambda}, p^{1*}_{\lambda})$ is adjoint biharmonic on $\complement K$, and therefore compatible, we have $p^{1*}_{\lambda} = 0$ on $\complement K$, and by assumption, $p^{1*}_{\lambda} = p^{2*}_{\lambda}$. We also know that $(\lambda, \lambda)^{\complement K} = (\lambda, 0)^{\complement K} + (0, \lambda)^{\complement K}$. Consequently, λ is 1- and 2-normal, while in view of Definition 1, (λ, λ) is binormal.

3 Some Examples

The functions u such that $\Delta^2 u = 0$ on an open set U of $\mathbb{R}^n$ satisfy a characteristic mean value property (see, [17]):

$$u(x) = \alpha_x \int u d\mu^{\omega_1}_x(z) - \beta_x \int u d\mu^{\omega_2}_x(z),$$

where $\omega_i(x_0, R_i)$, $(i = 1,2)$, are concentric balls with $0 < R_1 < R_2$, $\overline{\omega}_2 \subset U$, $\alpha_x = \frac{R_2^2-\rho^2}{R_2^2-R_1^2}$, $\beta_x = \frac{R_1^2-\rho^2}{R_2^2-R_1^2}$, $\rho = \|x - x_0\|$, $x \in \omega_1$ and $\mu^{\omega_i}_x$, $(i = 1,2)$, are the respective harmonic measures.

Let (w_y, p^2_y) be the Green's pair in $\mathbb{R}^n$ (cf., [19]); it is biharmonic on the open set $U = \mathbb{R}^n \backslash \{y\}$. If $\overline{\omega}_2 \subset U$ and $x \in \omega_1$, then we have

$$w_y(x) = \alpha_x \int w_y(z)d\mu^{\omega_1}_x(z) - \beta_x \int w_y(z)d\mu^{\omega_2}_x(z)$$

or equivalently

$$w^*_x(y) = \alpha_x \int w^*_z(y)d\mu^{\omega_1}_x(z) - \beta_x \int w^*_z(y)d\mu^{\omega_2}_x(z)$$

Example 6. We consider the compact set $K = \overline{\omega}_2$; the pair of measures $(\lambda, 0)$ with $\lambda = \lambda_1 - \lambda_2$, where $\lambda_1 = \epsilon_x + \beta_x\mu^{\omega_2}_x$, $\lambda_2 = \alpha_x\mu^{\omega_1}_x$ is a pure binormal pair of measures. We can also take the decomposition $\lambda = \lambda_1 - \lambda_2$, where $\lambda_1 = \alpha_x\epsilon_x + \beta_x\mu^{\omega_2}_x$, $\lambda_2 = \beta_x\epsilon_x + \alpha_x\mu^{\omega_1}_x$. Moreover, we observe that the pair (λ, λ) is a binormal pair for K.

Example 7. Let ν be a measure with compact support in ω_1. If $y \in \complement\overline{\omega}_2$, we obtain:

$$\begin{aligned}\int w^*_\nu(y) &= \int w^*_z(y)\int \alpha_x d\mu^{\omega_1}_x d\nu(x) \\ &\quad - \int w^*_z(y)\int \beta_x d\mu^{\omega_2}_x(z)d\nu(x) \\ &= \int w^*_z(y)d\sigma(z) - \int w^*_z d\tau(z).\end{aligned}$$

The pair $(\lambda, 0)$, where $\lambda = \nu + \tau - \sigma$ is pure biharmonic, while the pair (λ, λ) is a binormal pair.

Note. Obviously, since every compact set is contained in a ball, we can construct pure binormal (respectively binormal) pairs from a given measure.

Example 8. Starting from a measure λ supported by a compact set $E \subset \mathbb{R}^n$, G. Choquet and J. Deny (cf., [6]) have constructed another measure λ' such that $d\lambda' = U^\lambda d\tau$ on $\hat{E} = E \cup (\cup_i E_i)$, where the sets E_i are the connected components of $\complement E$, $\overline{E}_i$ is compact, U^λ is the potential generated by λ, and $d\tau$ is the volume element (and so on for the polyharmonic case). The potential $U^{\lambda'}$ is defined as follows:

$$\begin{aligned}U^{\lambda'}(x) &= \int G_1(x,y)d\lambda'(y) \\ &= \int G_1(x,y)U^\lambda(y)d\tau(y) \\ &= \int\int G_1(x,y)G_1(y,z)d\tau(y)d\lambda(z) \\ &= \int G_2(x,z)d\lambda(z),\end{aligned}$$

where G_1 is the Newtonian kernel, and $G_2(x,y) = \int G_1(x,z)G_1(z,y)d\tau(z)$ is the iterated kernel (see, [12]). If $U^{\lambda'}(x) = \int G_2(x,z)d\lambda(z) = 0$, on $\complement E$, then

$$\Delta_x \int G_2(x,z)d\lambda(z) = \int G_1(x,z)d\lambda(z) = 0 \text{ on } \complement E.$$

Therefore, the pair (λ, λ) is binormal.

Example 9. Let $(u_2^*, 1)$ be a strictly positive adjoint biharmonic pair and let V_2^* be the associated kernel of the potential part of u_2^*. If v_1^* is a nonnegative adjoint 1-hyperharmonic function, the adjoint pair $(V_2^* v_1^*, v_1^*)$ is a pure hyperharmonic pair; it will be an adjoint pure potential pair, if v_1^* is an adjoint 1-potential, continuous with a compact harmonic* support. Let λ be a measure supported by a compact set $K \subset \Omega$; we have $p_\lambda^{1*}(x) = \int p_z^{1*}(x) d\lambda(z)$, as well as $V_2^* 1(y) = \int p_x^{2*}(y) d\xi(x)$, where ξ is the nonnegative measure associated with the adjoint potential $V_2^* 1$. Now, let λ' be another measure with density p_λ^{1*} relative to ξ; we consider the following function:

$$\begin{aligned} q_2^*(y) &= \int p_x^{2*}(y) p_\lambda^{1*}(x) d\xi(x) \\ &= \int \left(\int p_z^{1*}(x) p_x^{2*}(y) d\xi(x)\right) d\lambda(z) \\ &= \int w_z^*(y) d\lambda(z) = V_2^* p_\lambda^{1*}(y) \\ &= w_\lambda^*(y). \end{aligned}$$

Therefore, if $V_2^* p_\lambda^{1*} = 0$ on $\complement K$, we also have that $p_\lambda^{1*} = 0$. Consequently, the pair $(\lambda, 0)$ is pure binormal for K. On the other hand, if μ is a 2-normal measure for K, then the pair (λ, μ) will be binormal for K.

4 Some Mean Values Properties of Biharmonic Pairs

Let us recall some further results on harmonic and biharmonic spaces (cf., [15], parts X, XI). In a harmonic space, we consider a potential P on Ω, which is finite, continuous, and strictly superharmonic. Let ξ be its associated nonnegative measure. We define Dynkin's operators L, and L' relative to P, as

$$L_P f(x) = \limsup_{\omega \searrow x} \frac{f(x) - \int f d\rho_x^\omega}{P(x) - \int P d\rho_x^\omega} \quad (1)$$

$$L'_P f(x) = \liminf_{\omega \searrow x} \frac{f(x) - \int f d\rho_x^\omega}{P(x) - \int P d\rho_x^\omega} \quad (2)$$

where $x \in \Omega$, ω is an open set with $\bar{\omega}$ compact, f is a numerical function on Ω such that the numerator in (1) and (2) is defined, and ρ_x^ω is the harmonic measure. We can see that $L_P f(x) = L_{p^\omega} f(x)$ on the harmonic space ω, where $p^\omega = P(x) - \int P d\rho_x^\omega$, and $x \in \omega$. Moreover, if V is the kernel associated with P, then we have $LV\phi = L'V\phi = \phi$, for $\phi \in C_b(\Omega)$. The following inequality $Lu(x) \geq 0$ (or $L'u(x) \geq 0$) on an open set $U \subset \Omega$ is also characteristic of hyperharmonic functions on U.

Let L^j, $L^{j'}$ be the operators in (1)-(2) associated to the space $(\Omega, \mathcal{H}_j)$ $(j = 1,2)$. We say that the pair (f_1, f_2) of finite and continuous functions in the open set $U \subset \Omega$, is *regular* if $L^1 f_1$ and $L^2 f_2$ (or equivalently $L^{1'} f_1$ and $L^{2'} f_2$) are finite and continuous in U.

Next, we define the operators:

$$\Gamma_1 f(x) = \limsup_{\omega \searrow x} \frac{f(x) - \int f d\mu_x^\omega}{\int d\nu_x^\omega},$$

$$\Gamma'_1 f(x) = \liminf_{\omega \searrow x} \frac{f(x) - \int f d\mu_x^\omega}{\int d\nu_x^\omega}.$$

Since on a relatively compact open set, there exists a strictly positive biharmonic pair (v_1, v_2), we can assume, without loss of generality, that $v_2 = 1$. The Riesz decomposition yields $v_1 = p_1 + h_1$, where p_1 is a 1-potential and h_1 is a 1-harmonic function on ω. We have $L_{p_1} f(x) = \Gamma_1 f(x)$, and $L'_{p_1} f(x) = \Gamma'_1 f(x)$. Moreover, the inequality $\Gamma_1 w_1 \geq w_2$ (or $\Gamma'_1 w_1 \geq w_2$), at the points where w_1 is finite, is a characteristic property of the hyperharmonic pairs (w_1, w_2).

Proposition 10. (λ, μ) be a binormal pair of measures supported by a compact set $K \subset U$, where U is an open subset of Ω, and (u_1, u_2) a biharmonic pair of functions on U. Then, $\int u_1 d\lambda = 0$, and $\int u_2 d\mu = 0$.

Proof. We know that $\int u_2 d\mu = 0$ if μ is a 2-normal measure relative to a compact set $K \subset U$ (cf., [14], Proposition 1]). Thus, it remains to prove the other equality. Let us consider a relatively compact open set ω, such that $K \subset \omega \subset \overline{\omega} \subset U$. By [18], Proposition 1.7, there exist continuous potential pairs (p_1, p_2), and (q_1, q_2), which are biharmonic on ω, and such that $(u_1, u_2) + (q_1, q_2) = (p_1, p_2)$. We have the decompositions: $(p_1, p_2) = (p'_1, p_2) + (s_1, 0)$, as well as $(q_1, q_2) = (q'_1, q_2) + (t_1, 0)$, where (p'_1, p_2), (q'_1, q_2) are pure potential pairs in Ω, biharmonic on ω, while s_1 and t_1 are 1-potentials in Ω (see, [18], Proposition 2.8 and Proposition 2.2); moreover, s_1 and t_1 are 1-harmonic on ω, since $\Gamma_1 p_1 = \Gamma_1 p'_1 = p_2$, $\Gamma_1 q_1 = \Gamma_1 q'_1 = q_2$ on Ω, while $\Gamma_1(p_1 - p'_1) = 0$, $\Gamma_1(q_1 - q'_1) = 0$ on ω, (cf., [15], Corollary 11.4). Therefore, we have on ω: $u_1 = p_1 - q_1 = p'_1 - q'_1 + h_1$, where $h_1 = s_1 - t_1$ is 1-

harmonic on ω. Finally, the nonnegative measures ζ and ξ associated with the pure pairs (p'_1, p_2) and (q'_1, q_2) (cf., [18], (3.13)), are supported by $\complement\omega$. As $\int h_1 d\lambda = 0$, (cf., [14], Proposition 1), we obtain:

$$\begin{aligned}\int u_1 d\lambda &= \int h_1 d\lambda + \int (p'_1 - q'_1) d\lambda \\ &= \int \int w_y(x) d\theta(y) d\lambda(x) \\ &= \int (\int w_y(x) d\lambda(x)) d\theta(y) = 0,\end{aligned}$$

where $\theta = \zeta - \xi$, since $\int w_y(x) d\lambda(x) = 0$ holds on $\complement K \supset \complement\omega$.

Next, we shall study the converse of Proposition 10.

Proposition 11. Let U be an open subset of Ω and let (u_1, u_2) be a pair of regular functions satisfying $\int u_1 d\lambda_i = 0, \int u_2 d\mu_i = 0$ for a family (λ_i, μ_i) of binormal pairs of measures relative to compact sets $K_i \subset \omega_i$ with $\lambda_i \neq 0$, $\mu_i \neq 0$, such that $P^{1*}_{\lambda_{i,1}} \geq P^{1*}_{\lambda_{i,2}}$, $P^{2*}_{\mu_{i,1}} \geq P^{2*}_{\mu_{i,2}}$, for all $i \in I$, the open sets ω_i forming a basis of U; then, the pair (u_1, u_2) is biharmonic on U.

Proof. Let ω be an open set with $\overline{\omega} \subset U$ and $\overline{\omega}$ compact. There is a strictly positive biharmonic pair (v_1, v_2) on ω (cf., [15], Theorem 6.9); without loss of generality, we may assume that $v_2 = 1$, and we may replace U with ω. In the associated 1-harmonic space, the Riesz decomposition implies that $v_1 = p_1 + h_1$; we consider the kernel V_1^ω associated with the potential p_1 and the associated operators L_1, Γ_1 (cf., [15], parts X, XI). The pair $(V_1^\omega u_2, u_2)$ is biharmonic since $L_1 V_1^\omega u_2 = \Gamma_1 V_1^\omega u_2 = u_2$, and u_2 is a 2-harmonic function (cf., [14], Proposition 2)[1]. It follows from Proposition 10 that $\int V_1^\omega u_2 d\lambda_i = 0$ holds for all λ_i satisfying the assumptions of Proposition 11. At this stage, we consider the function $\phi = V_1^\omega u_2 - u_1$ on ω; since the functions $V_1^\omega u_2$ and u_1 are continuous on ω, ϕ will also be continuous on ω. Therefore, we obtain $\int \phi d\lambda_i = 0$. In addition, since $p^{1*}_{\lambda_i} = 0$ on $\complement K_i$ (see the beginning of the proof of Theorem 3), ϕ is in view of [14], Proposition 3, a 1-harmonic function, that we denote by r_1. Therefore, $u_1 = V_1^\omega u_2 - r_1$ is the first component of a biharmonic pair on ω, namely, of the pair $(V_1^\omega u_2 - r_1, u_2)$. Finally, since the pair (u_1, u_2) is biharmonic on every open set $\omega \subset \overline{\omega} \subset U$, with $\overline{\omega}$ compact, it will also be biharmonic on U.

Corollary 12. Let L_j $(j = 1,2)$ be a second-order linear elliptic operator with regular coefficients defined on a domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$. We consider the biharmonic space of the solutions of the system $L_1 u_1 = -u_2$, $L_2 u_2 = 0$ on Ω. We suppose that there exists a positive potential pair; therefore, there exists a positive L_j-potential $(j = 1,2)$ (cf., [15], part XI, [9], Chap. VII). Then, $u_1(x) = \alpha_x \int u_1 d\mu_x^{\omega_1} - \beta_x \int u_1 d\mu_x^{\omega_2}$ holds for every $x \in \Omega$, where ω_1, ω_2 are concentric balls such that $x \in \omega_1 \subset \overline{\omega_2} \subset \Omega$ (cf. Section 3). This property is characteristic of biharmonic[2] functions on Ω. We notice that if $L_1 = L_2$, then we can also write $u_2(x) = \alpha_x \int u_2 d\mu_x^{\omega_1} - \beta_x \int u_2 d\mu_x^{\omega_2}$.

5 Properties of Binormal Pairs of Measures

Let λ, μ be measures, $\lambda = \lambda_1 - \lambda_2$, $\mu = \mu_1 - \mu_2$, with $\lambda_i \geq 0$, $\mu_i \geq 0$, $(i = 1,2)$, and consider the pairs $\Lambda := (\lambda, 0)$, $B_i := (\lambda_i, 0)$, as well as the pair $M := (0, \mu)$. Therefore, we have $\Lambda_i^{\complement K} = B_{1,i}^{\complement K} - B_{2,i}^{\complement K}$ and $M^{\complement K} = (0, \mu)^{\complement K}$ (cf. Section 1 and the proof of Theorem 3).

Theorem 13. The following are equivalent:

(i) The pair $\Lambda = (\lambda, 0)$ is pure binormal and the pair $M = (0, \mu)$ is 2-normal.

(ii) $\Lambda_i^{\complement K} = 0$, and $M_i^{\complement K} = 0$ $(i = 1,2)$.

(iii) $\int (p_1 - q_1) d\lambda = 0$ and $\int (p_2 - q_2) d\mu = 0$, where (p_1, p_2), (q_1, q_2) are potential pairs in Ω with support in $\complement K$.

(iv) The previous potential pairs could be pure potential pairs.

(v) $\int u_1 d\lambda = 0$, and $\int u_2 d\mu = 0$ hold for every biharmonic pair of functions (u_1, u_2) on an open set $\omega \supset K$.

[1] Analogous notions and results are available in the adjoint case.

[2] The function u_1 is called biharmonic on Ω, if it is the first component of a biharmonic pair on Ω.

(vi) $\lambda = \xi - \Xi_1^{\complement K}$, and $\Xi_2^{\complement K} = 0$, where $(\xi, 0)^{\complement K} = \left(\Xi_1^{\complement K}, \Xi_2^{\complement K}\right)$ with ξ the part of λ supported by the set of points of K where $\complement K$ is 1-thin; $\mu = \tau - T_2^{\complement K}$, where $(0, \tau)^{\complement K} = \left(T_1^{\complement K}, T_2^{\complement K}\right)$, with τ the part of μ supported by the set of points where $\complement K$ is 2-thin.

Proof. (i) ⇔ (ii). We have already established the first part of Theorem 13 in the proof of Theorem 3. Concerning the second part, we can see that these implications are well-known in harmonic spaces (cf., [14]).

(i) ⇒ (v). This is proved in Proposition 10.

(v) ⇒ (i). Suppose there exist points $y_1, y_2 \in \complement K$ where $w^*_{\lambda_1}(y_1) \neq w^*_{\lambda_2}(y_1)$, $p^{2*}_{\mu_1}(y_2) \neq p^{2*}_{\mu_2}(y_2)$; we take as (u_1, u_2) the Green pair $\left(w_y, p_y^2\right)$ and we have $\int w_{y_1}(x) d\lambda_1(x) = \int w_{y_1}(x) d\lambda_2(x)$ as well as $\int p^2_{y_2}(x) d\mu_1(x) = \int p^2_{y_2}(x) d\mu_2(x)$, therefore we get $w^*_{\lambda_1}(y_1) = w^*_{\lambda_2}(y_1)$ and $p^{2*}_{\mu_1}(y_2) = p^{2*}_{\mu_2}(y_2)$, which contradicts our assumptions (cf. Theorem 3).

(iii) ⇒ (v). By [18], Proposition 1.7, there are two continuous potential pairs (p_1, p_2), and (q_1, q_2), which are biharmonic on a relatively compact open set ω', with $K \subset \omega' \subset \overline{\omega'} \subset \omega$, and such that $u_i = p_i - q_i$ on ω', (i=1,2).

(v) ⇒ (iii). We choose an open set $U \supset K$ such that the supports of the potential pairs (p_1, p_2), and (q_1, q_2) are not contained in U; hence, these pairs are biharmonic on U.

(iii) ⇒ (iv). This is straightforward because (iv) is a particular case of (iii).

(iv) ⇒ (i). Suppose there exist two points $y_1, y_2 \in \complement K$ such that $w^*_{\lambda_1}(y_1) \neq w^*_{\lambda_2}(y_1)$, and $p^{2*}_{\mu_1}(y_2) \neq p^{2*}_{\mu_2}(y_2)$. We take as pure potential pairs supported on $\complement K$, the Green's pairs $\left(w_y, p_y^2\right)$, and $\left(kw_y, kp_y^2\right)$, where $k > 0$, $k \neq 1$. Therefore, we obtain

$$\int \left(kw_{y_1}(x) - w_{y_1}(x)\right) d\lambda_1(x) = \int \left(kw_{y_1}(x) - w_{y_1}(x)\right) d\lambda_2(x),$$

and $(k-1) p^{2*}_{\mu_1}(y_2) = (k-1) p^{2*}_{\mu_2}(y_2)$; clearly, this contradicts our assumptions (see also Theorem 3).

(ii)⇒(vi). $(\lambda, 0) = (\xi, 0) + (\sigma, 0)$, with σ the part of λ supported by the set of points where $\complement K$ is not 1-thin. Setting $\Sigma := (\sigma, 0)$, we have $(\lambda, 0)^{\complement K} = \left(\Lambda_1^{\complement K}, \Lambda_2^{\complement K}\right) = \left(\Xi_1^{\complement K}, \Xi_2^{\complement K}\right) + \left(\Sigma_1^{\complement K}, \Sigma_2^{\complement K}\right)$. On the other hand, we know that $\Sigma_1^{\complement K} = \sigma$. As $\Lambda_1^{\complement K} = 0$, we deduce that $\Sigma_1^{\complement K} + \Xi_1^{\complement K} = 0$; consequently, it follows that $\lambda = \xi + \sigma = \xi - \Xi_1^{\complement K}$. Furthermore, since $\Lambda_2^{\complement K} = 0$, we obtain $\Xi_2^{\complement K} + \Sigma_2^{\complement K} = 0$. Finally, in view of [16], Remark 2.12, we conclude that $\Sigma_2^{\complement K} = 0$ (see also, [15], Theorem 7.13).

(vi)⇒(i). We know that $\int P_1^{\complement K} d\xi = \int p_1 d\Xi_1^{\complement K} + \int p_2 d\Xi_2^{\complement K}$, where (p_1, p_2) is a potential pair; since $\Xi_2^{\complement K} = 0$, it follows that $\int P_1^{\complement K} d\xi = \int p_1 d\Xi_1^{\complement K}$. Now, if (p_1, p_2) is the Green's pair $\left(w_y, p_y^2\right)$, then we have $\int W_y^{\complement K}(x) d\xi(x) = \int w_y(z) d\Xi_1^{\complement K}(z)$. That is, $\int W_x^{*\complement K}(y) d\xi(x) = \int w_z^*(y) d\Xi_1^{\complement K}(z)$, in view of Lemma 1. As for $x \in K$, it holds that $W_x^{*\complement K} = w_x^*$ on $\complement K$, so we deduce that $\int w_x^*(y) d\xi(x) = \int w_z^*(y) d\Xi_1^{\complement K}(z)$; therefore, $w_\lambda^* = 0$ on $\complement K$.

Note. We point out that the implication (vi)⇒(ii) can be established, by reversing the arguments in the proof of (ii)⇒(vi). We can see in the proof of [4], Proposition 3, that $\sigma = -\Xi_1^{\complement K}$ holds for every 1-normal measure.

Theorem 14. Let K be a compact subset of Ω. The following are equivalent:

(i) There exists a pure binormal pair of measures $(\lambda, 0)$ for the compact set K, with $\lambda \neq 0$.
(ii) $\complement K$ is 1-thin for at least one point of K.

Proof. (i)⇒(ii). In view of Theorem 13, we have $\Lambda_i^{\complement K} = 0$, $(i = 1,2)$, and by assumption $\lambda \neq 0$. If $\complement K$ is not 1-thin at any point of K, then we will obtain $\lambda = \Lambda_1^{\complement K}$ (cf., [14], Proposition 3); since $\Lambda_1^{\complement K} = B_{1,1}^{\complement K} - B_{2,1}^{\complement K} = 0$, it follows that $\lambda = 0$. This is a contradiction.

(ii)⇒(i). Given a pure binormal pair $(\lambda, 0)$, suppose that $\lambda = 0$. By assumption and in view of Theorem 13, we will obtain $\lambda - \Lambda_1^{\complement K} = \xi - \Xi_1^{\complement K} = 0$ and $\xi \neq 0$ (since $\complement K$ is 1-thin for at least one point of K). As the measure ξ is supported by the set of unstable points of K, and $\Xi_1^{\complement K}$ is supported by the set of points where $\complement K$ is not 1-thin, we deduce that $\xi \neq \Xi_1^{\complement K}$ (see, [1], Proposition 4.6, [4], Lemma VIII, 2); therefore, $\lambda \neq 0$, which is a contradiction.

We denote by $\mathcal{M}$ the set of measures on Ω. We endow it with the vague topology, that is, the topology of the simple convergence on the space of continuous functions with compact support. Similarly, we consider the set $\mathcal{M}\times\mathcal{M}$ with the respective vague topology. We also denote by $\mathcal{K}_i$ the set of points of K, where $\complement K$ is i-thin, and by $\mathcal{N}$ (resp. $\mathcal{N}_i$), the set of pure binormal pairs of measures (resp. the set of i-normal measures, $i=1,2$) for K. Finally, we recall that $(\epsilon_x,0)^{\complement K}=(\mu_x^{\complement K},\nu_x^{\complement K})$, where $\mu_x^{\complement K}$ is the swept measure of ϵ_x on $\complement K$ in the 1-harmonic space, and $(0,\epsilon_x)^{\complement K}=(0,\lambda_x^{\complement K})$, where $\lambda_x^{\complement K}$ is the swept measure of ϵ_x on $\complement K$ in the 2-harmonic space.

Theorem 15.

(i) The pairs $(\epsilon_x-\mu_x^{\complement K},\nu_x^{\complement K})$, where $x\in\mathcal{K}_1$, form a total subset of $\mathcal{N}$.
(ii) The measures $(\epsilon_x-\mu_x^{\complement K})$, where $x\in\mathcal{K}_1$, form a total subset of $\mathcal{N}_1$.
(iii) The measures $(\epsilon_x-\lambda_x^{\complement K})$, where $x\in\mathcal{K}_2$, form a total subset of $\mathcal{N}_2$.

Proof. (i) First, it is well known that the swept pair $(\Xi_1^{\complement K},\Xi_2^{\complement K})$ of $(\xi,0)$ on $\complement K$, is expressed by $\Xi_1^{\complement K}(f)=\int\mu_x^{\complement K}(f)d\xi(x)$, $\Xi_2^{\complement K}(f)=\int\nu_x^{\complement K}(f)d\xi(x)$, where f is any continuous function with compact support. Next, we recall that by definition of the integral, there exist points x_n of $\mathcal{K}_1$ such that

$$|\int\mu_x^{\complement K}(f)d\xi(x)-\sum_{n=1}^{N}\lambda_n\mu_{x_n}^{\complement K}(f)|<\epsilon' \qquad (3)$$

with $\sum_{n=1}^{N}\lambda_n=\xi(\mathcal{K}_1)$. Note that by considering a suitable partition of $\mathcal{K}_1$, we can choose the (same) coefficients λ_j, such that relations (3) and (4) are satisfied (cf. [5, p. 109-109], [2] and [7, p. 126-127].

Moreover, according to [2], Theorem 1, chap. III, §2, No. 4, there exists a linear combination $\sum_{j=1}^{p}\lambda_j\epsilon_{x_j}$ such that

$$|\sum_{j=1}^{p}\lambda_j\epsilon_{x_j}(f)-\xi(f)|<\epsilon'' \text{ and } \sum_{j=1}^{p}\lambda_j=\xi(\mathcal{K}_1). \quad (4)$$

Consequently, by combining (3) and (4), we can write

$$\sum_{i=1}^{q}\lambda_i(\epsilon_{x_i}-\mu_{x_i}^{\complement K})(f)-\epsilon\le\xi(f)-\Xi_1^{\complement K}(f)$$
$$\le\sum_{i=1}^{q}\lambda_i(\epsilon_{x_i}-\mu_{x_i}^{\complement K})(f)+\epsilon,$$

and $-\epsilon\le\sum_{i=1}^{q}\lambda_i\nu_{x_i}^{\complement K}(f)\le\epsilon$ with $\sum_{i=1}^{q}l_i=\xi(\mathcal{K}_1)$. Since, by Theorem 13, $(\lambda,0)=(\xi,0)-(\Xi_1^{\complement K},\Xi_2^{\complement K})$, the result follows. Assertions (ii) and (iii) can be proved in the same way.

Finally, we shall examine how normal and binormal measures are connected.

Proposition 16.

(i) If $(\lambda,0)$ is a pure binormal pair for the compact set K, then the measure λ is 1-normal for K.
(ii) Conversely, suppose that λ is a 1-*normal* measure for K. Then the pair $(\lambda,0)$ is not necessarily a pure binormal pair, even if the j-harmonic spaces coincide *(*$j=1,2$*)*.

Proof. (i) Since the pair $(w_\lambda^*,p_\lambda^{1*})$ is biharmonic adjoint on $\complement K$, and therefore compatible, then the equality $w_{\lambda_1}^*=w_{\lambda_2}^*$ on $\complement K$ implies that $p_{\lambda_1}^{1*}=p_{\lambda_2}^{1*}$ holds there. Consequently, λ is 1-normal for K.
(ii) If $p_{\lambda_1}^{1*}=p_{\lambda_2}^{1*}$ on $\complement K$, then we assert that $w_{\lambda_1}^*=w_{\lambda_2}^*+h_2^*$ on $\complement K$, where h_2^* is an adjoint 2-harmonic function on $\complement K$. Indeed, since the pure potential pairs satisfy the relations $\Gamma_1^*w_{\lambda_1}^*=p_{\lambda_1}^{1*}$ and $\Gamma_1^*w_{\lambda_2}^*=p_{\lambda_2}^{1*}$ on $\complement K$, we obtain $\Gamma_1^*(w_{\lambda_1}^*-w_{\lambda_2}^*)=0$ on $\complement K$, that is, $w_{\lambda_1}^*-w_{\lambda_2}^*=h_2^*$, where h_2^* is an adjoint 2-harmonic function on the complement of K.

Let us now take for the elliptic operator, the Laplacian. We have the following inclusion:

$$\{\lambda:(\lambda,0)\text{ is pure binormal}\}\subset\{\lambda:\lambda\text{ is a }1-\text{normal measure}\}.$$

For instance, the measure $\lambda=\epsilon_x-\mu_x^{\omega}$, where ω is a ball, is normal for $K=\overline{\omega}$, but the pair $(\lambda,0)$ is not a pure binormal pair. On the other hand, the measure $\lambda=\epsilon_x-\alpha_x\mu_x^{\omega_1}+\beta_x\mu^{\omega_2}$ is 1-normal, and the pair $(\lambda,0)$ is also pure binormal (ω_1, ω_2 are concentric balls, and $\omega_1\subset\overline{\omega}_1\subset\omega_2$).

Nevertheless, in general, we have:

Proposition 17. Let λ be a 1-normal measure for K. Then, $(\lambda, 0)$ is a pure binormal pair if and only if $\Xi_2^{\complement K} = 0$.

Proof. This follows immediately from Theorem 13.

6 Conclusion and Future Work

In the present paper, we consider a biharmonic elliptic space, corresponding in $\mathbb{R}^n$ to the solutions of the system $L_1 h_1 = -h_2$, $L_2 h_2 = 0$, where the second-order linear elliptic differential operators L_i ($i = 1,2$) (cf., [15]), have adjoint operators L_i^* ($i = 1,2$) satisfying $L_2^* h_2 = -h_1$, $L_1^* h_1 = 0$ (cf., [20]). By introducing binormal pairs of measures, we have extended the normal measures from the harmonic case (cf., [6], [10], [14]) to the biharmonic context.

More specifically, by using pure adjoint potential pairs, we have studied the binormal pairs of measures satisfying the equivalent properties of Theorem 3. We have also established some characteristic properties of biharmonic pairs and binormal pairs of measures. In addition, we have pointed out in Theorems 13 and 14, the connection between binormal pairs of measures and the fine topologies of the associated harmonic spaces (corresponding in $\mathbb{R}^n$ to the solutions of equations $L_1 h = 0$ and $L_2 u = 0$ respectively). On the other hand, Theorem 15 provides an approximation of pure binormal pairs of measures by normal measures.

Finally, Example 6 generalizes a characteristic property of the classical biharmonic case for the equation $\Delta^2 u = 0$ (cf., [17]). It is an important result that may be useful to study some boundary value problems, for the above systems. For instance, it would be interesting to examine the following biharmonic problem in $\mathbb{R}^n$. Can we determine a biharmonic function in the interior of a smooth domain if its values and the values of its normal derivative are known on the boundary? Here, a biharmonic function is the first component of a biharmonic pair. Another interesting open problem would be the extension of the results in [13], in the context of the heat equation, to more general parabolic operators.

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