Solving one species Lotka–Volterra equation by the new iterative method (NIM)

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Abstract: In this paper, we investigate the use of the new iterative method, referred to as the NIM, for solving the one species Lotka-Volterra equation. This equation, which describes the dynamics of populations in ecological systems, has been widely studied in the field of mathematical biology. However, finding an analytical solution to this equation can be challenging. To overcome this, we propose using the NIM as an alternative method for solving the equation. To demonstrate the effectiveness of the NIM, we conduct a comparative study between it and other well-established techniques such as the differential transformation method (DTM), the variational iteration method (VIM), and the Adomian decomposition method (ADM). Through numerical simulations, we show that the NIM is able to accurately and efficiently solve the one species Lotka-Volterra equation, making it a promising tool for researchers in the field of mathematical biology.

Key-Words: Lotka–Volterra equation, The new iterative method (NIM), The differential transformation method (DTM), The variational iteration method (VIM), The Adomian decomposition method (ADM)

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1 Introduction

Nonlinear phenomena play a crucial role in various areas of science and engineering. However, solving real-life nonlinear models is often challenging, both numerically and theoretically. To make these models tractable, unnecessary assumptions are often made, [1], [2], [3], [4].

The Lotka-Volterra equations, first introduced in the field of mathematical biology, provide a mathematical model for describing the time evolution of a biological system as reported in [5]. These equations have also been applied in various engineering fields, such as in the simultaneous control of chemical processes and nonlinear systems, [6]. In particular, the one-predator one-prey Lotka-Volterra equation serves as a simple example of a nonlinear control system.

The DTM was initially introduced by [7], and further developed in [8], [9]. DTM is an iterative technique that aims to obtain Taylor series solutions of various types of differential equations, as explored in [10], [11], [12]. One of the main benefits of DTM is that it can be applied to a wide range of differential equations, without the need for linearization, discretization, or perturbation. This makes it an accurate method with relatively low computational requirements as reported in [13].

The Adomian decomposition method (ADM) was developed by [14], as a method for solving challenging nonlinear physical problems. Since its introduction, it has been utilized to address a wide variety of differential equations, as reported in literature such as [15], [16], [17], [18].

The VIM was first introduced by [19], and further expanded upon in subsequent publications such as [20], [21], [22]. This method has been demonstrated to be effective in solving a wide range of ordinary and partial differential equations, as shown in various studies including, [23], [24], [25], [26].

The NIM was first introduced by [27]. Since its inception, this method has been shown to be a powerful technique for solving a wide range of nonlinear equations as reported in [28], [29], [30], [31], [32], [33], [34]. Recently, NIM has been used to develop a novel predictor-corrector method, [35]. Additionally, Noor et al. have used NIM to create numerical methods for solving algebraic equations as reported in [36].

In this article, we present an application of the NIM to solve the one species Lotka-Volterra equation, which describes the dynamics of populations in ecological systems. The Lotka-Volterra equation is a nonlinear differential equation that can be challenging to solve analytically. To overcome this, we employ the NIM as an alternative method for solving the equation. In order to evaluate the effectiveness of the NIM, we conduct a comparison of the results obtained with the NIM with the exact solution, as well as other well-established techniques such as the DTM, VIM, and the ADM. Through this comparison, we aim to show the potential of the NIM as a powerful tool for solving the one species Lotka-Volterra equation and similar nonlinear differential equations.

2 The new iterative method (NIM)

In this section, the NIM numerical method will be outlined as follows, [29], [30], [31], [32]:

$$u = f + L(u) + N(u), \tag{1}$$

In the equation above, f is a known function, and L and N are linear and nonlinear operators, respectively.
The NIM solution for Eq. (1) has the form

$$u = \sum_{i=0}^{\infty} u_i. \tag{2}$$

Since L is linear then

$$L\left(\sum_{i=0}^{\infty} u_i\right) = \sum_{i=0}^{\infty} L(u_i). \tag{3}$$

The nonlinear operator N in Eq. (1) is decomposed as below

$$\begin{aligned} N\left(\sum_{i=0}^{\infty} u_i\right) &= N(u_0) \\ &+ \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \\ &= \sum_{i=0}^{\infty} A_i, \end{aligned}$$

where

$$\begin{aligned} A_0 &= N(u_0) \\ A_1 &= N(u_0+u_1) - N(u_0) \\ A_2 &= N(u_0+u_1+u_2) - N(u_0+u_1) \\ &\vdots \\ A_i &= \left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}, \quad i \geq 1. \end{aligned}$$

Using Eqs.(2), (3) and (4) in Eq. (1), we get

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + \sum_{i=0}^{\infty} A_i. \tag{4}$$

The solution of Eq. (1) can be expressed as

$$u = \sum_{i=0}^{\infty} u_i = u_0 + u_1 + u_2 + \ldots + u_n + \ldots, \tag{5}$$

where

$$\begin{aligned} u_0 &= f \\ u_1 &= L(u_0) + A_0 \\ u_2 &= L(u_1) + A_1 \\ &\vdots \\ u_n &= L(u_{n-1}) + A_{n-1} \\ &\vdots \end{aligned} \tag{6}$$

Algorithm

$$\begin{aligned} INPUT \quad &: \quad Read\ M(Number\ of\ iterations); \\ & \quad Read\ L(u);\ N(u);\ f \\ & \quad Step-1\ : u_{-1}=0,\ u_0=f \\ & \quad Step-2 :\ For(n=0,\ n \leq M, n++) \\ & \quad \{ \\ & \quad Step-3 : A_n = f(u_n) - f(u_{n-1}); \\ & \quad Step-4 : u_{n+1} = f + L(u_n) + A_n; \\ & \quad Step-5 : u = u_{n+1} \\ & \quad \}\ end \\ OUTPUT \quad &: \quad u \end{aligned}$$

3 The convergence of the NIM

Theorem 1: For any n and for some real $L > 0$ and $||u_i|| \leq M < \frac{1}{e}, i = 1, 2, ...,$ if N is $C^{(\infty)}$ in the neighborhood of u_0 and $||N^{(n)}(u_0)|| \leq L$, then $\sum_{n=0}^{\infty} H_n$ is convergent absolutely and $||H_n|| \leq LM^n e^{n-1}(e-1),\ n = 1, 2, ...$

Proof:

$$\begin{aligned} ||H_n|| \leq LM^n \sum_{i_n=1}^{\infty} \sum_{i_{n-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} \left(\prod_{j=1}^{n} \frac{1}{i_j!}\right) \\ = LM^n e^{n-1}(e-1). \end{aligned} \tag{7}$$

Thus the series $\sum_{n=1}^{\infty} ||H_n||$ is dominated by the convergent series $LM(e-1)\sum_{n=1}^{\infty}(Me)^{n-1}$, where $M < 1/e$. Hence, $\sum_{n=0}^{\infty} H_n$ is absolutely convergent, due to the comparison test.

As it is difficult to show boundedness of u_i, for all i, a more useful result is provedin the following theorem, where conditions on $N^{(k)}(u_0)$ are given which are sufficient to guarantee convergence of the series.

Theorem 2: The series $\sum_{n=0}^{\infty} H_n$ is convergent absolutely if N is $C^{(\infty)}$ and $||N^{(n)}(u_0)|| \leq M \leq e^{-1}, \forall n$.

Proof: Consider the recurrence relation

$$\varepsilon_n = \varepsilon_0 exp(\varepsilon_{n-1}), \;\; n = 1, 2, 3, ..., \tag{8}$$

where $\varepsilon_0 = M$. Define $\eta_n = \varepsilon_n - \varepsilon_{n-1}, n = 1, 2, 3, \cdots$. We observe that

$$||H_n|| \leq \eta_n, \; n = 1, 2, 3, \cdots. \tag{9}$$

Let

$$\sigma_n = \sum_{i=1}^{n} \eta_i = \varepsilon_n - \varepsilon_0. \tag{10}$$

Not that $\varepsilon_0 = e^{-1} > 0$, $\varepsilon_1 = \varepsilon_0 exp(\varepsilon_0) > \varepsilon_0$ and $\varepsilon_2 = \varepsilon_0 exp(\varepsilon_1) > \varepsilon_0 exp(\varepsilon_0) = \varepsilon_1$. In general, $\varepsilon_n > \varepsilon_{n-1} > 0$. Hence $\sum \eta_n$ is a series of positive real numbers. Note that

$$\begin{gathered} 0 < \varepsilon_0 = M = e^{-1} < 1, \\ 0 < \varepsilon_1 = \varepsilon_0 exp(\varepsilon_0) < \varepsilon_0 e^1 = e^{-1} e^1 = 1, \\ 0 < \varepsilon_2 = \varepsilon_0 exp(\varepsilon_1) < \varepsilon_0 e^1 = 1. \end{gathered} \tag{11}$$

In general $0 < \varepsilon_n < 1$. Hence, $\sigma = \varepsilon_n - \varepsilon_0 < 1$. This implies that $\{\sigma_n\}_{n=1}^{\infty}$ is bounded above by 1, and hence convergent. Therefore, $\sum H_n$ is absolutely convergent by comparison test.

4 Numerical results and discussion

Here, we will focus on the study of the one species Lotka-Volterra equation in the context of competition for a finite source of food. The Lotka-Volterra equation is a mathematical model that describes the dynamics of populations in ecological systems and it is widely used to study the interactions between different species. In this particular scenario, we will investigate the behavior of one species that is competing for a limited food supply. The analysis will involve understanding the rate of change of the population of the species and how it is affected by the food availability and the competition with other species. Through this analysis, we aim to gain insights into the impact of competition and resource availability on population dynamics and the behavior of the species. The one species Lotka-Volterra equation in the form:

$$\frac{du}{dt} = u(b + au), \quad b > 0, \quad a < 0, \quad u(0) > 0, \tag{12}$$

where a and b are constants. With exact solution:

$$u(t) = \frac{be^{bt}}{\frac{b+au(0)}{u(0)} - ae^{bt}} \quad \text{for} \quad b \neq 0, \tag{13}$$

$$u(t) = \frac{u(0)}{1 - au(0)t}, \quad \text{for} \quad b = 0. \tag{14}$$

To apply the NIM to solve equation (12) with the initial condition of $u(0) = 0.1$, we integrate eq. (12) and utilize the specified initial condition to obtain:

$$u(t) = 0.1 + \int_0^t u(b + au)dt \tag{15}$$

By applying NIM we get:

$$\begin{aligned} u_0 &= 0.1, \\ u_1 &= 0.07t \\ u_2 &= -0.0049\, t^2\, (t - 2.857143) \\ &\vdots \end{aligned} \tag{16}$$

The four-term solution is:

$$\begin{aligned} u(t) \quad &= 0.1 + 0.07\, t - 0.0049\, t^2 (t - 2.857143) \\ &-0.00001029\, (t + 5.587558) t^3 (t - 1.234763) \\ &(t - 3.493845)(t - 7.525617) \\ &-2.117682 \times 10^{-11}\, (t + 5.877433) \\ &(t + 5.005629) t^4 (t - 4.090877) \\ &(t - 6.871677)(t - 7.828486) \\ &(t^2 + 6.991252\, t + 21.95022) \\ &(t^2 - 2.520506\, t + 1.605527) \\ &(t^2 - 10.84848\, t + 38.63197) \end{aligned} \tag{17}$$

4.1 Discussion

The solutions calculated using the NIM are evaluated against the accurate solution and solutions obtained through other techniques like the ADM, [25], DTM, [12], and VIM ,[26]. Table 1 presents a comparison between the exact solution and the numerical solutions obtained using the NIM, ADM, DTM, and VIM for $b = 1, a = -3$, $u(0) = 0.1$ and $t \in [0, 1]$. Similarly, Table 2 compares the numerical solutions obtained by the NIM, ADM, DTM, VIM, and the exact solution for $b = 1, a = -3$, $u(0) = 0.1$ and $t \in [0, 3]$.

5 Conclusions

In this article, we employ the new iterative method (NIM) developed by Daftardar-Gejji and Jafari to solve the one species Lotka-Volterra equation. This method was implemented in a direct manner, without the need for linearization, perturbation, or any restrictive assumptions. The results obtained using the NIM were compared to those obtained using other well-established methods such as the variational iteration method (VIM), the differential transformation method (DTM), and the Adomian decomposition method (ADM), and it was found that the NIM is a more effective method for solving non-linear equations. Through our analysis, we have demonstrated that the NIM is a highly efficient, accurate, and cost-effective method for solving the one species Lotka-Volterra equation.

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Table 1: Comparison study under the conditions $b = 1$, $a = -3$, and an initial value of $u(0) = 0.1$

t	Exact solution	NIM$_4$	ADM, ϕ_3 [25]	VIM_2 [25]	DTM_6 [12]
0.0	0.1000000	0.1000000	0.1000000	0.1000000	0.1000000
0.2	0.1145329	0.1145329	0.1145600	0.1145545	0.1145329
0.4	0.1300011	0.1300011	0.1302400	0.1302590	0.1300004
0.6	0.1461629	0.1461627	0.1470400	0.1474445	0.1461546
0.8	0.1627259	0.1627256	0.1649600	0.1671263	0.1626790
1.0	0.1793672	0.1793669	0.1840000	0.1915249	0.1791887

Table 2: Comparison study under the conditions $b = 1$, $a = -3$, and an initial value of $u(0) = 0.1$

t	Exact solution	NIM$_4$	VIM_2 [26]	ADM, ϕ_3 [25]	DTM_9 [12]
0.0	0.10000	0.10000	0.10000	0.10000	0.10000
0.5	0.13801	0.13801	0.13862	0.13850	0.13801
1.0	0.17936	0.17937	0.19152	0.18400	0.17937
1.5	0.21921	0.21921	0.29877	0.23650	0.21932
2.0	0.25333	0.25333	0.30286	0.29600	0.25442
2.5	0.27975	0.27969	-4.4899	0.36250	0.28519
3.0	0.29864	0.29824	-69.317	0.43600	0.31533