A New Approach in Solving Regular and Singular Conformable Fractional Coupled Burger's Equations

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Abstract: - The conformable double ARA decomposition approach is presented in this current study to solve one-dimensional regular and singular conformable functional Burger's equations. We investigate the conformable double ARA transform's definition, existence requirements, and some basic properties. In this study, we introduce a novel interesting method that combines the double ARA transform with Adomian's decomposition method, in order to find the precise solutions of some nonlinear fractional problems. Moreover, we use the new approach to solve Burgers' equations for both regular and singular conformable fractional coupled systems. We also provide several instances to demonstrate the usefulness of the current study. Mathematica software has been used to get numerical results.

Key-Words: - Conformable ARA transform; Conformable double ARA decomposition method; Singular one-dimensional coupled Burgers' equation; Conformable partial fractional derivative.

1 Introduction

Fractional partial differential equations have drawn significant interest from a wide range of specialists in applied sciences and engineering, including acoustics, control, and viscoelasticity. In many areas of mathematics and physics, partial differential equations are crucial. To examine several time-fractional partial differential equations, the authors use a novel strategy termed the "simplest equation method", [1], [2].

In the context of applied sciences like mathematical modeling and fluid mechanics, this work focuses on Burger's equation. Burger's equation was initially brought up about steady-state solutions, in fact, [3]. Burger later changed the approach to characterize the viscosity of certain fluid types, [4]. The conformable double Laplace transform approach,

which was first proposed in [5], was improved and used to solve fractional partial differential equations. This method has been used by a number of academics to obtain precise and numerical solutions to this type of equation. To precisely solve time-fractional Burger's equations, other scholars used the first integral technique, [6]. Another set of researchers developed the coupled Burger's equation solution using the generalized two-dimensional differential transform approach, [7], [8], [9], [10].

The ARA transform is a revolutionary integral transform that Saadeh and others introduced in, [11]. ARA is a new transform; it is not an acronym. It has novel properties, including the ability to generate numerous transforms by varying the value of the index n, a duality with the Laplace transform, and the capacity to get around the singularity at time zero. Many researchers have studied the new approach and implemented it to solve many problems by merging it with other numerical methods or other transforms, such as ARA-Sumudu transform, [12], [13], Laplace-ARA transform, [14], double ARA transform, [15], [16], ARA residual power series method, [17], [18].

In this article, we choose to build a unique combination of Adomian's decomposition method and the double ARA transform, so that we obtain the advantages of these two methods and fully utilize these two potent techniques. The conformable double ARA transform method will be introduced in this research in combination with Adomian's decomposition method, [19], to solve systems of conformable fractional partial differential equations.

With the help of the conformable double ARA decomposition approach, this study aims to provide analytical solutions for the coupled, one-dimensional, singular, and regular conformable fractional Burger's equations (CDARADM). The following space-time fractional order coupled with Burger's equations were described in [20], and are given below:

$$\frac{\partial^q u}{\partial t^q} - \frac{\partial^{2p} u}{\partial x^{2p}} + \lambda\, u\, \frac{\partial^p u}{\partial x^p} + \alpha\, \frac{\partial^p}{\partial x^p}(uv)$$
$$= k\left(\frac{x^p}{p}, \frac{t^q}{q}\right),$$
$$\frac{\partial^q v}{\partial t^q} - \frac{\partial^{2p} v}{\partial x^{2p}} + \lambda\, v\, \frac{\partial^p v}{\partial x^p} + \beta\, \frac{\partial^p}{\partial x^p}(uv)$$
$$= l\left(\frac{x^p}{p}, \frac{t^q}{q}\right). \tag{1}$$

This article is organized as follows, in the following section, we present the ARA transform with the main characteristics. In Section 3, we introduce some preliminaries about the conformable fractional derivatives. The conformable ARA transform and some related results are presented in Section 4. In Section 5, we introduce some numerical experiments to prove the efficiency and applicability of the new method.

2 ARA Integral Transforms, [11]

Definition 1. If $h(x)$ is a continuous function on $(0, \infty)$, then the ARA transform of order n
$$\mathcal{G}_n[h(x)](r) = Q(n, r)$$
$$= r \int_0^\infty x^{n-1} e^{-rx} h(x)\, dx, \qquad r > 0, \tag{2}$$
and the inverse ARA transform is defined as
$$\mathcal{G}_{n+1}^{-1}\big[\mathcal{G}_{n+1}[h(x)]\big]$$
$$= \frac{(-1)^{2n}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{rx}\, Q(r)\, dr = h(x), \tag{3}$$
where
$$Q(r) = \int_0^\infty e^{-rx} h(x)\, dx.$$

Theorem 1. Let $h(x)$ be piecewise continuous in every finite interval $0 \le x \le \alpha$ and satisfies the condition
$$|x^{n-1} h(x)| \le M e^{\alpha x}.$$
where M is a positive constant. Then, ARA integral transform exists for all $r > \alpha$.

Proof. Using the definition of ARA transform, we get
$$|Q(n, r)| = \left| r \int_0^\infty x^{n-1} e^{-rx} h(x)\, dx \right|.$$
Thus, we get
$$|Q(n, r)| = \left| r \int_0^\beta x^{n-1} e^{-rx} h(x)\, dx \right.$$
$$\left. + r \int_\beta^\infty x^{n-1} e^{-rx} h(x)\, dx \right|$$
$$\le r \left| \int_\beta^\infty x^{n-1} e^{-rx} h(x)\, dx \right|$$
$$\le r \int_\beta^\infty e^{-rx} |x^{n-1} h(x)|\, dx$$
$$\le r \int_\beta^\infty e^{-rx} M e^{\alpha x}\, dx$$
$$= r M \int_\beta^\infty e^{-(r-\alpha)x}\, dx$$
$$= \frac{r M}{r - \alpha} e^{-\beta(r-\alpha)}.$$
Hence, the integral exists for all $r > \alpha$, and $\mathcal{G}_{n+1}[h(x)]$ exists.

Now, we present some properties of ARA transform.

If $H(n,r) = \mathcal{G}_n[h(x)]$ and $G(n,r) = \mathcal{G}_n[g(x)]$ and $a, b \in \mathbb{R}$, then

- $\mathcal{G}_n[a\,h(x) + b\,g(x)]$
$$= a\,\mathcal{G}_n[h(x)] \qquad (4)$$
$$+ b\,\mathcal{G}_n[g(x)].$$

- $\mathcal{G}_n^{-1}[a\,H(n,r) + b\,G(n,r)]$
$$= a\,\mathcal{G}_n^{-1}[H(n,r)] \qquad (5)$$
$$+ b\,\mathcal{G}_n^{-1}[G(n,r)].$$

- $\mathcal{G}_n[x^\alpha] = \dfrac{\Gamma(\alpha + n)}{r^{\alpha+n-1}}, \quad \alpha > 0. \qquad (6)$

- $\mathcal{G}_n[e^{ax}] = \dfrac{r\Gamma(n)}{(r-a)^n}, \quad a \in \mathbb{R}. \qquad (7)$

- $\mathcal{G}_n[\sin ax] = \dfrac{r}{2i}\Gamma(n)\left(\dfrac{1}{(r-ia)^n} - \dfrac{1}{(r+ia)^n}\right), a \in \mathbb{R}. \qquad (8)$

- $\mathcal{G}_n[h^{(n)}(x)]$
$$= (-1)^{n-1} r \frac{d^{n-1}}{ds^{n-1}}\left(r^{n-1}\mathcal{G}_1[h(x)] \right. \qquad (9)$$
$$\left. - \sum_{k=1}^{n} r^{n-k} h^{(k-1)}(0) \right),$$

where $\mathcal{G}_1[h(x)]$ is the ARA transform of a continuous function $h(x)$ of order one on $[0, \infty)$, and it is given by

$$\mathcal{G}_1[h(x)] = H(r) = r\int_0^\infty e^{-rx} h(x)\, dx, \qquad (10)$$
$$r > 0.$$

In this research, we denote $\mathcal{G}_1[h(x)]$ for $\mathcal{G}[h(x)]$.

3 Conformable Fractional Derivatives (CFD), [21], [22], [23], [24]

Conformable fractional derivatives are investigated and expanded in [15], [16], respectively. The following definitions of CFD are utilized in this study.

Definition 2. The CFD of $h\left(\frac{x^p}{p}\right)$ order p, where $h : (0, \infty) \to \mathbb{R}$, is given by

$$\frac{d^p}{dx^p} h\left(\frac{x^p}{p}\right)$$
$$= \lim_{\delta \to 0} \frac{h\left(\frac{x^p}{p} + \delta x^{1-p}\right) - h\left(\frac{x^p}{p}\right)}{\delta}, \qquad (11)$$
$$\frac{x^p}{p} > 0, \qquad 0 < p \le 1.$$

Definition 3. The conformable space fractional partial derivative of $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$ order p, where $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) : \mathbb{R} \times (0, \infty) \to \mathbb{R}$, is defined as

$$\frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$$
$$= \lim_{\delta \to 0} \frac{h\left(\frac{x^p}{p} + \delta x^{1-p}, \frac{t^q}{q}\right) - h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)}{\delta}, \qquad (12)$$
$$\frac{x^p}{p}, \frac{t^q}{q} > 0, \qquad 0 < p, q \le 1.$$

Definition 4. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) : \mathbb{R} \times (0, \infty) \to \mathbb{R}$. Then, the conformable fractional partial derivative of $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$ of order, q is given by

$$\frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$$
$$= \lim_{\varepsilon \to 0} \frac{h\left(\frac{x^p}{p}, \frac{t^q}{q} + \xi t^{1-q}\right) - h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)}{\xi}, \qquad (13)$$
$$\frac{x^p}{p}, \frac{t^q}{q} > 0, \qquad 0 < p, q \le 1.$$

3.1 Conformable Fractional Derivatives of Some Basic Functions

In the following arguments, we introduce the CFD for some basic functions.

i. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = \frac{x^p}{p}\frac{t^q}{q}$. Then
- $\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{t^q}{q}\right) = \frac{t^q}{q}$,
- $\frac{\partial^q}{\partial t^q}\left(\frac{x^p}{p}\frac{t^q}{q}\right) = \frac{x^p}{p}$.

ii. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = \left(\frac{x^p}{p}\right)^n$. Then
- $\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\right)^n = n\left(\frac{x^p}{p}\right)^{n-1}$,
- $\frac{\partial^q}{\partial t^q}\left(\frac{x^p}{p}\right)^n = 0$.

iii. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = \left(\frac{x^p}{p}\right)^n \left(\frac{t^q}{q}\right)^m$. Then
- $\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\right)^n \left(\frac{t^q}{q}\right)^m = n\left(\frac{x^p}{p}\right)^{n-1}\left(\frac{t^q}{q}\right)^m$,
- $\frac{\partial^q}{\partial t^q}\left(\frac{x^p}{p}\right)^n \left(\frac{t^q}{q}\right)^m = m\left(\frac{x^p}{p}\right)^n\left(\frac{t^q}{q}\right)^{m-1}$.

iv. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = \sin\lambda\left(\frac{x^p}{p}\right)\sin\beta\left(\frac{t^q}{q}\right)$. Then
- $\frac{\partial^p}{\partial x^p}\left(\sin\lambda\left(\frac{x^p}{p}\right)\sin\beta\left(\frac{t^q}{q}\right)\right) = \lambda\cos\lambda\left(\frac{x^p}{p}\right)\sin\beta\left(\frac{t^q}{q}\right)$,
- $\frac{\partial^q}{\partial t^q}\left(\sin\lambda\left(\frac{x^p}{p}\right)\sin\beta\left(\frac{t^q}{q}\right)\right) = \beta\sin\lambda\left(\frac{x^p}{p}\right)\cos\beta\left(\frac{t^q}{q}\right)$.

v. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = e^{\lambda \frac{x^p}{p} + \beta \frac{t^q}{q}}$. Then

- $\frac{\partial^p}{\partial x^p}\left(e^{\lambda \frac{x^p}{p} + \beta \frac{t^q}{q}}\right) = \lambda e^{\lambda \frac{x^p}{p} + \beta \frac{t^q}{q}}$,

- $\frac{\partial^q}{\partial t^q}\left(e^{\lambda \frac{x^p}{p} + \beta \frac{t^q}{q}}\right) = \beta e^{\lambda \frac{x^p}{p} + \beta \frac{t^q}{q}}$.

Property 1. If $h(x,t)$ is a differentiable function of order α and β at the points x and $t > 0$, where $0 < \alpha, \beta \le 1$. Then

$$\frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = x^{1-p} \frac{\partial}{\partial x} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right),$$

$$\frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = t^{1-q} \frac{\partial}{\partial t} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right).$$

Proof. Using Definition **3** and putting $k = \delta x^{1-p}$ in Equation (12), we get

$$\frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = \lim_{\delta \to 0} \frac{h\left(\frac{x^p}{p} + \delta x^{1-p}, \frac{t^q}{q}\right) - \left(\frac{x^p}{p}, \frac{t^q}{q}\right)}{\delta}$$

$$= \lim_{k \to 0} \frac{h\left(\frac{x^p}{p} + k, \frac{t^q}{q}\right) - \left(\frac{x^p}{p}, \frac{t^q}{q}\right)}{kx^{p-1}}$$

$$= x^{1-p} \lim_{k \to 0} \frac{h\left(\frac{x^p}{p} + k, \frac{t^q}{q}\right) - \left(\frac{x^p}{p}, \frac{t^q}{q}\right)}{k}$$

$$= x^{1-p} \frac{\partial}{\partial x} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right).$$

Similarly, we can easily prove that

$$\frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = t^{1-q} \frac{\partial}{\partial t} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right).$$

4 Conformable Double ARA Transform (CDARAT)

In this part of this study, we present the conformable double ARA transform using the following definitions.

Definition 5. Assume that $h(x)$ is a real-valued function defined on $[0, \infty)$ to $\mathbb{R}$, then the conformable ARA transform of $h\left(\frac{x^p}{p}\right)$ is given by

$$\mathcal{G}_x^p \left[h\left(\frac{x^p}{p}\right) \right]$$
$$= r \int_0^\infty e^{-r \frac{x^p}{p}} \left[h\left(\frac{x^p}{p}\right) \right] x^{p-1} dx, \qquad (14)$$
$$\forall r > 0,$$

If the integral exists.

Definition 6. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$ be a piecewise continuous function on the interval $[0, \infty) \times [0, \infty)$ of exponential order, that is considered for some

$a, b \in \mathbb{R}$, $\frac{x^p}{p}, \frac{t^q}{q} > 0$, $\frac{\left|h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)\right|}{e^{a\frac{x^p}{p} + b\frac{t^q}{q}}}$. Under these conditions the conformable double ARA transform is given by

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right] = G_{p,q}(r,s)$$
$$= rs \int_0^\infty \int_0^\infty e^{-r\frac{x^p}{p} - s\frac{t^q}{q}} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) x^{p-1} t^{q-1} dx\, dt,$$
(15)

where $r, s \in \mathbb{C}$, $0 < p, q \le 1$ and the integrals with respect to x^p and t^q respectively, are taken by conformable fractional derivative.

Property 2. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = f\left(\frac{x^p}{p}\right) g\left(\frac{t^q}{q}\right), x > 0, t > 0$. Then

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right] = \mathcal{G}_x[f(x)]\, \mathcal{G}_t[g(t)]. \qquad (16)$$

Proof. The definition of CDARAT implies

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[f\left(\frac{x^p}{p}\right) g\left(\frac{t^q}{q}\right) \right]$$
$$= rs \int_0^\infty \int_0^\infty e^{-r\frac{x^p}{p} - s\frac{t^q}{q}} f\left(\frac{x^p}{p}\right) g\left(\frac{t^q}{q}\right) x^{p-1} t^{q-1} dx\, dt \qquad (17)$$
$$= r \int_0^\infty e^{-r\frac{x^p}{p}} f\left(\frac{x^p}{p}\right) x^{p-1} dx\ s \int_0^\infty e^{-s\frac{t^q}{q}} g\left(\frac{t^q}{q}\right) t^{q-1} dt.$$

Substituting $u = \frac{x^p}{p}, v = \frac{t^q}{q}$, $du = x^{p-1} dx$ and $dv = t^{q-1} dt$ in Equation (17) and simplifying, we obtain

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[f\left(\frac{x^p}{p}\right) g\left(\frac{t^q}{q}\right) \right]$$
$$= r \int_0^\infty e^{-r u} [f(u)] du\ s \int_0^\infty e^{-s v} g(v)\, dv$$
$$= \mathcal{G}_x[f(x)]\, \mathcal{G}_t[g(t)].$$

4.1 CDARAT of Some Elementary Functions

In this section, we present the conformable Double ARA Transform for some basic functions.

i. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = 1$. Then

$$\mathcal{G}_x^p \mathcal{G}_t^q [1] = rs \int_0^\infty \int_0^\infty e^{-r\frac{x^p}{p} - s\frac{t^q}{q}} x^{p-1} t^{q-1} dx\, dt.$$

From Property 2 and Equation (6), we get

$$\mathcal{G}_x^p \mathcal{G}_t^q [1] = \mathcal{G}_x[1]\, \mathcal{G}_t[g(1)] = 1.$$

ii. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = \left(\frac{x^p}{p}\right)^n \left(\frac{t^q}{q}\right)^m$. Then

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\left(\frac{x^p}{p}\right)^n \left(\frac{t^q}{q}\right)^m \right]$$
$$= rs \int_0^\infty \int_0^\infty e^{-r\frac{x^p}{p} - s\frac{t^q}{q}} \left[\left(\frac{x^p}{p}\right)^n \left(\frac{t^q}{q}\right)^m \right] x^{p-1} t^{q-1} dx\, dt\,.$$

From Property 2 and Equation (6), we get

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\left(\frac{x^p}{p}\right)^n \left(\frac{t^q}{q}\right)^m \right] = \mathcal{G}_x[x^n]\, \mathcal{G}_t[t^m] = \frac{n!\, m!}{r^n s^m}.$$

iii. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = e^{\lambda \frac{x^p}{p} + \beta \frac{t^q}{q}}$. Then

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[e^{\lambda \frac{x^p}{p} + \beta \frac{t^q}{q}} \right]$$

$$= rs \int_0^\infty \int_0^\infty e^{-r\frac{x^p}{p} - s\frac{t^q}{q}} \left[e^{\lambda \frac{x^p}{p} + \beta \frac{t^q}{q}} \right] x^{p-1} t^{q-1} dx\, dt.$$

From Property 2 and Equation (7), we get

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[e^{\lambda \frac{x^p}{p} + \beta \frac{t^q}{q}} \right] = \mathcal{G}_x[e^{\lambda x}] \mathcal{G}_t[e^{\beta t}]$$

$$= \frac{rs}{(r - \lambda)(s - \beta)}.$$

iv. Let $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = \sin \lambda \left(\frac{x^p}{p}\right) \sin \beta \left(\frac{t^q}{q}\right)$. Then

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\sin \lambda \left(\frac{x^p}{p}\right) \sin \beta \left(\frac{t^q}{q}\right) \right]$$

$$= rs \int_0^\infty \int_0^\infty e^{-r\frac{x^p}{p} - s\frac{t^q}{q}} \sin \lambda \left(\frac{x^p}{p}\right) \sin \beta \left(\frac{t^q}{q}\right) x^{p-1} t^{q-1} dx\, dt.$$

From Property 2 and Equation (8), we get

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\sin \lambda \left(\frac{x^p}{p}\right) \sin \beta \left(\frac{t^q}{q}\right) \right] = \mathcal{G}_x[\sin \lambda x] \mathcal{G}_t[\sin \beta t]$$

$$= \frac{rs}{(r^2 + \lambda^2)(s^2 + \lambda^2)}.$$

4.2 Existence Condition for the Conformable Double ARA Transform

If $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$ is an exponential order a and b as $\frac{x^p}{p} \to \infty, \frac{t^q}{q} \to \infty$, If $\exists$ a constant $K > 0$, such that for all $x > X$ and $t > T$

$$\left| h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right| \le K\, e^{\gamma \frac{x^p}{p} + \tau \frac{t^q}{q}}.$$

it is easy to get,

$$h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = O\left(e^{a\frac{x^p}{p} + b\frac{t^q}{q}} \right) \text{ as } \frac{x^p}{p} \to \infty, \frac{t^q}{q} \to \infty.$$

Theorem 2. Let the function $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$ be continuous on the region $(0, X) \times (0, T)$ and are of exponential orders γ and , then the conformable double ARA transform of $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$ exists for all $Re(r) > \gamma, Re(s) > \tau$.

Proof. Using the definition of the CDARAT of $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$, we have

$$|G_{p.q}(r,s)|$$
$$= \left| rs \int_0^\infty \int_0^\infty e^{-r\frac{x^p}{p} - s\frac{t^q}{q}} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) x^{p-1} t^{q-1} dx\, dt \right|$$

$$\le rs \int_0^\infty \int_0^\infty e^{-r\frac{x^p}{p} - s\frac{t^q}{q}} \left| h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right| x^{p-1} t^{q-1} dx\, dt$$

$$\le rs\, K \int_0^\infty \int_0^\infty e^{-(r-\gamma)\frac{x^p}{p} - (s-\tau)\frac{t^q}{q}} x^{p-1} t^{q-1} dx\, dt$$

$$= \frac{rsK}{(r - \gamma)(s - \tau)}$$

For $Re(r) > \gamma, Re(s) > \tau$.

Theorem 3. Let $G_{p.q}(r,s) = \mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right]$, then

i. $\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{x^p}{p} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right] = -r\frac{\partial}{\partial r}\left(\frac{1}{r} \mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right] \right).$

ii. $\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{t^q}{q} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right] = -s\frac{\partial}{\partial s}\left(\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right] \right).$

iii. $\mathcal{G}_x^p \mathcal{G}_t^q \left[\left(\frac{x^p}{p}\right)^2 h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right] = \frac{\partial^2 G_{p.q}(r,s)}{\partial r^2} - \frac{2}{r}\frac{\partial G_{p.q}(r,s)}{\partial r} + \frac{2}{r^2} G_{p.q}(r,s).$

iv. $\mathcal{G}_x^p \mathcal{G}_t^q \left[\left(\frac{t^q}{q}\right)^2 h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right] = \frac{\partial^2 G_{p.q}(r,s)}{\partial s^2} - \frac{2}{s}\frac{\partial G_{p.q}(r,s)}{\partial s} + \frac{2}{s^2} G_{p.q}(r,s).$

v. $\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{x^p}{p}\frac{t^q}{q} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) \right] = \frac{\partial^2 G_{p.q}(r,s)}{\partial r \partial s} - \frac{1}{s}\frac{\partial G_{p.q}(r,s)}{\partial r} - \frac{1}{r}\frac{\partial G_{p.q}(r,s)}{\partial s} + \frac{1}{rs} G_{p.q}(r,s).$

Proof.
Proof of (i). Using the definition of CDARAT of $h\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$, we get

$$G_{p.q}(r,s)$$
$$= rs \int_0^\infty \int_0^\infty e^{-r\frac{x^p}{p} - s\frac{t^q}{q}} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) x^{p-1} t^{q-1} dx\, dt. \quad (18)$$

By differentiating both sides with respect to r in Equation (18), we have

$$\frac{\partial G_{p.q}(r,s)}{\partial r}$$
$$= s \int_0^\infty e^{-s\frac{t^q}{q}} t^{q-1} dt \int_0^\infty \frac{\partial}{\partial r}\left(re^{-r\frac{x^p}{p}} \right) h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) x^{p-1} dx.$$

Calculating the partial derivative of the second integral, we can get

$$\frac{\partial G_{p.q}(r,s)}{\partial r}$$
$$= s \int_0^\infty e^{-s\frac{t^q}{q}} t^{q-1} dt \int_0^\infty \left(-r\frac{x^p}{p} + 1 \right) e^{-r\frac{x^p}{p}} h\left(\frac{x^p}{p}, \frac{t^q}{q}\right) x^{p-1} dx. \quad (19)$$

Therefore,

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{x^p}{p} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right]$$

$$= -\frac{\partial G_{p.q}(r,s)}{\partial r} + \frac{1}{r} G_{p.q}(r,s) \qquad (20)$$

$$= -r \frac{\partial}{\partial r} \left(\frac{1}{r} \mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right] \right).$$

Proof of (iii). Differentiating the both sides with respect to r in Equation (19), we have

$$\frac{\partial^2 G_{p.q}(r,s)}{\partial r^2}$$

$$= s \int_0^\infty e^{-s\frac{t^q}{q}} t^{q-1} dt \int_0^\infty \frac{\partial}{\partial r} \left[\left(1 - r\frac{x^p}{p} \right) e^{-r\frac{x^p}{p}} \right] h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) x^{p-1} dx$$

$$= s \int_0^\infty e^{-s\frac{t^q}{q}} t^{q-1} dt \int_0^\infty \left(r\left(\frac{x^p}{p} \right)^2 - 2\frac{x^p}{p} \right) e^{-r\frac{x^p}{p}} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) x^{p-1} dx.$$

Thus,

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\left(\frac{x^p}{p} \right)^2 h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right]$$

$$= \frac{\partial^2 G_{p.q}(r,s)}{\partial r^2} \qquad (21)$$

$$+ \frac{2}{r} \mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{x^p}{p} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right].$$

From Equation (20), we have

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\left(\frac{x^p}{p} \right)^2 h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right]$$

$$= \frac{\partial^2 G_{p.q}(r,s)}{\partial r^2} + \frac{2}{r^2} G_{p.q}(r,s)$$

$$- \frac{2}{r} \frac{\partial G_{p.q}(r,s)}{\partial r}.$$

Similarly, we can easily prove that:

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{t^q}{q} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right]$$

$$= -s \frac{\partial}{\partial s} \left(\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right] \right).$$

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\left(\frac{t^q}{q} \right)^2 h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right]$$

$$= \frac{\partial^2 G_{p.q}(r,s)}{\partial s^2} + \frac{2}{s^2} G_{p.q}(r,s)$$

$$- \frac{2}{s} \frac{\partial G_{p.q}(r,s)}{\partial s}.$$

Proof of (v). Differentiating both sides with respect to s in Equation (19), we have

$$\frac{\partial^2 G_{p.q}(r,s)}{\partial r \partial s}$$

$$= \int_0^\infty \left(-s\frac{t^q}{q} + 1 \right) e^{-s\frac{t^q}{q}} \left(\int_0^\infty \left(-r\frac{x^p}{p} + 1 \right) e^{-r\frac{x^p}{p}} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) x^{p-1} dx \right) t^{q-1} dt.$$

Therefore,

$$\frac{\partial^2 G_{p.q}(r,s)}{\partial r \partial s}$$

$$= \int_0^\infty s\frac{t^q}{q} e^{-s\frac{t^q}{q}} \left(\int_0^\infty r\frac{x^p}{p} e^{-r\frac{x^p}{p}} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) x^{p-1} dx \right) t^{q-1} dt$$

$$- \int_0^\infty e^{-s\frac{t^q}{q}} \left(\int_0^\infty r\frac{x^p}{p} e^{-r\frac{x^p}{p}} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) x^{p-1} dx \right) t^{q-1} dt$$

$$- \int_0^\infty s\frac{t^q}{q} e^{-s\frac{t^q}{q}} \left(\int_0^\infty e^{-r\frac{x^p}{p}} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) x^{p-1} dx \right) t^{q-1} dt$$

$$+ \int_0^\infty e^{-s\frac{t^q}{q}} \left[\int_0^\infty e^{-r\frac{x^p}{p}} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) x^{p-1} dx \right] t^{q-1} dt.$$

From (i) and (ii), we have

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{x^p}{p} \frac{t^q}{q} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right]$$

$$= \frac{\partial^2 G_{p.q}(r,s)}{\partial r \partial s} + \frac{1}{rs} G_{p.q}(r,s)$$

$$- \frac{1}{s} \frac{\partial G_{p.q}(r,s)}{\partial r} - \frac{1}{r} \frac{\partial G_{p.q}(r,s)}{\partial s}.$$

The proof of (ii) and (iv) can be obtained by similar arguments of (i) and (iii).

Theorem 4. Let $G_{p.q}(r,s) = \mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right]$, then

i. $\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right] = r\, G_{p.q}(r,s) - r\, \mathcal{G}_t^q \left[h\left(0, \frac{t^q}{q} \right) \right].$

ii. $\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{\partial^{2p}}{\partial x^{2p}} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right] = r^2\, G_{p.q}(r,s) - r^2\, \mathcal{G}_t^q \left[h\left(0, \frac{t^q}{q} \right) \right] - r\, \mathcal{G}_t^q \left[\frac{\partial^p}{\partial x^p} h\left(0, \frac{t^q}{q} \right) \right].$

iii. $\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right] = s\, G_{p.q}(r,s) - s\, \mathcal{G}_x^p \left[h\left(\frac{x^p}{p}, 0 \right) \right].$

iv. $\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{\partial^{2q}}{\partial t^{2q}} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right] = s^2\, G_{p.q}(r,s) - s^2\, \mathcal{G}_x^p \left[h\left(\frac{x^p}{p}, 0 \right) \right] - s\, \mathcal{G}_x^p \left[\frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p}, 0 \right) \right].$

Proof.

i. Using the definition of CDARAT for $\frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right)$, we have

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) \right]$$

$$= rs \int_0^\infty \int_0^\infty e^{-r\frac{x^p}{p} - s\frac{t^q}{q}} \qquad (22)$$

$$\frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p}, \frac{t^q}{q} \right) x^{p-1} t^{q-1} dx dt$$

$$= s \int_0^\infty e^{-s\frac{t^q}{q}} t^{q-1}$$
$$\left(r \int_0^\infty e^{-r\frac{x^p}{p}} \frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p},\frac{t^q}{q}\right) x^{p-1} dx \right) dt.$$

Applying Property 1, $\frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p},\frac{t^q}{q}\right) = x^{1-p} \frac{\partial}{\partial x} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)$, then Equation (22) becomes

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right] = s \int_0^\infty e^{-s\frac{t^q}{q}} t^{q-1}$$
$$\left(r \int_0^\infty e^{-r\frac{x^p}{p}} \frac{\partial}{\partial x} h(x,t) dx \right). \quad (23)$$

Thus, the integral inside bracket is given by

$$r \int_0^\infty e^{-r\frac{x^p}{p}} \frac{\partial}{\partial x} h\left(\frac{x^p}{p},\frac{t^q}{q}\right) dx$$
$$= r\left(-h\left(0,\frac{t^q}{q}\right)\right. \quad (24)$$
$$\left. + r \int_0^\infty e^{-r\frac{x^p}{p}} h\left(\frac{x^p}{p},\frac{t^q}{q}\right) x^{p-1} dx \right).$$

Substituting Equation (24) into Equation (23), we obtain

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]$$
$$= r\, G_{p.q}(r,s) \quad (25)$$
$$- r\, \mathcal{G}_t^q \left[h\left(0,\frac{t^q}{q}\right)\right].$$

In the same manner, the CDARAT of $\frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p},\frac{t^q}{q}\right), \frac{\partial^{2p}}{\partial x^{2p}} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)$ and $\frac{\partial^{2q}}{\partial t^{2q}} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)$ can be obtained.

Theorem 5. Let $G_{p.q}(r,s) = \mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]$, then

   i. $\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{x^p}{p} \frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right] = -rs \frac{\partial}{\partial r}\left(\frac{1}{r}\mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right) + rs \frac{d}{dr}\left(\frac{1}{r}\mathcal{G}_x^p \left[h\left(\frac{x^p}{p},0\right)\right]\right).$

   ii. $\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{t^q}{q} \frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right] = -rs \frac{\partial}{\partial r}\left(\frac{1}{s}\mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right) + rs \frac{d}{ds}\left(\frac{1}{s}\mathcal{G}_t^q \left[h\left(0,\frac{t^q}{q}\right)\right]\right).$

Proof of i. The conformable double ARA transforms definition for fractional partial derivatives, implies

$$\frac{\partial}{\partial r}\left[\mathcal{G}_x^p \mathcal{G}_t^q \left(\frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right)\right]$$
$$= s \int_0^\infty e^{-s\frac{t^q}{q}} \frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p},\frac{t^q}{q}\right) t^{q-1} dt \quad (26)$$

$$\left(\int_0^\infty \frac{\partial}{\partial r}\left[r\, e^{-r\frac{x^p}{p}}\right] x^{p-1} dx\right).$$

we calculate the partial derivative in the second integral as follows

$$\int_0^\infty \frac{\partial}{\partial r}\left(r\, e^{-r\frac{x^p}{p}}\right) x^{p-1} dx$$
$$= \int_0^\infty e^{-r\frac{x^p}{p}} x^{p-1} dx \quad (27)$$
$$- r \int_0^\infty \frac{x^p}{p} e^{-r\frac{x^p}{p}} x^{p-1} dx.$$

Substituting Equation (27) into Equation (26), we get

$$\frac{\partial}{\partial r}\left[\mathcal{G}_x^p \mathcal{G}_t^q \left(\frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right)\right]$$
$$= s \int_0^\infty e^{-s\frac{t^q}{q}} \int_0^\infty e^{-r\frac{x^p}{p}} \frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p},\frac{t^q}{q}\right) x^{p-1} t^{q-1} dx dt$$
$$- rs \int_0^\infty e^{-s\frac{t^q}{q}}$$
$$\int_0^\infty e^{-r\frac{x^p}{p}} \frac{x^p}{p} \frac{\partial^q h\left(\frac{x^p}{p},\frac{t^q}{q}\right)}{\partial t^q} x^{p-1} t^{q-1} dx\, dt.$$

Thus,

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{x^p}{p} \frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]$$
$$= -\frac{\partial}{\partial r} \mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]$$
$$+ \frac{1}{r} \mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{\partial^q}{\partial t^q} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right].$$

Using Theorem 4, we have

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{x^p}{p} \frac{\partial^q h\left(\frac{x^p}{p},\frac{t^q}{q}\right)}{\partial t^q}\right]$$
$$= -rs \frac{\partial}{\partial r}\left(\frac{1}{r} \mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right)$$
$$+ rs \frac{d}{dr}\left(\frac{1}{r} \mathcal{G}_x^p \left[h\left(\frac{x^p}{p},0\right)\right]\right)$$

Similarly, one can prove that

$$\mathcal{G}_x^p \mathcal{G}_t^q \left[\frac{t^q}{q} \frac{\partial^p}{\partial x^p} h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]$$
$$= -rs \frac{\partial}{\partial s}\left(\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q \left[h\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right)$$
$$+ rs \frac{d}{ds}\left(\frac{1}{s} \mathcal{G}_t^q \left[h\left(0,\frac{t^q}{q}\right)\right]\right).$$

In the following, we introduce the previous results in the following table, Table 1, below:

Table 1. Analysis of the presented results

$h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$\mathcal{G}_x^p \mathcal{G}_t^q\left[h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)\right] = G_{p.q}(r,s)$
1	1
$\left(\dfrac{x^p}{p}\right)^n\left(\dfrac{t^q}{q}\right)^m$	$\dfrac{n!\,m!}{r^n s^m}.$
$e^{\lambda\frac{x^p}{p}+\beta\frac{t^q}{q}}$	$\dfrac{rs}{(r-\lambda)(s-\beta)}.$
$\sin\lambda\left(\dfrac{x^p}{p}\right)\sin\beta\left(\dfrac{t^q}{q}\right)$	$\dfrac{rs}{(r^2+\lambda^2)(s^2+\lambda^2)}.$
$\dfrac{x^p}{p}h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$-r\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)\right]\right).$
$\dfrac{t^q}{q}h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$-s\dfrac{\partial}{\partial s}\left(\dfrac{1}{s}\mathcal{G}_x^p\mathcal{G}_t^q\left[h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)\right]\right).$
$\left(\dfrac{x^p}{p}\right)^2 h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$\dfrac{\partial^2 G_{p.q}(r,s)}{\partial r^2}-\dfrac{2}{r}\dfrac{\partial G_{p.q}(r,s)}{\partial r}$ $+\dfrac{2}{r^2}G_{p.q}(r,s).$
$\left(\dfrac{t^q}{q}\right)^2 h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$\dfrac{\partial^2 G_{p.q}(r,s)}{\partial s^2}-\dfrac{2}{s}\dfrac{\partial G_{p.q}(r,s)}{\partial s}$ $+\dfrac{2}{s^2}G_{p.q}(r,s).$
$\dfrac{x^p}{p}\dfrac{t^q}{q}h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$\dfrac{\partial^2 G_{p.q}(r,s)}{\partial r\partial s}-\dfrac{1}{s}\dfrac{\partial G_{p.q}(r,s)}{\partial r}$ $-\dfrac{1}{r}\dfrac{\partial G_{p.q}(r,s)}{\partial s}$ $+\dfrac{1}{rs}G_{p.q}(r,s).$
$\dfrac{\partial^p}{\partial x^p}h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$r\,G_{p.q}(r,s)-r\,\mathcal{G}_t^q\left[h\left(0,\dfrac{t^q}{q}\right)\right].$
$\dfrac{\partial^q}{\partial t^q}h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$s\,G_{p.q}(r,s)-s\,\mathcal{G}_x^p\left[h\left(\dfrac{x^p}{p},0\right)\right].$
$\dfrac{\partial^{2p}}{\partial x^{2p}}h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$r^2\,G_{p.q}(r,s)-r^2\,\mathcal{G}_t^q\left[h\left(0,\dfrac{t^q}{q}\right)\right]$ $-r\,\mathcal{G}_t^q\left[\dfrac{\partial^p}{\partial x^p}h\left(0,\dfrac{t^q}{q}\right)\right]..$
$\dfrac{\partial^{2q}}{\partial t^{2q}}h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$s^2\,G_{p.q}(r,s)-s^2\,\mathcal{G}_x^p\left[h\left(\dfrac{x^p}{p},0\right)\right]$ $-s\,\mathcal{G}_x^p\left[\dfrac{\partial^q}{\partial t^q}h\left(\dfrac{x^p}{p},0\right)\right].$
$\dfrac{x^p}{p}\dfrac{\partial^q}{\partial t^q}h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$-rs\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)\right]\right)$ $+rs\dfrac{d}{dr}\left(\dfrac{1}{r}\mathcal{G}_x^p\left[h\left(\dfrac{x^p}{p},0\right)\right]\right).$
$\dfrac{t^q}{q}\dfrac{\partial^p}{\partial x^p}h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)$	$-rs\dfrac{\partial}{\partial s}\left(\dfrac{1}{s}\mathcal{G}_x^p\mathcal{G}_t^q\left[h\left(\dfrac{x^p}{p},\dfrac{t^q}{q}\right)\right]\right)$ $+rs\dfrac{d}{ds}\left(\dfrac{1}{s}\mathcal{G}_t^q\left[h\left(0,\dfrac{t^q}{q}\right)\right]\right).$

5 Applications

The CDARADM is used in this section of the study to solve regular and singular one-dimensional conformable fractional coupled Burger's equations. The goal problem is the same as the problem

examined in [1], when $p=1$ and $q=1$. This is what we mention here.

Example 1.
Consider the One-dimensional conformable fractional coupled Burgers' equation of the form

$$\frac{\partial^q u}{\partial t^q}-\frac{\partial^{2p}u}{\partial x^{2p}}+\lambda\,u\frac{\partial^p u}{\partial x^p}+\alpha\,\frac{\partial^p}{\partial x^p}(uv)$$
$$=k\left(\frac{x^p}{p},\frac{t^q}{q}\right)$$
$$\frac{\partial^q v}{\partial t^q}-\frac{\partial^{2p}v}{\partial x^{2p}}+\lambda\,v\frac{\partial^p v}{\partial x^p}+\beta\,\frac{\partial^p}{\partial x^p}(uv)$$
$$=l\left(\frac{x^p}{p},\frac{t^q}{q}\right),$$

(28)

subject to

$$u\left(\frac{x^p}{p},0\right)=k_1\left(\frac{x^p}{p}\right),$$
$$v\left(\frac{x^p}{p},0\right)=l_1\left(\frac{x^p}{p}\right),$$

(29)

for $t>0$. Here, $k\left(\frac{x^p}{p},\frac{t^q}{q}\right), l\left(\frac{x^p}{p},\frac{t^q}{q}\right)$, $k_1\left(\frac{x^p}{p}\right)$ and $l_1\left(\frac{x^p}{p}\right)$ are given functions, $\lambda, \alpha,$ and β are arbitrary parameters depending on the Peclet number, Stokes velocity of particles due to gravity and Brownian diffusivity, see, [9]. Now, operating the conformable double ARA transform to Equation (28) and the single conformable single ARA transform for Equation (29), to get

$$U(r,s)=K_1(r)+\frac{K(r,s)}{s}$$
$$+\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^{2p}u}{\partial x^{2p}}-\lambda\,u\frac{\partial^p u}{\partial x^p}-\alpha\,\frac{\partial^p}{\partial x^p}(uv)\right].$$

(30)

$$V(r,s)=L_1(r)+\frac{L(r,s)}{s}$$
$$+\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^{2p}v}{\partial x^{2p}}-\lambda\,v\frac{\partial^p v}{\partial x^p}-\beta\,\frac{\partial^p}{\partial x^p}(uv)\right].$$

(31)

The CDARADM defines the solution of the target problem $u\left(\frac{x^p}{p},\frac{t^q}{q}\right)$ and $v\left(\frac{x^p}{p},\frac{t^q}{q}\right)$ in the form of infinite series as

$$u\left(\frac{x^p}{p},\frac{t^q}{q}\right)=\sum_{n=0}^{\infty}u_n\left(\frac{x^p}{p},\frac{t^q}{q}\right),$$
$$v\left(\frac{x^p}{p},\frac{t^q}{q}\right)=\sum_{n=0}^{\infty}v_n\left(\frac{x^p}{p},\frac{t^q}{q}\right).$$

(32)

Define the Adomian's polynomials A_n , B_n and C_n as

$$A_n=\sum_{n=0}^{\infty}u_n u_{xn},$$

(33)

$$B_n = \sum_{n=0}^{\infty} v_n v_{xn},$$

$$C_n = \sum_{n=0}^{\infty} u_n v_{n.}$$

We can compute the Adomian polynomials of the nonlinear terms uu_x, vv_x and uv by the formulas

$$A_0 = u_0 u_{0x.}$$
$$A_1 = u_0 u_{1x} + u_1 u_{0x.}$$
$$A_2 = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x.}$$
$$A_3 = u_0 u_{3x} + u_1 u_{2x} + u_2 u_{1x} + u_3 u_{0x.}$$
$$A_3 = u_0 u_{4x} + u_1 u_{3x} + u_2 u_{2x} + u_3 u_{1x} + u_4 u_{0x.}$$
$$\vdots$$

$$B_0 = v_0 v_{0x.}$$
$$B_1 = v_0 v_{1x} + v_1 v_{0x.}$$
$$B_2 = v_0 v_{2x} + v_1 v_{1x} + v_2 v_{0x.}$$
$$B_3 = v_0 v_{3x} + v_1 v_{2x} + v_2 v_{1x} + v_3 v_{0x.}$$
$$B_3 = v_0 v_{4x} + v_1 v_{3x} + v_2 v_{2x} + v_3 v_{1x} + v_4 v_{0x.}$$
$$\vdots$$

$$C_0 = u_0 v_{0.}$$
$$C_1 = u_0 v_1 + u_1 v_{0.}$$
$$C_2 = u_0 v_2 + u_1 v_1 + u_2 v_{0.}$$
$$C_3 = u_0 v_3 + u_1 v_2 + u_2 v_1 + u_3 v_{0.}$$
$$C_3 = u_0 v_4 + u_1 v_3 + u_2 v_2 + u_3 v_1 + u_4 v_{0.}$$
$$\vdots$$

Operating the inverse double ARA transform to Equation (30) and Equation (31), utilizing Equation (33), we get

$$\sum_{n=0}^{\infty} u_n \left(\frac{x^p}{p}, \frac{t^q}{q} \right) = k_1(x)$$

$$\mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{K(r,s)}{s} \right]$$

$$+ \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q \left(\frac{\partial^{2p} u_n}{\partial x^{2p}} \right) \right] \qquad (34)$$

$$- \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q (\lambda A_n) \right]$$

$$- \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q (\alpha C_n) \right],$$

and

$$\sum_{n=0}^{\infty} v_n \left(\frac{x^p}{p}, \frac{t^q}{q} \right) = l_1(x)$$

$$+ \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{L(r,s)}{s} \right] +$$

$$\mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q \left(\frac{\partial^{2p} v_n}{\partial x^{2p}} \right) \right] - \qquad (35)$$

$$\mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q (\lambda B_n) \right] -$$

$$\mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q (\beta C_n) \right].$$

Now, we compare both sides of Equation (34) and Equation (35), to get

$$u_0 = k_1(x) + \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{F(r,s)}{s} \right],$$

$$v_0 = l_1(x) + \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{L(r,s)}{s} \right]. \qquad (36)$$

Following that, the recursive relation can be expressed as

$$u_n = \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q \left(\frac{\partial^{2p} v_n}{\partial x^{2\alpha}} \right) \right]$$

$$- \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q (\lambda A_n) \right] \qquad (37)$$

$$- \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q (\alpha C_n) \right],$$

and

$$v_n$$
$$= \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q \left(\frac{\partial^{2p} u_n}{\partial x^{2\alpha}} \right) \right]$$

$$- \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q (\lambda B_n) \right] \qquad (38)$$

$$- \mathcal{G}_r^{-1} \mathcal{G}_s^{-1} \left[\frac{1}{s} \mathcal{G}_x^p \mathcal{G}_t^q (\beta C_n) \right].$$

Herein, we should state that the solutions in (37) and (38) exist, provided the inverse double ARA transform exists $\forall p$ and s.

Putting $\lambda = -2$, $\alpha = \beta = 1$ and $k \left(\frac{x^p}{p}, \frac{t^q}{q} \right) = l \left(\frac{x^p}{p}, \frac{t^q}{q} \right) = 0$ in Equation (28) and $k_1 \left(\frac{x^p}{p} \right) = l_1 \left(\frac{x^p}{p} \right) = \sin \left(\frac{x^p}{p} \right)$ in Equation (29), we obtain the one-dimensional homogeneous coupled Burgers fractional equation is the conformable sense

$$\frac{\partial^q u}{\partial t^q} - \frac{\partial^{2p} u}{\partial x^{2p}} - 2u \frac{\partial^p u}{\partial x^p} + \frac{\partial^p}{\partial x^p}(uv) = 0,$$

$$\frac{\partial^q v}{\partial t^q} - \frac{\partial^{2p} v}{\partial x^{2p}} - 2v \frac{\partial^p v}{\partial x^p} + \frac{\partial^p}{\partial x^p}(uv) = 0. \qquad (39)$$

with initial condition

$$u \left(\frac{x^p}{p}, 0 \right) = \sin \left(\frac{x^p}{p} \right),$$

$$v \left(\frac{x^p}{p}, 0 \right) = \sin \left(\frac{x^p}{p} \right). \qquad (40)$$

By using Equations (36)–(38) , we have

$$u_0 = \sin \left(\frac{x^p}{p} \right), \qquad v_0 = \sin \left(\frac{x^p}{p} \right),$$

$$u_1 = \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^{2p}u_0}{\partial x^{2p}} + 2u_0\frac{\partial^p u_0}{\partial x^p}\right.\right.$$
$$\left.\left.- \frac{\partial^p}{\partial x^p}(u_0 v_0)\right]\right]$$
$$= \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[-\sin\left(\frac{x^p}{p}\right)\right]\right]$$
$$= \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{-r}{s(r^2+1)}\right]$$
$$= -\frac{t^q}{q}\,\sin\left(\frac{x^p}{p}\right),$$

$$v_1 = \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^{2p}v_0}{\partial x^{2p}} + 2v_0\frac{\partial^p v_0}{\partial x^p}\right.\right.$$
$$\left.\left.- \frac{\partial^p}{\partial x^p}(u_0 v_0)\right]\right]$$
$$= \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[-\sin\left(\frac{x^p}{p}\right)\right]\right]$$
$$= \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{-r}{s(r^2+1)}\right]$$
$$= -\frac{t^q}{q}\,\sin\left(\frac{x^p}{p}\right),$$

$$u_2 = \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^{2p}u_1}{\partial x^{2p}}\right.\right.$$
$$+ 2\left(u_0\frac{\partial^p u_1}{\partial x^p} + u_1\frac{\partial^p u_0}{\partial x^p}\right)$$
$$\left.\left.- \frac{\partial^p}{\partial x^p}(u_0 v_1 + u_1 v_0)\right]\right]$$
$$= \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{t^q}{q}\sin\left(\frac{x^p}{p}\right)\right]\right]$$
$$= \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s^2(r^2+1)}\right]$$
$$= \frac{\left(\frac{t^q}{q}\right)^2}{2}\,\sin\left(\frac{x^p}{p}\right),$$

$$v_2 = \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^{2p}v_1}{\partial x^{2p}}\right.\right.$$
$$+ 2\left(v_0\frac{\partial^p v_1}{\partial x^p} + v_1\frac{\partial^p v_0}{\partial x^p}\right)$$
$$\left.\left.- \frac{\partial^p}{\partial x^p}(u_0 v_1 + u_1 v_0)\right]\right]$$
$$= \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{t^q}{q}\sin\left(\frac{x^p}{p}\right)\right]\right]$$
$$= \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s^2(r^2+1)}\right]$$
$$= \frac{\left(\frac{t^q}{q}\right)^2}{2}\,\sin\left(\frac{x^p}{p}\right),$$

and

$$u_3 = \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^{2p}u_2}{\partial x^{2p}}\right.\right.$$
$$+ 2\left(u_0\frac{\partial^p u_2}{\partial x^p} + u_1\frac{\partial^p u_1}{\partial x^p}\right.$$
$$\left.+ u_2\frac{\partial^p u_0}{\partial x^p}\right)$$
$$\left.\left.- \frac{\partial^p}{\partial x^p}(u_0 v_2 + u_1 v_1 + u_2 v_0)\right]\right]$$
$$= -\frac{\left(\frac{t^q}{q}\right)^3}{6}\,\sin\left(\frac{x^p}{p}\right),$$

$$v_3 = \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\,\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^{2p}v_2}{\partial x^{2p}}\right.\right.$$
$$+ 2\left(v_0\frac{\partial^p v_2}{\partial x^p} + v_1\frac{\partial^p v_1}{\partial x^p}\right.$$
$$\left.+ v_2\frac{\partial^p v_0}{\partial x^p}\right)$$
$$\left.\left.- \frac{\partial^p}{\partial x^p}(u_0 v_2 + u_1 v_1 + u_2 v_0)\right]\right]$$
$$= -\frac{\left(\frac{t^q}{q}\right)^3}{6}\,\sin\left(\frac{x^p}{p}\right).$$

Therefore, using Equation (32), we can express the series solution as

$$u\left(\frac{x^p}{p},\frac{t^q}{q}\right) = u_0 + u_1 + u_2 + \cdots$$

$$= \left(1 - \left(\frac{t^q}{q}\right) + \frac{\left(\frac{t^q}{q}\right)^2}{2!} - \frac{\left(\frac{t^q}{q}\right)^3}{3!}\right.$$

$$\left. + \cdots \right) \sin\left(\frac{x^p}{p}\right),$$

$$\left(\frac{x^p}{p},\frac{t^q}{q}\right) = v_0 + v_1 + v_2 + \cdots$$

$$= \left(1 - \left(\frac{t^q}{q}\right) + \frac{\left(\frac{t^q}{q}\right)^2}{2!} - \frac{\left(\frac{t^q}{q}\right)^3}{6!}\right.$$

$$\left. + \cdots \right) v \sin\left(\frac{x^p}{p}\right),$$

and hence the exact solutions become

$$u\left(\frac{x^p}{p},\frac{t^q}{q}\right) = e^{-\frac{t^q}{q}} \sin\left(\frac{x^p}{p}\right),$$

$$v\left(\frac{x^p}{p},\frac{t^q}{q}\right) = e^{-\frac{t^q}{q}} \sin\left(\frac{x^p}{p}\right).$$

By taking $p = 1$ and $q = 1$, the fractional solution of Equation (39) becomes

$$u\left(\frac{x^p}{p},\frac{t^q}{q}\right) = e^{-t} \sin x,$$

$$v\left(\frac{x^p}{p},\frac{t^q}{q}\right) = e^{-t} \sin x.$$

The behavior of the velocity field of the two-CDARADM (28) and (29) is depicted in Figure 1 for (a) the approximate and exact solutions of $u\left(\frac{x^p}{p},\frac{t^q}{q}\right)$ for Example 1, when $p = q$, at $p = 0.8, 0.9, 1$, and (b) the approximate and exact solutions of $u\left(\frac{x^p}{p},\frac{t^q}{q}\right)$, for Example 1, when taking various values of fractional order q ($q = 0.8, 0.9, 1$) and $p = 1$.
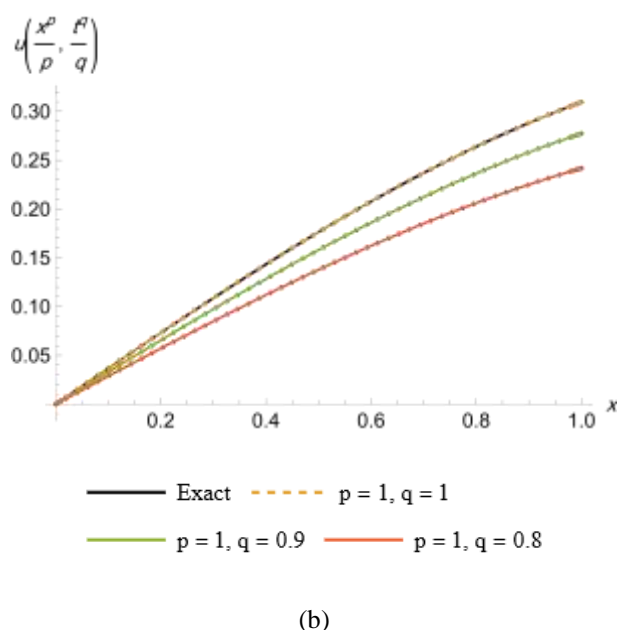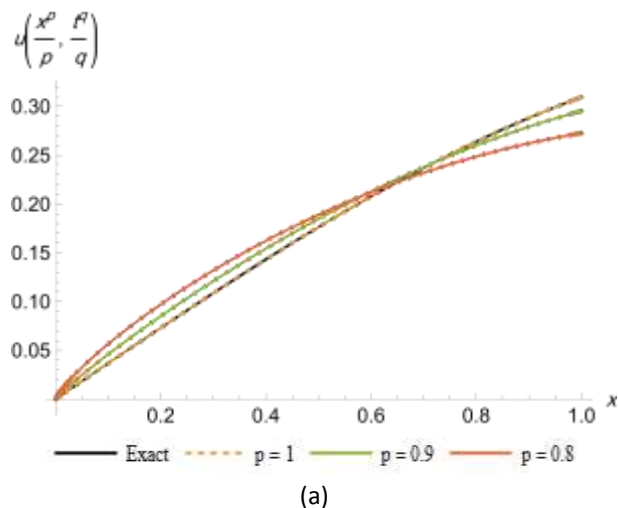

(a)

(b)

Fig. 1: The figure of (a) The approximate and exact solutions of $u(x^p/p, t^q/q)$ for Example 5.1, when $p = q$, (b) The approximate and exact solutions of $u(x^p/p, t^q/q)$, for Example 1, when taking various values of q and $p = 1$.

Table 2 below, presents the absolute errors considering $p = q = 1$, $x = 1, t \in [0.1, 0.5]$. We compare the obtained results by exact solution.

Table 2. Analysis of error to $u\left(\frac{x^p}{p},\frac{t^q}{q}\right)$ for Example 1, on $x=1, t\in[0.1,0.5]$

	Exact Solution	**Approximate Solution**	$\left|u\left(\frac{x^p}{p},\frac{t^q}{q}\right)-u_6\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right|$
0.1	0.761394	0.761394	1.64894×10^{-11}
0.15	0.724261	0.724261	2.80003×10^{-10}
0.2	0.688938	0.688938	2.08481×10^{-9}
0.25	0.655338	0.655338	9.88052×10^{-9}
0.3	0.623377	0.623377	3.51889×10^{-8}
0.35	0.592975	0.592975	1.02897×10^{-7}
0.4	0.564055	0.564055	2.60452×10^{-7}
0.45	0.536546	0.536546	5.90459×10^{-7}
0.5	0.510378	0.510379	1.22715×10^{-6}

Example 2.
Consider the singular one-dimensional conformable fractional coupled Burgers' equation with the Bessel operator of the form

$$\frac{\partial^q u}{\partial t^q}-\frac{p}{x^p}\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}u\right)+\lambda\,u\frac{\partial^p u}{\partial x^p}$$
$$+\alpha\frac{\partial^p}{\partial x^p}(uv)$$
$$=k\left(\frac{x^p}{p},\frac{t^q}{q}\right),$$
$$\frac{\partial^q v}{\partial t^q}-\frac{p}{x^p}\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}v\right)+\lambda\,v\frac{\partial^p v}{\partial x^p}$$
$$+\beta\frac{\partial^p}{\partial x^p}(uv)$$
$$=l\left(\frac{x^p}{p},\frac{t^q}{q}\right),\tag{41}$$

and with initial conditions

$$u\left(\frac{x^p}{p},0\right)=k_1\left(\frac{x^p}{p}\right),$$
$$v\left(\frac{x^p}{p},0\right)=l_1\left(\frac{x^p}{p}\right).\tag{42}$$

Where the linear terms $\frac{p}{x^p}\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}\right)$ is known as a conformable Bessel operator where λ,α, and β are real constants.

Multiplying both sides of Equation (41) by $\frac{x^p}{p}$, we get

$$\frac{x^p}{p}\frac{\partial^q u}{\partial t^q}-\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}u\right)+\lambda\frac{x^p}{p}u\frac{\partial^p u}{\partial x^p}$$
$$+\alpha\frac{x^p}{p}\frac{\partial^p}{\partial x^p}(uv)\tag{43}$$
$$=\frac{x^p}{p}k\left(\frac{x^p}{p},\frac{t^q}{q}\right),$$

$$\frac{x^p}{p}\frac{\partial^q v}{\partial t^q}-\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}v\right)+\lambda\frac{x^p}{p}v\frac{\partial^p v}{\partial x^p}$$
$$+\beta\frac{x^p}{p}\frac{\partial^p}{\partial x^p}(uv)$$
$$=\frac{x^p}{p}l\left(\frac{x^p}{p},\frac{t^q}{q}\right).$$

Applying conformable double ARA transform to both sides of Equation (43) and single conformable ARA transform for the initial condition, we get

$$\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{x^p}{p}\frac{\partial^q u}{\partial t^q}\right]-\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}u\right)\right.$$
$$-\lambda\frac{x^p}{p}u\frac{\partial^p u}{\partial x^p}$$
$$\left.-\alpha\frac{x^p}{p}\frac{\partial^p}{\partial x^p}(uv)\right]$$
$$=\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{x^p}{p}k\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right],$$
$$\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{x^p}{p}\frac{\partial^q v}{\partial t^q}\right]-\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}v\right)\right.\tag{44}$$
$$-\lambda\frac{x^p}{p}u\frac{\partial^p v}{\partial x^p}$$
$$\left.-\beta\frac{x^p}{p}\frac{\partial^p}{\partial x^p}(uv)\right]$$
$$=\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{x^p}{p}l\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right].$$

Applying Theorem 3 and Theorem 5, we have

$$-rs\frac{\partial}{\partial r}\left(\frac{1}{r}U(r,s)\right)+rs\frac{d}{dr}\left(\frac{1}{r}\mathcal{G}_x^p[k_1(x)]\right)$$
$$=\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}u\right)-\lambda\frac{x^p}{p}u\frac{\partial^p u}{\partial x^p}\right.$$
$$\left.-\alpha\frac{x^p}{p}\frac{\partial^p}{\partial x^p}(uv)\right]$$
$$-r\frac{\partial}{\partial r}\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[k\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right),$$
$$-rs\frac{\partial}{\partial r}\left(\frac{1}{r}V(r,s)\right)+rs\frac{d}{dr}\left(\frac{1}{r}\mathcal{G}_x^p[l_1(x)]\right)\tag{45}$$
$$=\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}v\right)-\lambda\frac{x^p}{p}v\frac{\partial^p u}{\partial x^p}\right.$$
$$\left.-\beta\frac{x^p}{p}\frac{\partial^p}{\partial x^p}(uv)\right]$$
$$-r\frac{\partial}{\partial r}\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[l\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right).$$

Simplifying Equation (45), we obtain

$$\frac{\partial}{\partial r}\left(\frac{1}{r}U(r,s)\right)$$
$$= \frac{d}{dr}\left(\frac{1}{r}\mathcal{G}_x^p[k_1(x)]\right)$$
$$- \frac{1}{rs}\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}u\right)\right.$$
$$\left.- \lambda\frac{x^p}{p}u\frac{\partial^p u}{\partial x^p} - \alpha\frac{x^p}{p}\frac{\partial^p}{\partial x^p}(uv)\right]$$
$$+ \frac{1}{s}\frac{\partial}{\partial r}\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[k\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right),$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}V(r,s)\right)$$
$$= \frac{d}{dr}\left(\frac{1}{r}\mathcal{G}_x^p[l_1(x)]\right)$$
$$- \frac{1}{rs}\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}v\right)\right.$$
$$\left.- \lambda\frac{x^p}{p}v\frac{\partial^p u}{\partial x^p} - \beta\frac{x^p}{p}\frac{\partial^p}{\partial x^p}(uv)\right]$$
$$+ \frac{1}{s}\frac{\partial}{\partial r}\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[l\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right).$$

(46)

Applying the definite integral $\int_0^r$ with respect to r to both sides of Equation (46)

$$\frac{1}{r}U(r,s)$$
$$= \int_0^r \frac{d}{dr}\left(\frac{1}{r}\mathcal{G}_x^p[k_1(x)]\right)dr$$
$$- \frac{1}{s}\int_0^r\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}u\right) - \lambda\frac{x^p}{p}N_1\right.\right.$$
$$\left.\left.- \alpha\frac{x^p}{p}N_2\right]\right)dr$$
$$+ \frac{1}{s}\int_0^r\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[k\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right)\right)dr,$$

$$\frac{1}{r}V(r,s)$$
$$= \int_0^r \frac{d}{dr}\left(\frac{1}{r}\mathcal{G}_x^p[l_1(x)]\right)dr$$
$$- \frac{1}{s}\int_0^r\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}v\right) - \lambda\frac{x^p}{p}N_3\right.\right.$$
$$\left.\left.- \beta\frac{x^p}{p}N_2\right]\right)dr$$
$$+ \frac{1}{s}\int_0^r\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[l\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right)\right)dr.$$

Multiplying both sides of the equations by r, we get

$$U(r,s)$$
$$= r\int_0^r \frac{d}{dr}\left(\frac{1}{r}\mathcal{G}_x^p[k_1(x)]\right)dr$$
$$- \frac{r}{s}\int_0^r\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}u\right)\right.\right.$$
$$\left.\left.- \lambda\frac{x^p}{p}N_1 - \alpha\frac{x^p}{p}N_2\right]\right)dr$$
$$+ \frac{r}{s}\int_0^r\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[k\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right)\right)dr,$$

$$V(r,s)$$
$$= r\int_0^r \frac{d}{dr}\left(\frac{1}{r}\mathcal{G}_x^p[l_1(x)]\right)dr$$
$$- \frac{r}{s}\int_0^r\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}v\right)\right.\right.$$
$$\left.\left.- \lambda\frac{x^p}{p}N_3 - \beta\frac{x^p}{p}N_2\right]\right)dr$$
$$+ \frac{r}{s}\int_0^r\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\mathcal{G}_x^p\mathcal{G}_t^q\left[l\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right)\right)dr.$$

(47)

Utilizing the CDARADM to present the solution of $u\left(\frac{x^p}{p},\frac{t^q}{q}\right)$ and $v\left(\frac{x^p}{p},\frac{t^q}{q}\right)$ by infinite series as

$$u\left(\frac{x^p}{p},\frac{t^q}{q}\right) = \sum_{n=0}^{\infty} u_n\left(\frac{x^p}{p},\frac{t^q}{q}\right),$$
$$v\left(\frac{x^p}{p},\frac{t^q}{q}\right) = \sum_{n=0}^{\infty} v_n\left(\frac{x^p}{p},\frac{t^q}{q}\right).$$

(48)

Define the nonlinear operators as

$$N_1 = \sum_{n=0}^{\infty} A_n, \qquad N_2 = \sum_{n=0}^{\infty} C_n,$$
$$N_3 = \sum_{n=0}^{\infty} B_n.$$

(49)

Operating the double inverse transform to Equation (47) and making use of Equation (48) and Equation (49), we have

$$\sum_{n=0}^{\infty} u_n\left(\frac{x^p}{p},\frac{t^q}{q}\right)$$
$$= k_1(x) + \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[k\left(\frac{x^p}{p},\frac{t^q}{q}\right)\right]\right)\right]$$
$$- \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r\frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\left(\sum_{n=0}^{\infty} u_n\right)\right)\right]\right)dr\right]$$
$$+ \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r\frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\lambda\frac{x^p}{p}\sum_{n=0}^{\infty} A_n\right]\right)dr\right]$$
$$+ \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r\frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\alpha\frac{x^p}{p}\sum_{n=0}^{\infty} C_n\right]\right)dr\right].$$

(50)

And

$$\sum_{n=0}^{\infty} v_n\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$$
$$= l_1(x) + \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[l\left(\frac{x^p}{p}, \frac{t^q}{q}\right)\right]\right)\right]$$
$$- \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\left(\sum_{n=0}^{\infty} v_n\right)\right)\right]\right)dr\right]$$
$$+ \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\lambda\,\frac{x^p}{p}\sum_{n=0}^{\infty} B_n\right]\right)dr\right]$$
$$+ G_r^{-1}G_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\beta\,\frac{x^p}{p}\sum_{n=0}^{\infty} C_n\right]\right)dr\right]. \quad (51)$$

Now, we can express the first few components as

$$u_0 = k_1(x)$$
$$+ \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[k\left(\frac{x^p}{p}, \frac{t^q}{q}\right)\right]\right)\right], \quad (52)$$
$$v_0 = l_1(x)$$
$$+ \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[l\left(\frac{x^p}{p}, \frac{t^q}{q}\right)\right]\right)\right].$$

And

$$u_{n+1}\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$$
$$= -\mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^r}{\partial x^r}\left(\frac{x^p}{p}\left(\sum_{n=0}^{\infty} u_n\right)\right)\right]\right)dr\right]$$
$$+ \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\lambda\,\frac{x^p}{p}\sum_{n=0}^{\infty} A_n\right]\right)dr\right]$$
$$+ \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\alpha\,\frac{x^p}{p}\sum_{n=0}^{\infty} C_n\right]\right)dr\right]. \quad (53)$$

And

$$v_{n+1}\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$$
$$= -\mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^r}{\partial x^r}\left(\frac{x^p}{p}\left(\sum_{n=0}^{\infty} v_n\right)\right)\right]\right)dr\right]$$
$$+ \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\lambda\,\frac{x^p}{p}\sum_{n=0}^{\infty} B_n\right]\right)dr\right]$$
$$+ \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[r\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\beta\,\frac{x^p}{p}\sum_{n=0}^{\infty} C_n\right]\right)dr\right]. \quad (54)$$

In addition, we assume that the double inverse transform in (53) and (54) exist, and substitution $\lambda = -2$, $\alpha = \beta = 1$ and $k\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = l\left(\frac{x^p}{p}, \frac{t^q}{q}\right) = \left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}} - 4e^{\frac{t^q}{q}}$ in Equation (41) and $k_1\left(\frac{x^p}{p}\right) = l_1\left(\frac{x^p}{p}\right) = \left(\frac{x^p}{p}\right)^2$ in Equation (42), we obtain the singular conformable coupled Burgers fractional equation of one dimensional

$$\frac{\partial^q u}{\partial t^q} - \frac{p}{x^p}\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}u\right) - 2u\frac{\partial^p u}{\partial x^p}$$
$$+ \frac{\partial^p}{\partial x^p}(uv)$$
$$= \left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}} - 4e^{\frac{t^q}{q}},$$
$$\frac{\partial^q v}{\partial t^q} - \frac{p}{x^p}\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p}{\partial x^p}v\right) - 2v\frac{\partial^p v}{\partial x^p}$$
$$+ \frac{\partial^p}{\partial x^p}(uv)$$
$$= \left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}} - 4e^{\frac{t^q}{q}}, \quad (55)$$

subject to

$$u\left(\frac{x^p}{p}, 0\right) = \left(\frac{x^p}{p}\right)^2,$$
$$v\left(\frac{x^p}{p}, 0\right) = \left(\frac{x^p}{p}\right)^2. \quad (56)$$

By following similar steps, we obtain

$$\sum_{n=0}^{\infty} u_n\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$$
$$= \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}} - 4e^{\frac{t^q}{q}}\right]\right)\right]$$
$$- \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\left(\sum_{n=0}^{\infty} u_n\right)\right)\right]\right)dr\right]$$
$$- \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[2\,\frac{x^p}{p}\sum_{n=0}^{\infty} A_n\right]\right)dr\right]$$
$$+ \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{x^p}{p}\sum_{n=0}^{\infty} C_n\right]\right)dr\right] + \left(\frac{x^p}{p}\right)^2, \quad (57)$$

$$\sum_{n=0}^{\infty} v_n\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$$
$$= \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}} - 4e^{\frac{t^q}{q}}\right]\right)\right]$$
$$- \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\left(\sum_{n=0}^{\infty} v_n\right)\right)\right]\right)dr\right]$$
$$- \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[2\,\frac{x^p}{p}\sum_{n=0}^{\infty} B_n\right]\right)dr\right]$$
$$+ \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r \frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{x^p}{p}\sum_{n=0}^{\infty} C_n\right]\right)dr\right] + \left(\frac{x^p}{p}\right)^2. \quad (58)$$

Using Equations (52) – (54) the components are given by

$$u_0 = \left(\frac{x^p}{p}\right)^2 + \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\mathcal{G}_x^p\mathcal{G}_t^q\left(\left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}} - 4e^{\frac{t^q}{q}}\right)\right]$$
$$= \left(\frac{x^p}{p}\right)^2 + \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\left(\frac{2}{r^2}\left(\frac{s}{s-1}\right) - \frac{4s}{s-1}\right)\right]$$

$$= \left(\frac{x^p}{p}\right)^2 + \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\left(\frac{2}{r^2}\left(\frac{s^2}{s-1}-s\right)+4s\left(1-\frac{s}{s-1}\right)\right)\right]$$

$$= \left(\frac{x^p}{p}\right)^2 + \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\left(\frac{2}{r^2}\left(\frac{s^2}{s-1}\right)-\frac{4s^2}{s-1}-\frac{2s}{r^2}+4s\right)\right]$$

$$= \left(\frac{x^p}{p}\right)^2 + \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\left(\frac{2}{r^2}\left(\frac{s}{s-1}\right)-\frac{4s}{s-1}-\frac{2}{r^2}+4\right)\right]$$

$$= \left(\frac{x^p}{p}\right)^2 + \left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}} - 4e^{\frac{t^q}{q}} - \left(\frac{x^p}{p}\right)^2 + 4$$

$$= \left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}} - 4e^{\frac{t^q}{q}} + 4,$$

u_1
$$= -\mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r\frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\frac{\partial^p}{\partial x^p}\left(\frac{x^p}{p}\frac{\partial^p u_0}{\partial x^p}\right)+2\frac{x^p}{p}u_0\frac{\partial^p u_0}{\partial x^p}\right.\right.\right.$$
$$\left.\left.\left.-\frac{x^p}{p}\frac{\partial^p}{\partial x^p}(u_0\,v_0)\right]\right)dr\right]$$

$$= -\mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r\frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[4\frac{x^p}{p}e^{\frac{t^q}{q}}\right.\right.\right.$$

$$+4\left(\left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}} - 4e^{\frac{t^q}{q}}+4\right)$$

$$-4\left(\left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}} - 4e^{\frac{t^q}{q}}+4\right)\right]\right)dr\right]$$

$$= -\mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{r}{s}\int_0^r\frac{1}{r}\left(\mathcal{G}_x^p\mathcal{G}_t^q\left[\left(4\frac{x^p}{p}e^{\frac{t^q}{q}}\right)\right]\right)dr\right]$$

$$= \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{1}{s}\int_0^r\left(\frac{4s}{s-1}\right)dr\right] = \mathcal{G}_r^{-1}\mathcal{G}_s^{-1}\left[\frac{4}{s}\left(\frac{s^2}{s-1}-s\right)\right]$$

$$= 4e^{\frac{t^q}{q}} - 4,$$
$$u_2 = 0 = u_3 = \cdots$$

In a similar way, we obtain
$$v_0 = \left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}} - 4e^{\frac{t^q}{q}} + 4.$$

$$v_1 = 4e^{\frac{t^q}{q}} - 4.$$
$$v_2 = 0 = v_3 = \cdots.$$

Following that, one can express the solution as
$$u\left(\frac{x^p}{p},\frac{t^q}{q}\right) = u_0 + u_1,$$
$$v\left(\frac{x^p}{p},\frac{t^q}{q}\right) = v_0 + v_1.$$

Therefore, the exact solution is given by
$$u\left(\frac{x^p}{p},\frac{t^q}{q}\right) = \left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}},$$
$$v\left(\frac{x^p}{p},\frac{t^q}{q}\right) = \left(\frac{x^p}{p}\right)^2 e^{\frac{t^q}{q}}.$$

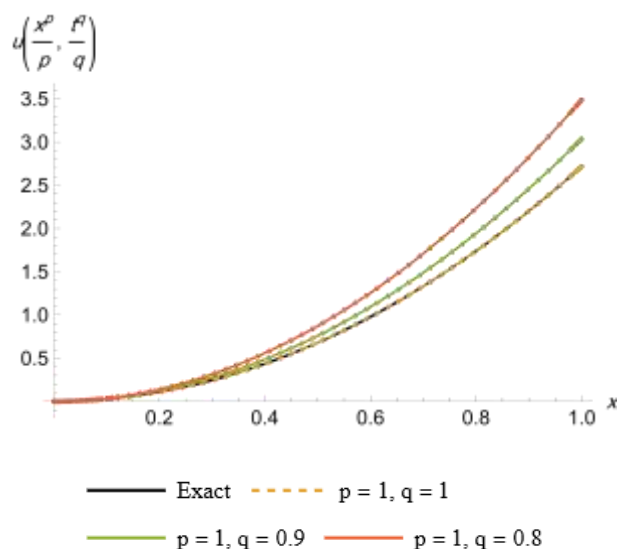By taking $p = 1$ and $q = 1$, the fractional solution becomes
$$u(x,t) = x^2 e^t \quad , \quad v(x,t) = x^2 e^t.$$

The behavior of the velocity field of the two-CDARADM (41) and (42) is depicted in Figure 2 for (a) the approximate and exact solutions of u for

Example 2, at $p = q = 0.8, 0.9, 1$, and (b) the approximate and exact solutions of $u\left(\frac{x^p}{p},\frac{t^q}{q}\right)$, for Example 2, when taking various values of q ($q = 0.8, 0.9, 1$) and $p = 1$.
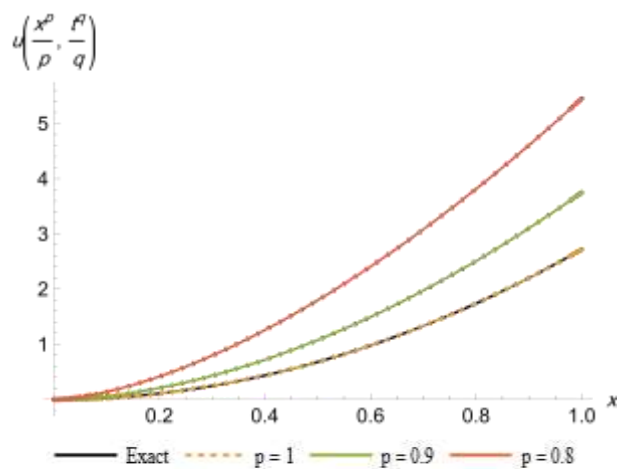
(a)

(b)

Fig. 2: The figure of (a) The approximate and exact solutions of $u\left(\frac{x^p}{p},\frac{t^q}{q}\right)$ for Example 5.2, when $p = q$, (b) The approximate and exact solutions of $u\left(\frac{x^p}{p},\frac{t^q}{q}\right)$, for Example 2, when we take different values of fractional order q and $p = 1$.

Table 3 below, presents the absolute errors with respect to $p = q = 1$, $x = 1, t \in [0.1, 0.5]$. We compare the obtained results by exact solution.

Table 3. Error analysis of $u\left(\frac{x^p}{p}, \frac{t^q}{q}\right)$ for Example 2 on $x = 1, t \in [0.1, 0.5]$

	Exact Solution	Approximate Solution	$\left\|u\left(\frac{x^p}{p}, \frac{t^q}{q}\right) - u_6\left(\frac{x^p}{p}, \frac{t^q}{q}\right)\right\|$
0.1	1.10517	1.10517	2.00924×10^{-11}
0.15	1.16183	1.16183	3.45471×10^{-10}
0.2	1.2214	1.2214	2.60461×10^{-9}
0.25	1.28403	1.28403	1.24994×10^{-8}
0.3	1.34986	1.34986	4.5076×10^{-8}
0.35	1.41907	1.41907	1.33467×10^{-7}
0.4	1.49182	1.49182	3.42086×10^{-7}
0.45	1.56831	1.56831	7.85295×10^{-7}
0.5	1.64872	1.64872	1.65264×10^{-6}

6 Conclusion

In the current study, we defined and went over some of the characteristics of the conformable double ARA transform. The conformable double ARA decomposition method is a novel approach that we present for the solution of nonlinear conformable partial differential equations. We used the proposed approach, a novel amalgamation of the conformable double ARA transform and Adomian decomposition methods, to present solutions to the one-dimensional regular and singular conformable fractional coupled Burgers' problem. Additionally, two intriguing examples were given to demonstrate the applicability of the novel approach. Different types of nonlinear time-fractional differential equations with conformable derivatives can be solved using this technique. We want to answer more fractional integral equations and fractional nonlinear problem classes in the future.

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