On the Period-Amplitude Relation by Reduction to Liénard Quadratic Equation

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Abstract: We apply Sabatini's transformation for the study of a class of nonlinear oscillators, dependent on quadratic terms. As a result, an initial equation is reduced to Newtonian form, for which in a standard way the period-amplitude relation can be established.

Key-Words: Differential equations, Oscillation, Period-Amplitude relation.
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1 Introduction

The problem of vibrations in mechanical systems is important from a practical point of view and interesting as a descent object of investigation for theoreticians. The mathematical models often are formulated in terms of the second-order ordinary differential equations. The second-order oscillators are of great importance in mechanics, engineering, and other practical areas. The oscillatory behavior of solutions was and continues to be an object of intensive studies (books, [19], [7], [15]). The main characteristics of these oscillators are the amplitude and periods of solutions, as well as the amplitude-period relations, [2], [3], [4], [5], [6], [8], [9], [10], [11], [12], [14], [16], [17], [21]. This theory arose from elementary harmonic oscillations represented by equation

$$x'' + k^2 x = 0. \quad (1)$$

An interesting question arises immediately. What happens if the coefficient k^2 is not constant? If it depends on the independent variable *t*, then the equation is still linear, and the linear theory applies. But if k^2 is dependent on x or x' (or both), the equation can become nonlinear. The remarkable feature of nonlinear equations is that the oscillation amplitude is dependent on the period of a solution, and vice versa. In a series of papers, [8], [9], [10], [11], [12], [14], this problem was treated using the "ancient Chines" technique.

For instance, in the work, [11], the nonlinear oscillator

$$x'' + (1 + x'^2)x = 0 \quad (2)$$

was investigated. The trial solution in the form $x(t) = A\cos\omega t$ was used, where ω is the frequency to be determined. For equation (2) the frequency-amplitude relation was found in the form

$$\omega = \frac{2}{\sqrt{4-A^2}} \quad (3)$$

Another approximate relation was found using "the ancient Chinese inequality called Chengtian's inequality", [13]. The resulting formula is

$$\omega = \sqrt{\frac{8}{8-A^2}} \quad (4)$$

Let us look at this problem from a different point of view. Rewrite equation (2) as

$$x'' + x\,x'^2 + x = 0 \tag{5}$$

It resembles the classical Liénard equation

$$x'' + \mu(x^2 - 1)x' + x = 0 \tag{6}$$

and the generalized one

$$x'' + f(x)x' + g(x) = 0. \tag{7}$$

The second term in (5), however, is quadratic.

In the work, [20], a special transformation was invented which can be used for the reduction of the equation

$$x'' + f(x)x'^2 + x = 0 \tag{8}$$

to the Newtonian form

$$u'' + h(u) = 0. \tag{9}$$

We will apply this transformation to the study of our selected cases. We are interested in establishing the relation period versus amplitude.

2 The Scheme of the Study

Consider the equation

$$x'' + f(x)x'^2 + g(x) = 0. \tag{10}$$

Introduce the new variable *u* by the relation

$$u = \int_0^x e^{F(s)} ds, \tag{11}$$

where $F(x) = \int_0^x f(s)ds$. Then $\frac{du}{dx} = e^{F(x)} > 0$.

Therefore $u(x)$ is a monotone function and, as a consequence, the inverse function $x(u)$ exists. If $x(t)$ is an arbitrary solution of (10), then the corresponding function

$$u(t) = \int_0^{x(t)} e^{F(s)} ds \tag{12}$$

satisfies the second-order conservative equation

$$u'' + h(u) = 0, \tag{13}$$

where $h(u) = x(u)e^{F(x(u))}$. Equation (13) has an integral

$$u'^2(t) + 2H\big(u(t)\big) = const, \tag{14}$$

where $H(u) = \int_0^u h(\xi)d\xi$.

The purpose of this article is to use the described approach to equations of the form (8). Several cases will be considered.

Since we are interested in periodic solutions of differential equations, the following assertion is important.

Proposition 2.1. *If $x(t)$ is a periodic solution of the equation (15), then the corresponding function $u(t)$, obtained by the formula (11), is the periodic solution of (13).*

Proof. If $x(t)$ is a periodic solution of (15), then $x(t_1) = x(t_2)$, $x'(t_1) = x'(t_2)$ for some $t_1 \neq t_2$. The respective trajectory in the (x, x')-phase plane is closed. The respective solution $u(t)$, defined by (11) is also periodic, because $u(t_1) = \int_0^{x(t_1)} e^{F(s)}ds = \int_0^{x(t_2)} e^{F(s)}ds = u(t_2)$. Due to the autonomy of both equations (15) and (13), these solutions on a phase plane are represented by closed trajectories.

3 Equation $x'' + (1 + x'^2)x = 0$

First, notice that this equation can be written in the form

$$x'' + xx'^2 + x = 0. \tag{15}$$

This is a Liénard type equation with quadratic dependence on x'. The variable change, described above, is applicable.

Equation (15), written as $x'' + (1 + x'^2)x = 0$, can be considered as a perturbation of the harmonic equation, which is known to have periodic solutions. Does equation (15) have a periodic solution? Let us write equation (15) in the form (13). We get $f(x) = x$, $F(x) = \int_0^x f(s)ds = \int_0^x s\,ds = \frac{s^2}{2}$, $u(x) = \int_0^x e^{\frac{s^2}{2}} ds$. The latter can be written as the differential equation, given the initial condition,

$$\frac{du}{dx} = e^{\frac{x^2}{2}},\ \ u(0) = 0. \tag{16}$$

Evidently, $u = u(x)$ is strictly monotonically increasing function with the graph, symmetrical with respect to the origin and passing through the origin. It is known as the function $\sqrt{\frac{\pi}{2}}\, erfi\, \frac{u}{\sqrt{2}}$, [22]. The inverse function $x = x(u)$ exists and has similar properties. Both functions are depicted in Figure 1.

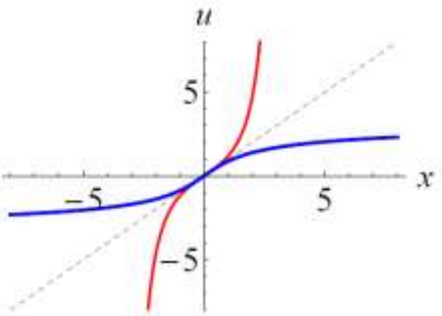

Fig. 1: Blue: $x = x(u)$, red: $u = u(x)$.

The function $x(u)$ is the solution of the Cauchy problem

$$\frac{dx}{du} = e^{-\frac{x^2}{2}}, \ \ x(0) = 0. \qquad (17)$$

The function $h(u)$ in equation (13) is

$$h(u) = x(u)e^{F(x(u))} = x(u)e^{\frac{x^2(u)}{2}}. \qquad (18)$$

The functions $x(u)$ and $h(u)$ are depicted in Figure 2.

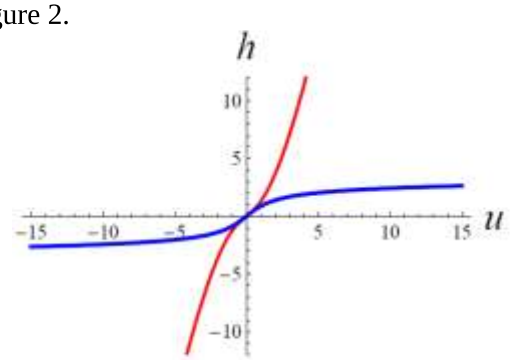

Fig. 2: Blue: $x = x(u)$, red: $h(u)$.

Equation (13) has the only equilibrium $u = 0$ and, therefore, the periodic solution ought to oscillate around it.

Let us apply the above transformation to equation (2) written in the form (15).

$$u = \int_0^x e^{\frac{s^2}{2}} ds. \qquad (19)$$

The graphs of $u(x)$ and the inverse function $x(u)$ are depicted in the Figure 3 below (blue – $u(x)$, red - $x(u)$).

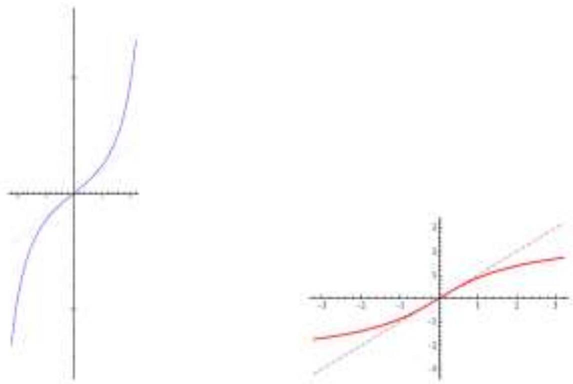

Fig. 3: Left: *u(x)*, right: *x(u).*

The inverse function $x(u)$ exists and the graph is symmetrical with respect to the bisectrix.

Equation (13) in this case is

$$u'' + x(u)e^{\frac{x(u)^2}{2}} = 0. \qquad (20)$$

Periodic solutions of the equation (2) are in one-to-one correspondence to periodic solutions of (20).

3.1 Period-amplitude Relation

Our goal in this subsection is to state the frequency-amplitude, or, which is almost the same, period-amplitude relation for the equation (15). For this, we have the function $u(x)$, defined in (19), or as a solution to the Cauchy problem (16). We have the inverse function $x(u)$, which can be obtained explicitly or numerically as in (17). The solution $x(t)$ of the initial value problem

$$x'' + xx'^2 + x = 0, \ \ x(0) = 0, x'(0) = \alpha \qquad (21)$$

has a counterpart $u(t)$, which solves the initial value problem

$$u'' + h(u) = 0, \ \ u(0) = 0, u'(0) = \alpha, \qquad (22)$$

since $u(0) = \int_0^{x(0)} e^{\frac{s^2}{2}} ds = \int_0^0 e^{\frac{s^2}{2}} ds = 0,$

$$\left.\frac{du}{dt}\right|_{t=0} = \left.\left(e^{\frac{x(t)^2}{2}} x'(t)\right)\right|_{t=0} = \alpha.$$

It follows that if $x(t)$ is a periodic solution of the problem (21), the same is a solution $u(t)$ of the problem (22), and their periods are equal. As an illustration, $u(t)$ and $x(t)$ are depicted in Figure 4.

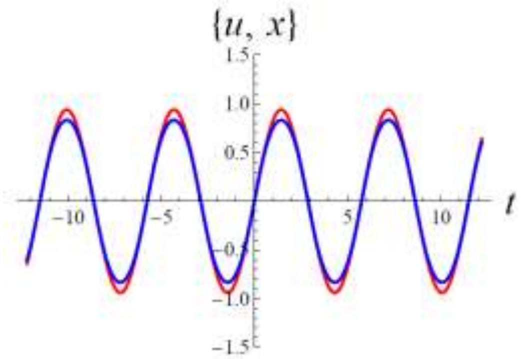

Fig. 4: Red: $u(t)$, blue: $x(t)$, $\alpha = 1$.

It follows that

$$u'^2(t) + 2H(u(t)) = const,$$

where $H(u)$ is the primitive for $h(u) = x(u)e^{\frac{x(u)^2}{2}}$. One has, in a standard manner, that $const = u'^2(0) = \alpha^2$ and, at the same time, $2H(M) = \alpha^2$, where M is the maximal va+lue of $u(t)$. Hence formula for the amplitude of a solution $u(t)$ of the initial value problem (19) is

$$M = H^{-1}\left(\frac{\alpha^2}{2}\right) \quad (23)$$

or, equivalently,

$$2\int_0^M h(s)ds = \alpha^2. \quad (24)$$

Periods of solutions $x(t)$ of the problem (21) and $u(t)$ of the problem (22) are the same. The period T of a solution to the problem (22) can be found in the relation

$$M = \int_0^A e^{F(s)}ds, \ \ 4\int_0^M \frac{ds}{\sqrt{\alpha^2 - 2H(s)}} = T. \quad (25)$$

The amplitudes of solutions to the problem (21) and (22) may differ. The relation between the amplitude $A(\alpha)$ of $x(t)$ and $M(\alpha)$ of $u(t)$ is

$$M = \int_0^A e^{\frac{s^2}{2}}ds. \quad (26)$$

Table 1. The amplitudes A of *x(t)* versus M of *u(t)*.

α	Period T	Amplitude A	Amplitude M
0.1	6.32	0.10	0.10
0.5	6.11	0.47	0.49
1.0	5.72	0.82	0.92
1.5	5.40	1.06	1.30
2.0	5.16	1.26	1.69

4 Equation $x'' + f(x)x'^2 + kx = 0$

The equation under investigation is

$$x'' + f(x)x'^2 + kx = 0. \quad (27)$$

Suppose that

$$f(x) = -f(-x). \quad (28)$$

We get, using the transformation

$$u = \int_0^x e^{F(s)}ds, \ \ F(x) = \int_0^x f(s)ds, \quad (29)$$

that the equation (27) takes the Newtonian form

$$u'' + h(u) = 0, \quad (30)$$

where $h(u) = kx(u)e^{F(x(u))}$, $x(u)$ is the inverse function of $u(x)$.

Both functions $u(x)$ and $x(u)$ are odd functions, in view of (28). Their graphs are symmetrical with respect to the origin. Then $h(u) = -h(-u)$.

Therefore, if $u(t)$ is a solution of the equation (30), then $u(c - t)$ also, c is an arbitrary constant. The function $-u(t)$ is a solution also, since $(-u)'' + h(-u) = -u'' - h(u) = 0$

The positive and negative amplitudes of $u(t)$ have the same absolute value, denote it M again. One has for M, using the integral relation

$$u'^2(t) + 2H(u(t)) = const, \quad (31)$$

that for a solution $u(t)$ with the initial conditions

$$u(0) = 0, \ \ u'(0) = \alpha > 0, \quad (32)$$

holds

$$\int_0^M \frac{ds}{\sqrt{\alpha^2 - 2H(s)}} = \frac{1}{4}T, \quad (33)$$

where T is the period.

Let $x(t)$ be a solution of (27) with the initial conditions

$$x(0) = 0, \ \ x'(0) = \alpha > 0. \quad (34)$$

Both solutions $x(t)$ and $u(t)$ have the same period $T(\alpha)$. Let A be the amplitude of $x(t)$. In view of (29), the relation between T and A is

$$M = \int_0^A e^{F(s)}ds, \ \ 4\int_0^M \frac{ds}{\sqrt{\alpha^2 - 2H(s)}} = T, \quad (35)$$

for α given. The relations (33) and (35) fully describe the period-amplitude relation.

5 Generalizations

Using the same approach, more general oscillatory equations can be investigated.

More general equations of the form

$$x'' + f(x, x')x = 0 \quad (36)$$

can be studied by using Sabatini's transformation. The function f in (36) can be interpreted as the stiffness coefficient, dependent generally on (x, x'). The equation with a "generalized" stiffness coefficient may be in the form

$$x'' + (g(x) + f(x)x'^2)x = 0. \quad (37)$$

It can be represented as

$$x'' + xf(x)x'^2 + xg(x) = 0. \quad (38)$$

In this form, it is convenient to study using Sabatini's transformation.

There are many articles on the subject, for instance, [16], [18], [11]. The main problem solved is the relation between oscillation frequency and amplitude. Equations of the form (38) can be shown to have multiple embedded period annuli. Investigation of oscillation of solutions, in that case, deserves special attention.

Multiple cases are possible, depending on whether the function $u(x)$ has an asymptote for $x > 0$, or $x < 0$, or two asymptotes.

The equation in the next section can be studied in the generalized form

$$x'' + \frac{x}{1 - x^2}x'^2 + g(x) = 0, \qquad |x| < 1.$$

In particular, it was shown in the work, [1], that a period annulus may appear in this equation, while in the shortened equation without the middle-term period, annuli are absent. The function *g(x)* was chosen as a polynomial of the 5^{th} degree.

In the work, [23], the above equation was studied together with the Dirichlet boundary conditions. The number of solutions was estimated provided that *g(x)* is a cubic polynomial.

6 Equation $x'' + \frac{x}{1-x^2}x'^2 + kx = 0$

Consider equation

$$x'' + \frac{x}{1-x^2}x'^2 + kx = 0, \ |x| < 1. \quad (39)$$

This equation and its generalization were studied in the paper, [1]. The effectiveness of Sabatini's transformation was tested and confirmed. This equation exhibits an especially simple relation between the original one and its counterpart in terms of the variable *u*.

Proposition 6.1. *Equation (39) by Sabatini's transformation turns into an equation*

$$u'' + h(u) = 0, \ h(u) = \frac{kx(u)}{\sqrt{1-x^2(u)}}, \quad (40)$$

Proof. Indeed,

$$f(x) = \frac{x}{1-x^2},$$

$$F(x) = \int_0^x f(s)ds = \ln\frac{1}{\sqrt{1-x^2}}$$

$$e^{F(x)} = \frac{1}{\sqrt{1-x^2}},$$

$$u = \Phi(x) = \int_0^x e^{F(s)}ds = \int_0^x \frac{1}{\sqrt{1-s^2}}ds =$$

$$= \arcsin x, \ |x| < 1,$$

$$x = x(u) = \sin u, \ \ |u| < \frac{\pi}{2}.$$

Then

$$u'' + h(u) = u'' + k\, tgu = 0, \ |u| < \frac{\pi}{2}. \quad (41)$$

Since $\int_0^u tg\, s\, ds = -\ln\cos u$, $|u| < \frac{\pi}{2}$, the integral $u'^2 + 2H$ of (41) is

$$u'^2(t) - 2k\ln\cos u = const = \alpha^2 =$$

$$= -2k\ln\cos M, \ 0 < M < \frac{\pi}{2} \quad (42)$$

for a solution of (41) with the initial conditions

$$u(0) = 0, \ u'(0) = \alpha > 0. \quad (43)$$

One has, as before, that

$$\int_0^M \frac{ds}{\sqrt{\alpha^2 + 2k\ln\cos s}} = \frac{1}{4}\, T, \quad (44)$$

and the relation between amplitude A of $x(t)$ ad the period T is

$$T = 4\int_0^{arcsinA} \frac{ds}{\sqrt{\alpha^2 + 2k\ln\cos s}}, \quad (45)$$

for α given.

7 Conclusion

A relatively broad class of nonlinear oscillators can be treated using Sabatini's transformation. Relations between period/frequency and the amplitudes of oscillation can be established with the accuracy, allowed by used computational instruments. In further studies of nonlinear oscillators represented by the equations of the form $x'' + f(x)x'^2 + g(x) = 0$ focus can be made on the coefficient *f(x)*. The new variable *u* can be defined on a bounded interval, in contrast with the variable *x*. This is the case if the integral in (11) is convergent. A great variety of variants are possible if *f(x)* is a somewhat arbitrary polynomial with multiple zeros. If *g(x)* is a polynomial of sufficiently high degree, period annuli can appear in a related equation $x'' + g(x) = 0$. The interrelation between period annuli in this shortened equation and the above one with the middle term is interesting for practical purposes and challenging for theoreticians' problems. An interesting problem is to compare the equation in question with its dissipative counterpart $x'' + f(x)x' + g(x) = 0$.

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