Multiplication Operators on Weighted Grand Lorentz Spaces with Various Properties

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Abstract: - The concept of Lebesgue space has been generalized to the grand Lebesgue space with non-weight and weight, and the classical Lorentz space concept has been generalized to grand Lorentz spaces with a similar logic. In this study, instead of using rearrangement for a measurable function, weighted Grand Lorentz spaces are defined by using the maximal function $1\leq p,q\leq\infty$ where the weight function is measurable, complex valued, and locally bounded. In addition, multiplication operators on weighted grand Lorentz spaces are defined and the fundamental properties of these operators such as boundedness, closed range, invertibility, compactness, and closedness are characterized.

Key-Words: - Weighted Grand Lorentz space, Multiplication operator, Compact operator

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1 Introduction

Iwaniec and Sbordone generalized the concept of Lebesgue spaces and introduced a new space of measurable, almost everywhere equal integrable function classes, which they called grand Lebesgue spaces. According to [12], grand Lebesgue spaces are the space of equivalence classes of all measurable functions defined on $(0,1)$ and denoted by $L^{p)}$ with $1<p<\infty$. For any $f\in L^{p)}$, the function

$$\|f\|_{p)}=\sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1|f(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}$$

defines a norm on $L^{p)}$ and makes the space a Banach function space. Also $L^p\subset L^{p)}\subset L^{p-\varepsilon}$ if $0<\varepsilon\leq p-1$. New results on grand Lebesgue spaces can be observed in current studies, [8], [10], [14], [15], [16], [21], [22], [27]. Presented in terms of the Jacobian integrability problem, these works have proven useful in various applications of partial differential equations and variational problems, where they are used in the study of maximum functions, extrapolation theory, etc. The harmonic analysis of these spaces, and the related small Lebesgue spaces, has been intensively developed in recent years and continues to attract the attention of researchers due to various applications.

Let $1<p<\infty$ and w be a weight function. Weighted grand Lebesgue spaces denoted by $L_w^{p)}$ are the space of measurable functions defined on $(0,1)$ such that

$$\|f\|_{p),w}=\sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1|f(x)|^{p-\varepsilon}w(x)dx\right)^{\frac{1}{p-\varepsilon}}$$

is finite for any $f\in L_w^{p)}$. These spaces were defined in the study, [9], and they also examined the boundedness of the maximal operator on the space $L_w^{p)}$ in there. The boundedness of the Riesz potential on weighted grand Lebesgue spaces is characterized in [16]. In addition to these, the classical weighted Lorentz and grand Lorentz spaces were compared and the boundedness of the maximal operator was examined in [7], [25]. The basic properties of grand Lorentz sequence spaces and the multiplication operators on these spaces are examined in [24].

Let (X,Σ,μ) and (Y,Γ,υ) be two σ-finite measure spaces. A measurable transformation $T:Y\to X$ is said to be non-singular if $\upsilon(T^{-1}(A))=0$, whenever $A\in\Sigma$ with $\mu(A)=0$. In this case, we write $\upsilon\circ T^{-1}\ll\mu$. Let $u:Y\to\mathbb{C}$ be a measurable function. Then, the linear transformation

$$W=W_{u,T}:L(X,\Sigma,\mu)\to L(Y,\Gamma,\upsilon)$$

is defined as

$$W(f)(x)=W_{u,T}(f)(x)=u(T(x))\cdot f(T(x))$$

for all $x\in Y$ and for each $f\in L(X,\Sigma,\mu)$ where $L(X,\Sigma,\mu)$ and $L(Y,\Gamma,\upsilon)$ are the linear spaces of all $\mu-$measurable and $\upsilon-$measurable functions on X and Y, respectively. Here, the non-singularity of T guarantees that the operator W is well defined as a mapping of equivalence classes of functions. In the case when W maps $L(\mu)$ into $L(\upsilon)$, we call $W=W_{u,T}$ a weighted composition operator induced by the pair (u,T). If $u\equiv 1$, then $W\equiv C_T$ is called a composition operator induced by T. If T is the identity mapping, then $W\equiv M_u: f\to u\cdot f$ is a multiplication operator induced by u.

These operators are simple but have a wide range of applications in ergodic theory, dynamical systems, etc. The studies of (weighted) composition and multiplication operators have a very long history in Mathematics. From books on functional analysis and papers related to these operators, one can learn many properties of these operators on various function spaces including Lebesgue and Lorentz spaces. The study of these operators acting on Lebesgue and Lorentz spaces has been made in [5], [11], [13], [19], [20], [28], [29], [30], [1], [2], [3], [17], [18], respectively.

2 Preliminaries

Throughout the paper $X=(X,\Sigma,\mu)$ will stand a $\sigma-$finite measure space, $L(\mu)$ will denote the linear space of all equivalence classes of Σ-measurable functions on X and χ_A will be used for the characteristic function of a set A. For any two non-negative expressions (i.e. functions or functionals), A and B, the symbol $A\prec B$ means that $A\le cB$ for some positive constant c independent of the variables in the expressions A and B. If $A\prec B$ and $B\prec A$, we write $A\approx B$ and say that A and B are equivalent.

Let w be a weight function, i.e. a measurable, complex valued, and locally bounded function on X, satisfying $w(x)\ge 1$ for all $x\in X$. Weighted Lorentz spaces or Lorentz spaces over weighted measure spaces $L(p,q,wd\mu)$ are studied and discussed in [6], [23], by taking the measure $wd\mu$ instead of the measure μ. Then the distribution function f which is considered complex-valued, measurable, and defined on the measure space $(X,wd\mu)$

$$\lambda_{f,w}(y)=w\{x\in X:|f(x)|>y\} = \int_{\{x\in X:|f(x)|>y\}} w(x)\,d\mu(x),\ y\ge 0$$

is found. The nonnegative rearrangement f is given by

$$f_w^*(t)=\inf\{y>0:\lambda_{f,w}(y)\le t\} =\sup\{y>0:\lambda_{f,w}(y)>t\},\ t\ge 0$$

where we assume that $\inf\varnothing=\infty$ and $\sup\varnothing=0$. Also, the average (maximal) function of f on $(0,\infty)$ is given by

$$f_w^{**}(t)=\frac{1}{t}\int_0^t f_w^*(s)\,ds.$$

Note that $\lambda_{f,w}(\cdot)$, $f_w^*(\cdot)$ and $f_w^{**}(\cdot)$ are non-increasing and right continuous functions. The weighted Lorentz space $L(p,q,wd\mu)$ is the collection of all the functions f such that $\|f\|_{p,q}^*<\infty$, where

$$\|f\|_{p,q,w}^*=\begin{cases}\left(\dfrac{q}{p}\displaystyle\int_0^\infty t^{\frac{q}{p}-1}\left[f_w^*(t)\right]^q dt\right)^{\frac{1}{q}}, & 0<p,q<\infty\\ \sup\limits_{t>0} t^{\frac{1}{p}} f_w^*(t), & 0<p<q=\infty\end{cases}. \tag{1}$$

In general, however, $\|\cdot\|_{p,q,w}^*$ is not a norm since the Minkowski inequality may fail. But by replacing f_w^* with f_w^{**} in (1), we get that $L(p,q,wd\mu)$ is a normed space, with the functional $\|\cdot\|_{p,q,w}$ defined by

$$\|f\|_{p,q,w}=\begin{cases}\left(\dfrac{q}{p}\displaystyle\int_0^\infty t^{\frac{q}{p}-1}\left[f_w^{**}(t)\right]^q dt\right)^{\frac{1}{q}}, & 0<p,q<\infty\\ \sup\limits_{t>0} t^{\frac{1}{p}} f_w^{**}(t), & 0<p<q=\infty\end{cases}.$$

If $1<p\le\infty$ and $1\le q\le\infty$, then

$$\|f\|_{p,q,w}^*\le\|f\|_{p,q,w}\le\frac{p}{p-1}\|f\|_{p,q,w}^*$$

where the first inequality is an immediate consequence of the fact that $f_w^* \leq f_w^{**}$ the second follows from the Hardy inequality, [4]. For more on weighted Lorentz spaces, one can refer to [6], [23], and references therein.

Using the maximal function $f_w^{**}(\cdot)$, instead of the nonnegative rearrangement $f_w^*(\cdot)$ used in the definition of grand Lorentz space defined in [25], we defined the weighted grand Lorentz spaces as follows.

Definition 1 The weighted grand Lorentz space $L_w^{p,q)}$ is the collection of all the complex-valued, measurable functions which are defined on the measure space $((0,1), wd\mu)$ such that $\| f \|_{p,q)}^w < \infty$ where

$$\| f \|_{p,q)}^w = \begin{cases} \sup\limits_{0<\varepsilon<q-1} \left(\frac{q}{p} \varepsilon \int_0^1 t^{\frac{q}{p}-1} \left[f_w^{**}(t) \right]^{q-\varepsilon} dt \right)^{\frac{1}{q-\varepsilon}}, 1 < p, q < \infty \\ \sup\limits_{0<t<1} t^{\frac{1}{p}} f_w^{**}(t), \qquad 1 < p < \infty, q = \infty \end{cases}$$

for any $f \in L_w^{p,q)}$. In particular, if $1 < p < \infty$ any $1 \leq q \leq \infty$; $p = q = 1$ or $p = q = \infty$, then the normed space $L_w^{p,q)}$ is a Banach space.

3 Multiplication Operators

Let $u : X \to \mathbb{C}$ be a measurable function such that $u \cdot f \in L(\mu)$ whenever $f \in L(\mu)$. This gives rise to a linear transformation $M_u : L(\mu) \to L(\mu)$ defined by $M_u(f) = u \cdot f$, where the product is pointwise. In case if $L(\mu)$ is a topological vector space and M_u is a continuous (bounded) operator, then it is called a multiplication operator induced by u.

Remark 1. In general, the multiplication operators on measurable function's spaces are not injective. For example, let $G = X - supp(u)$ where $supp(u) = \{ x \in X : u(x) \neq 0 \}$. Then $\mu(G) > 0$ and $(\chi_G \cdot u)(x) = \chi_G(x) \cdot u(x) = 0$ for all $x \in X$. This implies that $M_u(\chi_G) = 0$ and $KerM_u \neq \{0\}$. Hence M_u is not injective.

On the contrary, if M_u is injective, then $\mu(X - supp(u))$ must be 0. If $\mu(X - supp(u)) = 0$ and μ is a complete measure, then $M_u(f) = 0$ implies that $f(x) \cdot u(x) = 0$ for all $x \in X$ and so $\{ x \in X : f(x) \neq 0 \} \subseteq X - supp(u)$ and $f = 0$ (a.e.) on X.

Proposition 1. The operator M_u is injective on $K = L_w^{p,q)}(supp(u)) = \left\{ \tilde{f} = f \chi_{supp(u)} : f \in L_w^{p,q)}(X) \right\}$.

Proof. To show that the operator M_u is injective, it is enough to show that $KerM_u = \{0\}$. Indeed, if $M_u(\tilde{f}) = 0$ with $\tilde{f} \in K$, then

$$\tilde{f}(x) \cdot u(x) = f(x) \cdot \chi_{supp(u)}(x) \cdot u(x) = 0$$

for all $x \in X$. From this, we get $f(x) \cdot u(x) = 0$ for all $x \in supp(u)$ and so $f(x) = 0$. Therefore $\tilde{f} = 0$ and $KerM_u = \{0\}$.

Theorem 1. The linear operator $M_u(f) = u \cdot f$ on weighted grand Lorentz spaces $L_w^{p,q)}$ is bounded for $1 < p, q \leq \infty$ if and only if u is essentially bounded. Moreover $\|u\|_\infty \leq \|M_u\| \leq \|u\|_\infty^{\frac{q}{p}}$.

Proof. Suppose that $f \in L_w^{p,q)}$ and u is essentially bounded, i.e. $u \in L_\infty(\mu)$. Since $|u(x)| \leq \|u\|_\infty$ for all $x \in X$, it can be written that $|(u \cdot f)(x)| \leq \|u\|_\infty |f(x)|$ and so

$$\lambda_{M_u f, w}(y) \leq \lambda_{f,w}\left(\frac{y}{\|u\|_\infty} \right)$$

for all $y \geq 0$. Also, it is easy to see that $(M_u f)_w^*(t) \leq \|u\|_\infty f_w^*(t)$ and

$$(M_u f)_w^{**}(t) \leq f_w^{**}\left(\frac{t}{\|u\|_\infty} \right).$$

Therefore

$$\|M_u(f)\|_{p,q)}^{w} = \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left((M_u f)^{**}(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}$$
$$\leq \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(f^*\left(\frac{t}{\|u\|_\infty}\right)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}$$
$$= \|u\|_\infty^{\frac{q}{p}}\sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(f^*(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}$$
$$= \|u\|_\infty^{\frac{q}{p}}\|f\|_{p,q)}^{w}$$

can be obtained. For $q=\infty$, we have

$$\|M_u f\|_{p,\infty)}^{w} = \sup_{0<t<1} t^{\frac{1}{p}}(M_u f)_w^{**}(t) \leq \sup_{0<t<1} t^{\frac{1}{p}} f_w^{**}\left(\frac{t}{\|u\|_\infty}\right) = \|u\|_\infty^{\frac{1}{p}}\|f\|_{p,\infty)}^{w}.$$

Thus, M_u is bounded for all $1<p,q\leq\infty$.

Conversely, suppose that M_u is a bounded operator on weighted grand Lorentz spaces for $1<q<\infty$. If u is not an essentially bounded function, then we can write a set $G_k=\{x\in(0,1): |u(x)|>k\}$ that has a positive measure for all $k\in\mathbb{N}$. Since the non-increasing rearrangement of the characteristic function χ_{G_k} is $(\chi_{G_k})_w^*(t)=\chi_{[0,w(G_k))}(t)$, we can easily get that

$$\lambda_{k\chi_{G_k},w}(y)\leq\lambda_{M_u\chi_{G_k},w}(y)\Rightarrow$$
$$(M_u\chi_{G_k})_w^*(t)\geq k(\chi_{G_k})_w^*(t)$$

and

$$(M_u\chi_{G_k})_w^{**}(t)\geq(\chi_{G_k})_w^{**}\left(\frac{t}{k}\right).$$

Therefore

$$\|M_u(\chi_{G_k})\|_{p,q)}^{w} = \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left((M_u(\chi_{G_k}))_w^{**}(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}$$
$$\geq \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left((k\cdot\chi_{G_k})_w^{**}(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}$$
$$= k^{\frac{q}{p}}\|\chi_{G_k}\|_{p,q)}^{w}.$$

Besides these, $q=\infty$ we have

$$\|M_u(\chi_{G_k})\|_{p,q)}^{w} = \sup_{0<t<1} t^{\frac{1}{p}}(M_u(\chi_{G_k}))_w^{**}(t)$$
$$\geq \sup_{0<t<1} t^{\frac{1}{p}}k(\chi_{G_k})_w^{**}(t) = k^{\frac{1}{p}}\|\chi_{G_k}\|_{p,q)}^{w}$$
$$= k^{\frac{1}{p}}(q-1)w(G_k)^{\frac{q}{p}}.$$

This contradicts the boundedness of M_u. Hence u must be essentially bounded. Now for any $\delta>0$, let

$$F=\{x\in X: |u(x)|\geq\|u\|_\infty-\delta\}.$$

Then
$\{x\in X:(\|u\|_\infty-\delta)\chi_F(x)>\lambda\}\subseteq\{x\in X:(u\cdot\chi_F)(x)>\lambda\}$
and so $\lambda_{(\|u\|_\infty-\delta)\chi_F,w}(y)\leq\lambda_{u\chi_F,w}(y)$. Therefore, we get

$$(M_u(\chi_F))_w^*(t)\geq(\|u\|_\infty-\delta)(\chi_F)_w^*(t)$$

and $\|M_u\|\geq(\|u\|_\infty-\delta)$. As a result, $\|M_u\|\geq\|u\|_\infty$.

Remark 2. If X and Y are Banach spaces and $T\in\mathcal{B}(X,Y)$, then T is bounded below if and only if T is 1-1 and has a closed range. According to this knowledge, we can give the following corollaries.

Corollary 1. M_u has a closed range on $L_w^{p,q)}(supp(u))$ if and only if M_u is bounded below on $L_w^{p,q)}(supp(u))$.

Corollary 2. If μ is a complete measure and $u\neq0$ a.e., then $M_u: L_w^{p,q)}(X,\Sigma,\mu)\to L_w^{p,q)}(X,\Sigma,\mu)$ has closed range if and only if M_u is bounded below on $L_w^{p,q)}(X,\Sigma,\mu)$.

Theorem 2. The set of all multiplication operators on weighted grand Lorentz spaces $L_w^{p,q)}$ for $1<p,q<\infty$ is a maximal Abelian sub algebra of $\mathcal{B}\left(L_w^{p,q)},L_w^{p,q)}\right)$, Banach algebra of all bounded linear operators on $L_w^{p,q)}$.

Proof. Let $H=\{M_u: u\in L_\infty\}$. Then H is a vector space under addition and scaler multiplication. Also, it is a sub algebra of $\mathcal{B}\left(L_w^{p,q)},L_w^{p,q)}\right)$ according to multiplication. Let T be any operator on $L_w^{p,q)}$ such that $T \circ M_u = M_u \circ T$ for every $u \in L_\infty(\mu)$ and $e : X \to \mathbb{C}$ be the unit function defined by $e(x) = 1$ for all $x \in X$ with $v = Te$. Then

$$T(\chi_E) = T(M_{\chi_E} e) = M_{\chi_E}(T(e)) = \chi_E v = M_v(\chi_E)$$

and so $T = M_v$ for all measurable set E. If possible, the set $G_k = \{x \in X : |v(x)| > k\}$ has a positive measure for each $k \in \mathbb{N}$, then

$$\left\| T(\chi_{G_k}) \right\|_{p,q)}^{w} = \left\| M_v(\chi_{G_k}) \right\|_{p,q)}^{w} \geq k^{\frac{q}{p}} \left\| \chi_{G_k} \right\|_{p,q)}^{w}.$$

Therefore T is an unbounded operator that is a contradiction to the fact that T is bounded. Therefore $v \in L_\infty(\mu)$ and M_v is bounded by Theorem 1. Now, let $f \in L_w^{p,q)}$ and $(s_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of measurable simple functions such that $\lim s_n = f$. Then

$$\begin{aligned} T(f) &= T\left(\lim_{n\to\infty} s_n\right) = \lim_{n\to\infty} T(s_n) = \lim_{n\to\infty} M_v(s_n) \\ &= M_v \lim_{n\to\infty}(s_n) = M_v(f) \end{aligned}.$$

Therefore, we can conclude that $T \in H$.

Corollary 3. The multiplication operator M_u on $L_w^{p,q)}$ for $1 < p, q < \infty$ is invertible if and only if u is invertible in L_∞.

Proof. Let M_u be invertible. Then there exists a $T \in \mathcal{B}\left(L_w^{p,q)}, L_w^{p,q)}\right)$ such that $T \circ M_u = M_u \circ T = I$. Let $M_v \in H$. Then $M_u \circ M_v = M_v \circ M_u$ and

$$\begin{aligned} T \circ M_v &= T \circ M_v \circ I \\ &= T \circ M_v \circ (M_u \circ T) = (T \circ M_u) \circ M_v \circ T \\ &= I \circ M_v \circ T = M_v \circ T \end{aligned}.$$

Therefore, we can conclude that T commute with M_v and so $T \in H$ by Theorem 2. Then there exists a $w \in L_\infty$ such that $T = M_w$ and

$$M_u \circ M_w = M_w \circ M_u = I.$$

This implies that $uw = wu = 1$ a.e, which means that u is invertible on L_∞. On the other hand, assume that u is invertible on L_∞, that is $u^{-1} \in L_\infty$. Then $M_u \circ M_{u^{-1}} = M_{u^{-1}} \circ M_u = I$ which means that M_u is invertible on $\mathcal{B}\left(L_w^{p,q)}, L_w^{p,q)}\right)$.

4 Compact Multiplication Operators

In this section, compact multiplication operators are characterized.

Definition 2. Let T be an operator on a normed space X. A subspace K of X is said to be invariant under T (or simply T-invariant) if $T(K) \subseteq K$.

Lemma 1. Let $T : X \to X$ be an operator. If T is compact and N is a closed T-invariant subspace of X, then $T|_N$ is also compact.

Proof. Let $(g_k)_{k\in\mathbb{N}}$ be a sequence in $N \subset X$. Then compactness property of T implies that there exists a subsequence $\left(g_{k_n}\right)_{n\in\mathbb{N}}$ of $(g_k)_{k\in\mathbb{N}}$ such that $\left(T\left(g_{k_n}\right)\right)_{n\in\mathbb{N}}$ converges in X. Since $\left(g_{k_n}\right) \subset N$ and $\left(T\left(g_{k_n}\right)\right)_{n\in\mathbb{N}} \subset T(N)$, then $\left(T\left(g_{k_n}\right)\right)_{n\in\mathbb{N}}$ converges on N. Hence $T|_N$ is compact.

Theorem 4. Let M_u be a compact operator, $G_\delta(u) = \{x \in X : |u(x)| \geq \delta\}$ and $L_w^{p,q)}(G_\delta(u)) = \left\{ f\chi_{G_\delta(u)} : f \in L_w^{p,q)} \right\}$ for any $\delta > 0$. Then $L_w^{p,q)}(G_\delta(u))$ is a closed invariant subspace of $L_w^{p,q)}$ under M_u and M_u is a compact operator on $L_w^{p,q)}(G_\delta(u))$.

Proof. We first show that $L_w^{p,q)}(G_\delta(u))$ is a subspace of $L_w^{p,q)}$. Let $\tilde{f}, \tilde{g} \in L_w^{p,q)}(G_\delta(u))$ and $a, b \in \mathbb{C}$. Since $\tilde{f} = f\chi_{G_\delta(u)}$ and $\tilde{g} = g\chi_{G_\delta(u)}$ for any $f, g \in L_w^{p,q)}$, we have

$$a\tilde{f} + b\tilde{g} = af\chi_{G_\delta(u)} + bg\chi_{G_\delta(u)} = (af + bg)\chi_{G_\delta(u)}.$$

By the definition of $M_u : L_w^{p,q)}(G_\delta(u)) \to L_w^{p,q)}(X)$, we have $M_u(\tilde{f}) = u\tilde{f} = u \cdot f\chi_{G_\delta(u)}$. Therefore $L_w^{p,q)}(G_\delta(u))$ is an invariant subspace of $L_w^{p,q)}(X, \Sigma, \mu)$ under M_u. Now, let us show that $\overline{L_w^{p,q)}(G_\delta(u))} \subset L_w^{p,q)}(G_\delta(u))$. Let $\tilde{g} \in \overline{L_w^{p,q)}(G_\delta(u))}$. Then there exists a sequence $\tilde{g}_k$ in $L_w^{p,q)}(G_\delta(u))$ such that $\tilde{g}_k \to \tilde{g}$ where $\tilde{g}_k = g_k\chi_{G_\delta(u)}$ for each

$k \in \mathbb{N}$. Since $\tilde{g}_k$ is a Cauchy sequence in $L_w^{p,q)}(G_\delta(u))$, it can be written that for all $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that $\|\tilde{g}_k - \tilde{g}_r\|_{p,q)}^w < \varepsilon$ for all $k, r > k_0$. Hence for all $k, r > k_0$, we can find a $\delta > 0$ such that

$$\delta(g_k - g_r) \le (g_k - g_r)\chi_{G_\delta(u)}$$

and

$$\delta(g_k - g_r)_w^*(t) \le (g_k - g_r)_w^*(t)\chi_{[0, w(G_\delta(u)))}(t).$$

Then,

$\|g_k - g_r\|_{p,q)}^w \le w(G_\delta(u))^{\frac{q}{p}}(q-1)\delta^{-\frac{q}{p}}\|\tilde{g}_k - \tilde{g}_r\|_{p,q)}^w$ can be written. Therefore $\{g_k\}_{k\in\mathbb{N}}$ is also a Cauchy sequence in $L_w^{p,q)}$. Since $L_w^{p,q)}$ is a Banach space, we can write that $g_k \to g$ for an element $g \in L_w^{p,q)}$. Thus, we have

$$\left\|g_k\chi_{G_\delta(u)} - g\chi_{G_\delta(u)}\right\|_{p,q)}^w \le \|g_k - g\|_{p,q)}^w$$

and $\tilde{g}_k \to \tilde{g}$. Consequently $\tilde{g} \in L_w^{p,q)}(G_\delta(u))$ and $M_u|_{L_w^{p,q)}(G_\delta(u))}$ is a compact operator by Lemma 1.

Theorem 5. A multiplication operator M_u on $L_w^{p,q)}$ is compact if and only if $L_w^{p,q)}(G_\delta(u))$ is finite dimensional for each $\delta > 0$, where $G_\delta(u)$ and $L_w^{p,q)}(G_\delta(u))$ as in Theorem 4.

Proof. If M_u is a compact operator, then $L_w^{p,q)}(G_\delta(u))$ is a closed invariant subspace of $L_w^{p,q)}$ under M_u and $M_u|_{L_w^{p,q)}(G_\delta(u))}$ is a compact operator by Theorem 4. Let's take any $x \in X$. If $x \notin G_\delta(u)$ then for each $f \in L_w^{p,q)}$, we can obtain $\left(M_u|_{L_w^{p,q)}(G_\delta(u))}(f)\right)^*(t) = \left(u \cdot f\chi_{G_\delta(u)}\right)^*(t) = 0$ and so $M_u|_{L_w^{p,q)}(G_\delta(u))} = 0$. If $x \in G_\delta(u)$, then we have $|u(x)| \ge \delta$ and $\left|\left(u \cdot f\chi_{G_\delta(u)}\right)(x)\right| \ge \delta\left|\left(f\chi_{G_\delta(u)}\right)(x)\right|$, $\lambda_{f\chi_{G_\delta(u)},w}\left(\frac{y}{\delta}\right) \le \lambda_{\left(u\cdot f\chi_{G_\delta(u)}\right),w}(y)$. Therefore

$$\left\{y > 0 : \lambda_{\left(u\cdot f\chi_{G_\delta(u)}\right),w}(y) \le t\right\} \subset \left\{y > 0 : \lambda_{f\chi_{G_\delta(u)},w}\left(\frac{y}{\delta}\right) \le t\right\}$$

for all $t > 0$. By using this, we have

$$\delta \cdot \left(f\chi_{G_\delta(u)}\right)_w^*(t) \le \left(u \cdot f\chi_{G_\delta(u)}\right)_w^*(t) \Rightarrow$$
$$\delta \cdot \left(f\chi_{G_\delta(u)}\right)_w^{**}(t) \le \left(u \cdot f\chi_{G_\delta(u)}\right)_w^{**}(t)$$

and

$$\left\|M_u\left(f\chi_{G_\delta(u)}\right)\right\|_{p,q)}^w = \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(\left(M_u\left(f\chi_{G_\delta(u)}\right)\right)_w^{**}(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}$$
$$\ge \sup_{0<\varepsilon<q-1}\left(\frac{q}{p}\varepsilon\int_0^1 t^{\frac{q}{p}-1}\left(\delta \cdot \left(f\chi_{G_\delta(u)}\right)_w^{**}(t)\right)^{q-\varepsilon}dt\right)^{\frac{1}{q-\varepsilon}}$$
$$\ge \delta \cdot \left\|f\chi_{G_\delta(u)}\right\|_{p,q)}^w.$$

Thus, in either case $M_u|_{L_w^{p,q)}(G_\delta(u))}$ has a closed range in $L_w^{p,q)}(G_\delta(u))$ and invertible. Being compact implies that $L_w^{p,q)}(G_\delta(u))$ is finite dimensional.

Conversely, suppose that $L_w^{p,q)}(G_\delta(u))$ is finite dimensional for each $\delta > 0$. In particular, $L_w^{p,q)}(G_{1/n}(u))$ is finite dimensional for each $n \in \mathbb{N}$. Define a sequence $u_n : X \to \mathbb{C}$ as

$$u_n(x) = \begin{cases} u(x), & |u(x)| \ge 1/n \\ 0, & |u(x)| < 1/n \end{cases}$$

for all $n \in \mathbb{N}$. Since $u \in L_\infty$, it's easy to see that $u_n \in L_\infty$ for each $n \in \mathbb{N}$. Moreover for any $f \in L_w^{p,q)}$,

$$\lambda_{(u_n-u)\cdot f,w}(y) = w\left(\left\{x \in X : \left|\left((u_n - u)f\right)(x)\right| > y\right\}\right)$$

and

$$\left((u_n - u)f\right)_w^*(t) = \inf\left\{y > 0 : \lambda_{(u_n-u)\cdot f,w}(y) \le t\right\}.$$

If $x \in G_{1/n}(u)$ then $\left((u_n - u)f\right)_w^*(t) = 0$ and $(u_n - u)f = 0$. If $x \notin G_{1/n}(u)$, then we get $\left((u_n - u)f\right)_w^*(t) \le \frac{1}{n}f_w^*(t) \Rightarrow \left((u_n - u)f\right)_w^{**}(t) \le \frac{1}{n}f_w^{**}(t)$

and $\left\|M_{(u_n-u)}(f)\right\|_{p,q)}^{w} \leq \frac{1}{n}\|f\|_{p,q)}^{w}$. This implies that M_{u_n} converges to M_u uniformly. Since $L_w^{p,q)}\left(G_{\frac{1}{n}}(u)\right)$ is finite-dimensional, M_{u_n} is a finite rank operator. Therefore, M_{u_n} is a compact operator's sequence and so M_u is.

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