On the Solution of Equations with Linear-Fractional Shifts

ANNA TARASENKO, OLEKSANDR KARELIN, MANUEL GONZÁLEZ-HERNÁNDEZ, DARYA KARELINA
Institute of Basic Sciences and Engineering,
Hidalgo State Autonomous University,
Carretera Pachuca-Tulancingo, km.4.5,
Pachuca, Hidalgo, C.P. 42184,
MEXICO

Abstract: - This work represents a continuation of the studies relating to nonlinear equations, carried out by the authors. Special attention is paid to the operators with linear-fractional shifts that act on the argument of the unknown function, but also on the unknown function itself. In this work, we study homogeneous equations with such operators. The main classes of functions for which non-linear equations are considered are Hölder class real functions. Solutions of the equations have the form of infinite products or the form of infinite continued fractions; an abstract description of the solutions is also offered. The developed mathematical methods can be applied to finding the conditions of invertibility of certain operators found in modelling, as well as for the construction of their inverse operators. Subsequently, we suggest using these results for the modelling of renewable systems with elements that can be in different states: sick, healthy, immune, or vaccinated. These results can also be applied to the analysis of balance equations of the model and for finding equilibrium states of the system.

Key-Words: - Operator with linear-fractional shifts**,** Non-linear equations, Homogeneous equation, infinite continued fraction.

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I Introduction

Previously, attention to functional operators with shift was initiated due to the development of the theory of solvability of boundary value problems and singular integral equations with Carleman and non-Carleman shifts, [1], [2], [3].

Now, the interest and motivation for the study of such operators are growing in connection with the problems of depletion of natural resources and research on the possibility of using renewable resources.

So, in the modelling of systems with renewable resources, [4], [5], equations that contain functional operators with shift appear in balance relations. We consider the simplest functional operator with shift $A\varphi(x) = a(x)\varphi(x) + b(x)\varphi[\alpha(x)]$. We found conditions of invertibility of the operator A in Hölder space with weight and constructed operator A^{-1} inverse to it, [6], [7]. In the analysis of equations of balance relations of such systems, these inverse operators are used. In this way, due to the obtained results, equilibrium states of the considered renewable systems were found, in addition to formulating the corresponding economic-ecological problems, [8].

We consider an operator with a more complex form $C\varphi(x) = k(x)\varphi(x) + AB\varphi(x)$, where the operator B has the same structure as the operator A $B\varphi(x) = c(x)\varphi(x) + d(x)[\alpha(x)]$. The operator C appears in the investigation of systems with elements that can be in multiple states, for instance, be sick, have immunity, or be vaccinated. The conditions of invertibility of the composition of operators AB can be found as the union of invertibility conditions for the operator A and for the operator B. The inverse operator $\left(AB\right)^{-1}$ is constructed as the superposition of inverse operators A^{-1} B^{-1}.

For the operator $W = kI + AE$, which arises in the modelling of systems with renewable resources with elements that can be in different states, the conditions of invertibility in the Hölder space with weight, in the general case, are unknown, as are the types of the inverse operator. The interest and the motivation for the study of such operators are growing.

Studying the possibility of reducing the operator $W = kI + AB$ to the composition of more simple functional operators with shift, a system of nonlinear equations that describes the connections between the coefficients emerges:

$$\begin{cases} k(x) + a(x)c(x) = \delta(x)\rho(x) \\ a(x)d(x) + b(x)B_\alpha c(x) = \delta(x)\eta(x) + \mu(x)B_\alpha \rho(x) \\ b(x)B_\alpha d(x) = \mu(x)B_\alpha \eta(x) \end{cases}$$

Substituting $\delta(x)$ from the first equation of the system and $\mu(x)$ from the third equation of the system to the second equation, we obtain $\varphi(x) - G(x)/B_\alpha\varphi(x) = g(x)$, where $G(x)$ and $g(x)$ are expressed through the known functions $k(x)$, $a(x)$, $b(x)$, $c(x)$, $d(x)$ and $\varphi(x) = \eta(x)/\rho(x)$. Substituting $\rho(x)$ from the first equation of the system as well as $B_\alpha\eta(x)$ from the third equation of the system to the second equation, we obtain a non-linear equation. As we can observe, the solubility of non-linear equations plays an especially important part.

The present work is dedicated to the study of such nonlinear equations and their various generalizations.

Special attention is dedicated to operators with a linear-fractional shift, which acts not only on the argument of the unknown function

$$B_\alpha\varphi(x) = \varphi[\alpha(x)], \quad \alpha(x) = \frac{a_1x + a_2}{a_3x + a_4},$$

but also on the unknown function itself:

$$\Gamma\varphi(x) = \frac{a_1\varphi(x) + a_2}{a_3\varphi(x) + a_4}.$$

To obtain conditions for the invertibility of functional operators with a shift in weighted Hölder spaces and to construct inverse operators, the theory of functional series was used as the main mathematical apparatus, [6], [7]. In this work, homogeneous non-linear equations that contain operators with the linear-fractional shift are considered. Other mathematical tools, such as infinite products and infinite continued fractions, were used to describe solutions of non-linear equations with a shift.

2 Equation with Two Shifts in Hölder Space

Let us remember the definition of a Hölder space $H_\mu(J)$. The function $\varphi(x)$, which satisfies the following, conditions on $J = [0,1]$,

$$|\varphi(x_1) - \varphi(x_2)| \le C|x_1 - x_2|^\mu,$$

$x_1 \in J$, $x_2 \in J$, $\mu \in (0,1)$, is called a Hölder function with an exponent μ and a constant C. The functions of Hölder class form a set $H_\mu(J)$. The norm in $H_\mu(J)$ is defined by:

$\|f(x)\|_{H_\mu(J)} = \|f(x)\|_C + \|f(x)\|_\mu$, where $\|f(x)\|_C = \max|f(x)| \quad x \in J$, and

$$\|f(x)\|_\mu = \sup_{x_1 \ne x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\mu}, \quad x \in J .$$

Let us construct a linear-fractional shift $\alpha(x) = \dfrac{a_1x + a_2}{a_3x + a_4}$ with the properties: to be bijective and to preserve orientation on J: if $x_1 < x_2$, then $\alpha(x_1) < \alpha(x_2)$ for any $x_1 \in J$, $x_1 \in J$, , and let c $\alpha(x)$ have only two fixed points: $\alpha(0) = 0$, $\alpha(1) = 1$ and $\alpha(x) \ne x$ when $x \ne 0,1$.

This implies that $a_2 = 0$ $\dfrac{a_1}{a_3 + a_4} = 1$ that is to say, $a_1 = a_3 + a_4$. Additionally, let $\alpha(x)$ be a differentiable function with $\frac{d}{dx}\alpha(x) \ne 0$ and require that the derivative of the shift be positive:

$$\frac{d}{dx}\alpha(x) = \frac{a_1(a_3x + a_4) - a_3a_1x}{(a_3x + a_4)^2} = \frac{a_1a_4}{(a_3x + a_4)^2},$$

that is to say $a_1a_4 > 0$. In this way, we obtain $\alpha(x) = \dfrac{a_{34}x}{a_3x + a_4}$, where $a_{34} = a_3 + a_4$ and $a_{34}a_4 > 0$.

In what follows, by $B_\alpha\varphi(x)$ or by $(B_\alpha\varphi)(x)$ we are going to understand precisely this functional operator of shift, $B_\alpha\varphi(x) = \varphi[\alpha(x)]$, with the function $\varphi(x)$, which has exactly these properties. At the same time as the shift $\alpha(x)$, we consider the linear-fractional shift $\gamma(x) = \dfrac{b_1x + b_2}{b_3x + b_4}$.

Let us search what should be the ratios of coefficients $\alpha(x)$ y $\gamma(x)$ for these shifts to commute. We write the equality $\alpha[\gamma(x)] = \gamma[\alpha(x)]$ and carry out some arithmetic operations:

$$\frac{b_1\dfrac{a_{34}x}{a_3x + a_4} + b_2}{b_3\dfrac{a_{34}x}{a_3x + a_4} + b_4} = \frac{a_{34}\dfrac{b_1x + b_2}{b_3x + b_4}}{a_3\dfrac{b_1x + b_2}{b_3x + b_4} + a_4},$$

$$\frac{b_1a_{34}x+b_2\left(a_3x+a_4\right)}{b_3a_{34}x+b_4\left(a_3x+a_4\right)}=\frac{a_{34}\left(b_1x+b_2\right)}{a_3\left(b_1x+b_2\right)+a_4\left(b_3x+b_4\right)},$$

$$[b_1a_{34}x+b_2(a_3x+a_4)][a_3(b_1x+b_2)+a_4(b_3x+b_4)]=$$
$$[b_3a_{34}x+b_4(a_3x+a_4)][a_{34}(b_1x+b_2)].$$

Furthermore,

$$a_{34}b_1(b_3a_{34}+b_4a_3)x^2+$$
$$[a_{34}b_1b_4a_4+a_{34}b_2(b_3a_{34}+b_4a_3)]x+a_{34}b_2b_4a_4=$$
$$(a_3b_1+a_4b_3)(b_1a_{34}+b_2a_3)x^2+$$
$$[(a_3b_1+a_4b_3)b_2a_4+(a_3b_2+a_4b_4)(b_1a_{34}+b_2a_3)]x+$$
$$(a_3b_2+a_4b_4)b_2a_4,$$
$$a_{34}b_2b_4a_4=(a_3b_2+a_4b_4)b_2a_4.$$

Comparing the coefficients in the corresponding power functions, we obtain:

$$a_{34}b_1(b_3a_{34}+b_4a_3)=(a_3b_1+a_4b_3)(b_1a_{34}+b_2a_3),\quad (1)$$
$$[a_{34}b_1b_4a_4+a_{34}b_2(b_3a_{34}+b_4a_3)]=$$
$$[(a_3b_1+a_4b_3)b_2a_4+(a_3b_2+a_4b_4)(b_1a_{34}+b_2a_3)],\quad (2)$$
$$a_{34}b_2b_4a_4=(a_3b_2+a_4b_4)b_2a_4.\quad (3)$$

The constants $a_3,\ a_4,\ b_1,\ b_2,\ b_3,\ b_4$, connected by the relations (1-3), form two shifts that commute with each other, $\alpha(x)=\frac{\left(a_3+a_4\right)x}{a_3x+a_4}$ and $\gamma(x)=\frac{b_1x+b_2}{b_3x+b_4}$.

After analysing the system (1), (2), (3), it turns out that the linear-fractional commutates for B_α are functional operators with shifts of the same type $B_\gamma\varphi(x)=\frac{\left(b_3+b_4\right)x+b_2}{b_3x+b_4}$, but without the mandatory requirement $\left(b_3+b_4\right)b_4>0$, which is satisfied for $\alpha(x)=\frac{\left(a_3+a_4\right)x}{a_3x+a_4}$: $\left(a_3+a_4\right)a_4>0$.

Let $b_2\neq 0$ and $b_1\neq 0, b_3\neq 0, b_4\neq 0$.

The relation (3) is consolidated and transforms to $a_{34}b_4=(a_3b_2+a_4b_4)$, we obtain $(a_3+a_4)b_4=a_3b_2+a_4b_4$ and $b_4=b_2$.

Let us turn now to (1). We are going to multiply this relation by $\frac{1}{a_3}\cdot\frac{1}{a_3}$:

$$\frac{a_{34}}{a_3}b_1(b_3\frac{a_{34}}{a_3}+b_4)=(b_1+\frac{a_4}{a_3}b_3)(b_1\frac{a_{34}}{a_3}+b_2)$$

and transcribe the obtained equality relative to the unknown $Z=\frac{a_{34}}{a_3}$.

Considering that $\frac{a_4}{a_3}=Z-1$ $b_4=b_2$, we have:

$$Zb_1(b_3Z+b_4)=(b_1+\left(Z-1\right)b_3)(b_1Z+b_2)\ \text{ and}$$
$$Z(b_1b_2-b_1b_1-b_3b_2+b_3b_1)=b_1b_2-b_3b_2.$$

We obtain $Z=\frac{b_2}{b_2-b_1}$ or $\frac{a_3+a_4}{a_3}=\frac{b_2}{b_2-b_1}$ as well as $Z-1=\frac{a_4}{a_3}=\frac{b_1}{b_2-b_1}$.

We now turn the relation (2) into the form

$$a_{34}b_2b_3a_{34}=a_3b_1b_2a_{34}+a_4b_2b_3a_4+a_3b_1b_2a_{34}$$

and multiply it by $\frac{1}{a_3}\cdot\frac{1}{a_3}$, turning it into

$$\frac{a_{34}}{a_3}b_1(b_3\frac{a_{34}}{a_3}+b_4)=(b_1+\frac{a_4}{a_3}b_3)(b_1\frac{a_{34}}{a_3}+b_2),$$

or well,

$$b_2b_3Z^2=b_1b_2(Z-1)+b_2b_3(Z-1)^2+b_1b_2Z$$

and finally we obtain,

$$(2b_1b_2-2b_2b_3)Z=b_1b_2-b_2b_3,\quad \text{as well as}$$
$$Z=\frac{1}{2}\ \text{ or }\ \frac{a_{34}}{a_3}=\frac{1}{2}\ \text{ and }\ Z-1=\frac{a_4}{a_3}=\frac{b_1}{b_2-b_1}.$$

We have fundamentally simplified the relations between the coefficients and we put them together:

$$b_4=b_2,\ \frac{a_4}{a_3}=\frac{b_1}{b_2-b_1},\frac{a_4}{a_3}=-\frac{1}{2},\ \left(a_3+a_4\right)a_4>0.$$

Here we recall that the last equality must hold.

So we get $a_3a_4>-a_4^2, a_3^2\frac{a_4}{a_3}>-a_4^2, a_3^2\left(-\frac{1}{2}\right)>-a_4^2.$

We come to a contradiction: $-\frac{1}{2}>-\frac{a_4^2}{a_3^2}=-\frac{1}{4}$.

Now, let $b_2=0$, $b_1\neq 0,\ b_3\neq 0,\ b_4\neq 0$. The relation (3) degenerates into $b_3a_{34}+b_4a_3=a_3b_1+a_4b_3$. From here, $b_3+b_4=b_1$.

We consider the equation

$$\varphi(x)-G(x)\cdot B_\alpha B_\gamma\varphi(x)=0,\quad (4)$$

with the initial condition $\varphi(1)=\varphi_0$ and the condition of concordance $\varphi(1)=G(1)\left(B_\alpha B_\gamma\varphi\right)(1)$. The equation will be considered in the space $H_\mu(J)$, which means that the coefficient $G(x)$ belongs to $H_\mu(J)$ and the function $\varphi(x)$ is searched for in

$H_\mu(J)$. We assume the functions under consideration to be positive, which corresponds to the equations that arise in applications.

We write the reductive representation:

$$\varphi(x) = G(x) \cdot B_\alpha B_\gamma \varphi(x).$$

There is a representation of the solution in the form of an infinite product:

$$\varphi(x) = G(x) \cdot B_\alpha B_\gamma [G(x) \cdot B_\alpha B_\gamma \varphi(x)] =$$

$$G(x) \cdot B_\alpha B_\gamma [G(x) \cdot B_\alpha B_\gamma [G(x) \cdot B_\alpha B_\gamma \varphi(x)]] =$$

$$G(x) \cdot B_\alpha B_\gamma B_\alpha^{-1} B_\alpha G(x) \cdot$$

$$\cdot B_\alpha B_\gamma B_\alpha^{-1} B_\alpha^2 B_\gamma B_\alpha^{-2} G(x) \cdot (B_\alpha B_\gamma)^2 [\varphi(x)] = \cdots \quad (5)$$

The shifts possess multiplicative properties:

$B_\alpha[a(x) \cdot b(x)] = B_\alpha[a(x)] \cdot B_\alpha[b(x)]$,

$B_\gamma[a(x) \cdot b(x)] = B_\gamma[a(x)] \cdot B_\gamma[b(x)]$,

hence their composition has the multiplicative property $B_\alpha B_\gamma[a(x) \cdot b(x)] = B_\alpha B_\gamma[a(x)] \cdot B_\alpha B_\gamma[b(x)]$, which was used in the transformations carried out above.

We derive the multiplier $(B_\alpha B_\gamma)^n[\varphi(x)]$ at the beginning of the representation

$$\varphi(x) = (B_\alpha B_\gamma)^n [\varphi(x)] \cdot G(x) \cdot B_\alpha B_\gamma G(x) \cdots (B_\alpha B_\gamma)^n G(x) = \cdots$$

Note that

$(B_\alpha B_\gamma)^n = B_\alpha B_\gamma B_\alpha^{-1} B_\alpha^2 B_\gamma B_\alpha^{-2} \cdots B_\alpha^n B_\gamma B_\alpha^{-n} [B_\alpha^n]$.

We get

$$\varphi(x) = B_\alpha B_\gamma B_\alpha^{-1} B_\alpha^2 B_\gamma B_\alpha^{-2} \cdots B_\alpha^n B_\gamma B_\alpha^{-n} [B_\alpha^n \varphi(x)] \cdot$$

$$G(x) \cdot B_\alpha B_\gamma G(x) \cdots (B_\alpha B_\gamma)^n G(x) = \cdots$$

For some functional operators with fractional linear shifts $B_\gamma \varphi(x) = \varphi[\gamma(x)]$, $\gamma(x) = \dfrac{b_1 x + b_2}{b_3 x + b_4}$ and B_α, it is possible to calculate

$\lim_{n \to \infty} \Omega_n [B_\alpha^n \varphi(x)]$, where

$\Omega_n = B_\alpha B_\gamma B_\alpha^{-1} B_\alpha^2 B_\gamma B_\alpha^{-2} \cdots B_\alpha^n B_\gamma B_\alpha^{-n}$

And $\lim_{n \to \infty} G(x) \cdot B_\alpha B_\gamma G(x) \cdots (B_\alpha B_\gamma)^n G(x)$.

As it was shown, for B_α, $\alpha(x) = \dfrac{(a_3 + a_4)x}{a_3 x + a_4}$, $(a_3 + a_4)a_4 > 0$ the fractional linear commutates will be $B_\gamma \varphi(x) = \varphi[\gamma(x)]$, $\gamma(x) = \dfrac{(b_3 + b_4)x}{b_3 x + b_4}$. The representation (5) of the equation is simplified and will take the form

$$\varphi(x) = [B_\alpha^2 B_\gamma^2 \varphi(x)] \cdot G(x) \cdot B_\alpha B_\gamma G(x) \cdot B_\alpha^2 B_\gamma^2 G(x) = \cdots$$

If the operator B_γ is still requested to be $(b_3 + b_4) b_4 > 0$, then we obtain that not only is $\lim_{n \to \infty} B_\alpha^n \varphi(x) = \varphi(1)$ but also $\lim_{n \to \infty} B_\gamma^n \varphi(x) = \varphi(1)$. From here, $\lim_{n \to \infty} B_\gamma^n B_\alpha^n \varphi(x) = \varphi(1) = \varphi_0$.

We obtain

$$\varphi(x) = [\varphi(1)] \cdot G(x) \cdot B_\alpha B_\gamma G(x) \cdot B_\alpha^2 B_\gamma^2 G(x) \cdot B_\alpha^3 B_\gamma^3 G(x) \cdots$$

Theorem 1

Solution of equation (4) in the class of functions $H_\mu(J)$ with the initial condition $\varphi(1) = \varphi_0 > 0$, when $\alpha(x) = \dfrac{(a_3 + a_4)x}{a_3 x + a_4}$, $\gamma(x) = \dfrac{(b_3 + b_4)x}{b_3 x + b_4}$, $a_3 > 0, a_4 > 0,\ b_3 > 0, b_4 > 0$, is represented by

$$\varphi(x) = \varphi_0 \cdot G(x) \cdot B_\alpha B_\gamma G(x) \cdot B_\alpha^2 B_\gamma^2 G(x) \cdot B_\alpha^3 B_\gamma^3 G(x) \cdots$$

The convergence of the infinite product ensures the existence of a unique solution.

3 Equation with an External Operator with Shift and an Internal Operator with Shift

Let us consider the non-linear equation

$$\varphi(x) - G(x) \cdot (\Gamma B_\alpha) \varphi(x) = 0, \quad (6)$$

where the coefficient $G(x) > 0$ and belongs to $H_\mu(J)$. The unknown function $\varphi(x)$ is also searched for in $H_\mu(J)$. The operator $B_\alpha \varphi(x)$ is the shift operator, described in section 2. Equation (6) has two shifts, Γ is an external nonlinear operator, described by the formula $\Gamma \varphi(x) = \psi[\varphi(x)]$, where $\psi(x)$ is a Hölder class function.

Let us note that $\Gamma \neq B_\gamma$. The operator $\Gamma \varphi(x)$ transforms the whole function φ as a single entity, while the shift operator $B_\alpha \varphi(x)$ transforms only the argument x. Commutativity $\Gamma B_\alpha = B_\alpha \Gamma$ has a place. The operator Γ does not have the multiplicative property $\Gamma(a \cdot b) \neq \Gamma(a) \cdot \Gamma(b)$ and $\Gamma(\lambda \cdot b) \neq \Gamma(\lambda) \cdot \Gamma(b)$

when λ is constant. Let us write out the concordance condition $\varphi(1) = G(1)(\Gamma\varphi)(1)$, which follows directly from equation (6). Note that in the example from [5], the value of $\varphi(1)$ is calculated through the value of $G(1)$.

We are going to write the recurring relation and use it to express the solution $\varphi(x) =$

$$G(x)\cdot(\Gamma B_\alpha\varphi)(x) = G(x)\cdot(\Gamma B_\alpha[G(x)\cdot(\Gamma B_\alpha\varphi(x))]) =$$

$$G(x)\cdot\Gamma(B_\alpha G(x)\cdot\Gamma B_\alpha^2[G(x)\cdot\Gamma B_\alpha\varphi(x)]) =$$

$$G(x)\cdot\Gamma(B_\alpha G(x)\cdot\Gamma(B_\alpha^2 G(x)\cdot B_\alpha^2\Gamma B_\alpha\varphi(x))) =$$

$$G(x)\cdot\Gamma(B_\alpha G(x)\cdot\Gamma(B_\alpha^2 G(x)\cdot\Gamma B_\alpha^3\varphi(x))) =$$

$$G(x)\cdot\Gamma(B_\alpha G(x)\cdot\Gamma(B_\alpha^2 G(x)\cdot\Gamma B_\alpha^3[\cdots])).$$

As an example that provides the realization of the operator Γ, we are going to take $\Gamma\varphi(x) = \dfrac{b_1\varphi(x)+b_2}{b_3\varphi(x)+b_4}$. We will assume that the constants that define the operator Γ are positive. The equation (6) obtains the form

$$\varphi(x) - G(x)\cdot B_\alpha\frac{b_1\varphi(x)+b_2}{b_3\varphi(x)+b_4} = 0.$$

Let's emphasize again. Here, the internal operator $B_\alpha\varphi(x) = \varphi[\alpha(x)]$ acts on the argument; it is defined by the composite function $\varphi\circ\alpha$. The external operator is defined by the composite function $(\Gamma\circ\varphi)(x) = \dfrac{b_1\varphi(x)+b_2}{b_3\varphi(x)+b_4}$.

The solution of the equation has the form of an infinite continued fraction, [9]:

$$\varphi(x) = G(x)\cdot\frac{b_1 B_\alpha\varphi(x)+b_2}{b_3 B_\alpha\varphi(x)+b_4} =$$

$$G(x)\cdot\frac{b_1\left(B_\alpha G(x)\cdot\dfrac{b_1 B_\alpha^2\varphi(x)+b_2}{b_3 B_\alpha^2\varphi(x)+b_4}\right)+b_2}{b_3\left(B_\alpha G(x)\cdot\dfrac{b_1 B_\alpha^2\varphi(x)+b_2}{b_3 B_\alpha^2\varphi(x)+b_4}\right)+b_4} = \cdots$$

Theorem 2

Solution of equation (6) in the class of functions $H_\mu(J)$ is represented by

$$G(x)\cdot\frac{b_1\left(B_\alpha G(x)\cdot\dfrac{b_1 B_\alpha^2[\cdots]+b_2}{b_3 B_\alpha^2[\cdots]+b_4}\right)+b_2}{b_3\left(B_\alpha G(x)\cdot\dfrac{b_1 B_\alpha^2[\cdots]+b_2}{b_3 B_\alpha^2[\cdots]+b_4}\right)+b_4}.$$

The convergence of the infinite product ensures the existence of a unique solution.

In the work, [10], the homogeneous non-linear equation

$$\nu(x) - G(x)\cdot\frac{1}{B_\alpha\nu(x)} = 0 \qquad (7)$$

was studied in the Hölder space $H_\mu(J)$. In our notation, $\Gamma\nu(x) = \dfrac{1}{\nu(x)}$. To simplify, it is assumed that all the considered functions are positive. Briefly, we will write the results obtained in solving this equation, as this material will serve as a demonstrative example of the method proposed for the solution of the nonlinear equation (7). We write the recurrent relation $\nu(x) = \dfrac{G(x)}{(B_\alpha\nu)(x)}$. From here, it follows that:

$$\nu(x) = \frac{G(x)}{B_\alpha\left(\dfrac{G(x)}{B_\alpha[\nu(x)]}\right)} = \frac{G(x)}{B_\alpha G(x)}\cdot B_\alpha^2([\nu(x)]) =$$

$$\frac{G(x)}{(B_\alpha G)(x)}\cdot B_\alpha^2\left(\frac{G(x)}{B_\alpha[\nu(x)]}\right) = \frac{G(x)}{(B_\alpha G)(x)}\cdot\frac{B_\alpha^2 G(x)}{B_\alpha^3[\nu(x)]} =$$

$$\frac{G(x)}{B_\alpha G(x)}\cdot\frac{B_\alpha^2 G(x)}{B_\alpha^3\dfrac{G(x)}{B_\alpha[\nu(x)]}} = B_\alpha^4[\nu(x)]\cdot\frac{G(x)}{B_\alpha G(x)}\cdot\frac{B_\alpha^2 G(x)}{B_\alpha^3 G(x)}\cdots$$

The condition for the solvability of equation (7) is the condition for the convergence of the infinite product according to the Hölder norm to the function from $H_\mu(J)$. The solution of the non-linear non-homogeneous equation (7) is going to be:

$$\nu(x) = \sqrt{G(1)}\,\frac{G(x)\cdot(B_\alpha^2 G(x))\cdot(B_\alpha^4 G(x))}{(B_\alpha G(x))\cdot(B_\alpha^3 G(x))\cdot(B_\alpha^5 G(x))}\cdots$$

The fact that $\varphi(1) = \sqrt{G(1)}$ follows from the equality obtained from the equation when $x = 1$: $\nu(1) = \dfrac{G(1)}{\nu(1)}$. Of the two values $\nu(1) = \pm\sqrt{G(1)}$, the value $\nu(1) = +\sqrt{G(1)}$ chosen corresponds to the positivity of the solutions $\nu(x)$.

4 Conclusions

In this work, certain homogeneous non-linear equations that contain operators with the linear-fractional shift are considered. The main method is the representation of solutions through recurrent relations. The proposed approach can serve as a means of constructing inverse operators to operators with linear-fractional shifts. In certain cases, this method is easier. So, in the works, [6], [7], the construction of inverse operators for linear functional operators looks cumbersome. The authors plan to further investigate non-homogeneous non-linear equations and apply the proposed method to other types of nonlinear equations, particularly to equations that contain conjunction operator.

In addition, it is planned to develop a mathematical model of renewable systems with elements in various states and subsequently use the obtained results in the analysis of balance relations obtained in modelling.

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