A New Computation Approach: ARA Decomposition Method

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Abstract: - In this study, we present a novel combination between the ARA transformation and the decomposition method, termed the ARA decomposition approach. We present the method in a simple algorithm and use it to solve nonlinear integro-differential equations. To test the efficiency of the new approach, we solve some examples and calculate the absolute errors and sketch the approximate and exact solutions.

Key-Words: - Decomposition method; ARA transform; ARA decomposition method; Volterra integro–differential equations

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1 Introduction

Integral equations are presented in numerous areas of engineering, physics, and mathematics, used in initial and boundary value problems, and transferred to Fredholm and Volterra integro–differential equations (VIE), e.g. Dirichlet problems in astrophysics, conformal mapping, mathematics, physical models, diffusion problems, water, [1].

Nonlinear integral equations are used in many fields of study, e.g. queuing theory, chemical kinetics, fluid dynamics, etc., [2], [3], [4], are also used in numerical solution by various methods such as Galerkin, decomposition, quadrature, cubic spline polynomials, etc., [5], [6], [7]. One of the useful and important methods that have received a lot of attention is the Adomian decomposition method (ADM). In this method, more emphasis is placed on finding reliable and efficient solution methods in various fields of science and technology, [8], [9], [10], [11], [12].

The ARA transformation is introduced in 2020, [13]. It is defined by the improper integral.

$$\mathcal{G}_n[\psi(\tau)] = \Psi(n,u) = u\int_0^\infty \tau^{n-1}e^{-u\tau}\psi(\tau)d\tau, \quad u>0.$$

This transformation has attracted a lot of attention from researchers due to its ability to produce multiple transformations of index n, and it could also easily overcome the challenges of having singular points in differential equations. Despite all these merits, it could be used to solve different types of problems. In this work, we use the first-order ARA transform $\mathcal{G}_1[\psi(\tau)]$, which we denote by $\mathcal{G}[\psi(\tau)]$ for the sake of simplicity.

This work aims to develop a combined form of the ARA transformation method with the ADM, called the ARA-decomposition method (ARA-DM), to obtain exact solutions or high-precision approximations for the nonlinear VIE. The advantage of this method is its ability to combine the two powerful methods for obtaining exact solutions to nonlinear integral equations.

In this study, we investigate the solutions of the nonlinear VIE of the second kind is

$$\psi^{(m)}(\tau) = \varphi(\tau) + \int_0^\tau k(\tau-v)Q(\psi(v))dv,$$

where the kernel $k(\tau-v)$ and $\varphi(\tau)$ are real-valued functions, and $Q(\psi(v))$is a nonlinear function of $\psi(v)$, such as $\psi^3(v)$, $\sin\psi(v)$, $\cos\psi(v)$.

The rest of the paper is constructed as follows. Section 2 defines the basic definitions of the ARA transform and ADM. Section 3 introduces the concept of applying the ARA transform in combination with the ADM to solve the second type of nonlinear VIE. By solving significant examples in Section 4, the effectiveness and efficiency of the proposed method are illustrated. Finally, in Section 5, the conclusion of the work is presented.

2 Preliminaries

In this section, we present the basic definitions and properties of the ARA transform. In addition, the basic idea of the ADM method is presented.

2.1. ARA Integral Transform, [13]

Definition 1. Let $\psi(\tau)$ be a piecewise continuous function defined on $(0,\infty)$. Then ARA transforms for $\psi(\tau)$ denoted and defined by

$$\mathcal{G}[\psi(\tau)] = \Psi(u) = u\int_0^\infty e^{-u\tau}\psi(\tau)d\tau, \qquad u > 0.$$

The inverse ARA transform for $\Psi(u)$ denoted and defined by

$$\mathcal{G}^{-1}[\Psi(u)] = \psi(\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{u\tau}}{u}\Psi(u)du.$$

Theorem 1.(Existence Condition) If $\psi(\tau)$ is a piecewise continuous function on $[0,\infty)$ and satisfies the condition

$$|\psi(\tau)| \le Me^{\beta\tau}, \text{ for some } m > 0.$$

Then, ARA transform $\mathcal{G}[\psi(\tau)] = \Psi(u)$ exists for $\text{Re}(u) > \beta$.

Proof. Using the definition of ARA transform, we obtain

$$|\Psi(u)| = \left|u\int_0^\infty e^{-u\tau}\psi(\tau)d\tau\right| \le u\int_0^\infty e^{-u\tau}|\psi(\tau)|d\tau$$
$$\le u\int_0^\infty e^{-u\tau}Me^{\alpha\tau}d\tau$$
$$= uM\int_0^\infty e^{-\tau(u-\alpha)}d\tau = \frac{uM}{u-\beta},$$
$$\text{Re}(u) > \beta > 0.$$

Hence, ARA integral transform exists for $\text{Re}(u) > \beta > 0$. □

Now, we mention some properties of ARA transform to the basic functions. Suppose that $\Psi_1(u) = \mathcal{G}[\psi_1(\tau)]$ and $\Psi_2(u) = \mathcal{G}[\psi_2(\tau)]$ and $\alpha, \beta \in \mathbb{R}$, then

$$\mathcal{G}[\alpha\psi_1(\tau) + \beta\psi_2(\tau)] = \alpha\Psi_1(u) + \beta\Psi_2(u).$$

$$\mathcal{G}^{-1}[\alpha\Psi_1(u) + \beta\Psi_2(u)] = \alpha\psi_1(\tau) + \beta\psi_2(\tau).$$

Now the following table (Table 1) introduces some values of ARA transform to some elementary functions.

Table 1. ARA transform for some functions.

$\psi(\tau)$	$\mathcal{G}[\psi(\tau)] = \Psi(u)$
1	1
τ^a	$\frac{\Gamma(a+1)}{u^a}$
$e^{a\tau}$	$\frac{u}{u-a}$
$\sin a\tau$	$\frac{au}{u^2+a^2}$
$\cos a\tau$	$\frac{u^2}{u^2+a^2}$
$\sinh a\tau$	$\frac{au}{u^2-a^2}$
$\cosh a\tau$	$\frac{u^2}{u^2-a^2}$
$\psi'(\tau)$	$u\Psi(u) - u\psi(0)$
$\psi^{(n)}(\tau)$	$u^n\Psi(u) - \sum_{j=1}^{n} u^{n-j+1}\psi^{(j-1)}(0)$
$(\psi * \phi)(\tau)$	$\frac{\mathcal{G}[\psi(\tau)]\mathcal{G}[\phi(\tau)]}{u}$

2.2 Adomian Decomposition Method, [2]

In this section, we introduce the main idea of ADM, which is a powerful technique used to handle a large class of nonlinear ordinary differential equations and partial differential equations.

The ADM is a very powerful approach used to solve broad classes of nonlinear partial and ordinary differential equations. It has wide applications in engineering, physics, and applied mathematics.

The ADM depends on decomposing the unknown equation into the sum of some components to be determined. The sum of these components represents the solution with high accuracy. ADM's algorithm is illustrated in the following steps:

- Assume that the target problem has the following series solution represented as

$$\psi(\tau) = \sum_{n=0}^{\infty}\psi_n(\tau) = \psi_0(\tau) + \psi_1(\tau) + \cdots.$$

- Establish a recursive relation of the nonlinear term in the target problem and substitute the value of the series solution depending on the relation

$$A_i(\tau) = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left(l\left(\sum_{j=0}^{i} \lambda^j \psi_j(\tau) \right)\Bigg|_{\lambda=0} \right), \quad i = 0,1,2,\cdots.$$

3 Solving Nonlinear VIE by ARA-DM

In this section, we apply the ARA transform in combination with the ADM to solve the nonlinear VIE of the second type. Also, we assume that the given kernel is of a different kind, which could be expressed in the form $k(x-\tau)$, such as $\sinh(x-\tau)$, $(x-\tau)^2$, $\cosh(x-\tau)$.

Now let us consider the nonlinear VIE equation of the form

$$\psi^{(m)}(\tau) = \varphi(\tau) + \int_0^\tau k(\tau - v) Q(\psi(v)) dv, \quad (1)$$

subject to the initial conditions (ICs)

$$\psi^{(i)}(0) = \xi_i, \qquad i = 0,1,\dots,m-1, m \in \mathbb{N}, \quad (2)$$

where $Q(\psi(v))$ is a nonlinear function on $\psi(v)$.

To obtain the solution of Equation (1) by ARA-DM, we firstly apply ARA transform to both sides of Equation (1)

$$\mathcal{G}[\psi^{(m)}(\tau)] = \mathcal{G}[\varphi(\tau)] + \mathcal{G}\left[\int_0^\tau k(\tau - v) Q(\psi(v)) dv\right].$$

Applying the differential and the convolution properties of the ARA transform, we can rewrite Equation (1) as

$$u^m \mathcal{G}[\psi(\tau)] - u^m \xi_0 - u^{m-1}\xi_1 - \cdots - u\xi_{m-1} = \mathcal{G}[\varphi(\tau)] + \frac{1}{u}\mathcal{G}[k(\tau)]\mathcal{G}[Q(\psi(\tau))]. \quad (3)$$

Thus, substituting the ICs (2) and simplifying Equation (3), we obtain

$$\mathcal{G}[\psi(\tau)] = \xi_0 + \frac{\xi_1}{u} + \cdots + \frac{\xi_{m-1}}{u^{m-1}} + \frac{1}{u^m}\mathcal{G}[\varphi(\tau)] + \frac{1}{u^{m+1}}\mathcal{G}[k(\tau)]\mathcal{G}[Q(\psi(\tau))]. \quad (4)$$

Now, utilizing the ADM to handle the nonlinear term $Q(\psi(\tau))$, we need to express $\psi(\tau)$ as an infinite series with components as

$$\psi(\tau) = \sum_{i=0}^{\infty} \psi_i(\tau) = \psi_0(\tau) + \psi_1(\tau) + \psi_2(\tau) + \cdots. \quad (5)$$

The components $\psi_i(\tau)$, $\tau = 0,1,\dots$, can be obtained from a recurrence relation, and the nonlinear term $Q(\psi(\tau))$ can be presented as

$$Q(\psi(\tau)) = \sum_{i=0}^{\infty} A_i(\tau), \quad (6)$$

where $A_i(\tau), i = 0,1,2,\dots$ are defined as

$$A_i(\tau) = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left(l\left(\sum_{j=0}^{i} \lambda^j \psi_j(\tau) \right)\Bigg|_{\lambda=0} \right), i = 0,1,2,\cdots. \quad (7)$$

where A_i's are called the Adomian polynomials for the nonlinear function $H(\psi(\tau))$, the Adomian polynomial can be determined by

$$\begin{aligned}
A_0 &= Q(\psi_0),\\
A_1 &= \psi_1 Q'(\psi_0),\\
A_2 &= \psi_2 Q'(\psi_0) + \frac{1}{2!}\psi_1^2 Q''(\psi_0),\\
A_3 &= \psi_3 Q'(\psi_0) + \psi_1\psi_2 Q''(\psi_0) + \frac{1}{3!}\psi_1^3 Q'''(\psi_0),\\
A_4 &= \psi_4 Q'(\psi_0) + \left(\frac{1}{2!}\psi_2^2 + \psi_1\psi_3\right) Q''(\psi_0) + \frac{1}{2!}\psi_1^2\psi_2 Q'''(\psi_0) + \frac{1}{4!}\psi_1^4 Q^{(4)}(\psi_0).
\end{aligned} \quad (8)$$

Thus, by substituting Equations (5) and (6) in Equation (4), we get

$$\mathcal{G}\left[\sum_{i=0}^{\infty}\psi_i(\tau)\right] = \xi_0 + \frac{\xi_1}{u} + \cdots + \frac{\xi_{m-1}}{u^{m-1}} + \frac{1}{u^m}\mathcal{G}[\varphi(\tau)] + \frac{1}{u^{m+1}}\mathcal{G}[k(\tau)]\mathcal{G}\left[\sum_{i=0}^{\infty} A_i(\tau)\right]. \quad (9)$$

The recursive relation from ADM implies

$$\mathcal{G}[\psi_0(\tau)] = \xi_0 + \frac{1}{u}\xi_1 + \cdots + \frac{1}{u^{m-1}}\xi_m(0) + \frac{1}{u^m}\mathcal{G}[\varphi(\tau)]. \quad (10)$$

From Equation (9), one can get

$$\mathcal{G}[\psi_{n+1}(\tau)] = \frac{1}{u^{m+1}}\mathcal{G}[k(\tau)]\mathcal{G}[A_n(\tau)]. \quad (11)$$

Remark 1. A necessary condition for Equation (11) to be well-defined is that

$$\lim_{u\to\infty}\frac{1}{u^{m+1}}\mathcal{G}[k(\tau)]=0.$$

Operating the inverse ARA transform to the equations in (11) recursively, we can obtain the values of the components $\psi_0(\tau),\psi_1(\tau),\cdots$.

The solution of the VID Equation (1) is

$$\psi(\tau)=\psi_0(\tau)+\psi_1(\tau)+\cdots.$$

The proposed method is efficient in finding approximate solutions of nonlinear VIEs. To measure the accuracy of the method, we solve some problems and use the maximum absolute error, given as

$$AbsErr=\max\left|\psi_{exact}-\psi_{app}\right|,$$

which is given in some intervals.

4 Numerical Applications

In this section, we apply ARA-DM to solve some applications of VIEs, and we use absolute error to determine the efficiency of our results.

Problem 1.

Consider the following nonlinear VIE of the form

$$\psi(\tau)=2\tau-\frac{\tau^4}{12}+\frac{1}{4}\int_0^\tau(\tau-u)\psi^2(u)du. \quad (12)$$

Solution. The exact solution of Equation (12) is $\psi(\tau)=2\tau$.

To get the solution by the proposed method, we again apply ARA transform to Equation (12), to get

$$\begin{aligned}\Psi(s)&=\mathcal{G}\left[2\tau-\frac{\tau^4}{12}\right]+\frac{1}{4u}\mathcal{G}[\tau]\mathcal{G}[\psi^2(\tau)]\\&=\frac{2}{u}-\frac{5!}{12\,u^4}\\&+\frac{1}{4u^2}\mathcal{G}[\psi^2(\tau)].\end{aligned} \quad (13)$$

For the nonlinear term $\psi^2(u)$, it can be decomposed using the formula in Equation (7), one can obtain the following components

$$\begin{aligned}A_0&=\psi_0^2,\\A_1&=2\psi_0\psi_1,\\A_2&=\psi_1^2+2\psi_0\psi_2,\\A_3&=2\psi_1\psi_2+2\psi_0\psi_3,\\A_4&=\psi_2^2+2\psi_1\psi_3+2\psi_0\psi_4.\end{aligned} \quad (14)$$

Making comparisons in the iterative form of Equation (7) and applying the inverse ARA transform, to obtain

$$\psi_0(\tau)=2\tau-\frac{\tau^4}{12},$$

$$\psi_1(\tau)=\frac{\tau^4}{12}-\frac{\tau^7}{126}+\frac{\tau^{10}}{51840},$$

$$\psi_2(\tau)=\frac{\tau^7}{504}-\frac{\tau^{10}}{181440}+\frac{127\tau^{13}}{56609280}-\frac{\tau^{16}}{298598400},$$

$$\psi_3(\tau)=\frac{\tau^4}{12}-\frac{\tau^7}{504}+\frac{\tau^{10}}{2792}-\frac{19\tau^{13}}{14152320}+\frac{71\tau^{16}}{2264371200}-\frac{7893\tau^{19}}{575787643000000}.$$

Thus, the approximate solution can be expressed as

$$\begin{aligned}\psi(\tau)&=\psi_0(\tau)+\psi_1(\tau)+\psi_2(\tau)+\psi_3(\tau)+\cdots\\&=2\tau+\frac{\tau^4}{12}-\frac{\tau^7}{126}-\frac{\tau^{10}}{362880}\\&+\frac{51\tau^{13}}{56609280}+\cdots.\end{aligned}$$

Table 2 below presents the values of the exact and ARA-DM solutions of Problem 1, and to test the efficiency we compute the absolute error as follows.

Table 2. The exact and ARA-DM solution of Problem 1, and the absolute error.

	Exact Solution	ARA-DM Solution	Absolute Error
0.0	0.0	0.0000000000	0.0000000000
0.1	0.2	0.2000083325	0.0000083325
0.2	0.4	0.4001332317	0.0001332317
0.3	0.6	0.6006732643	0.0006732643
0.4	0.8	0.8021203322	0.0021203322
0.5	1.0	1.0051463480	0.0051463480
0.6	1.2	1.2105779450	0.0105779450
0.7	1.4	1.4193552730	0.0193552730
0.8	1.6	1.6328034650	0.0328034650
0.9	1.8	1.8508857190	0.0508857190
1.0	2.0	1.8833526250	0.1166473750

In the following figure below, we sketch the exact and approximate solutions in Figure 1 below. Also, we sketch the absolute error of Problem 1 in Figure 2.

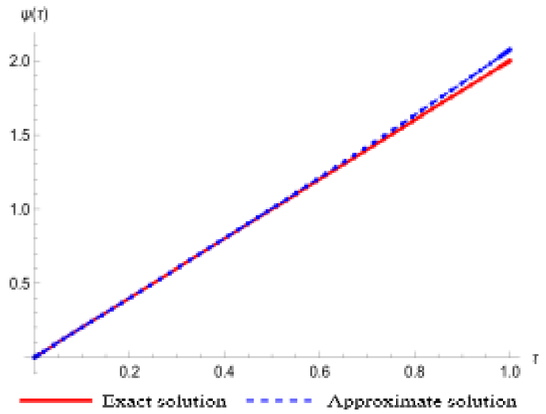

Fig. 1: The exact and approximate solutions of the nonlinear VIE in Problem 1.

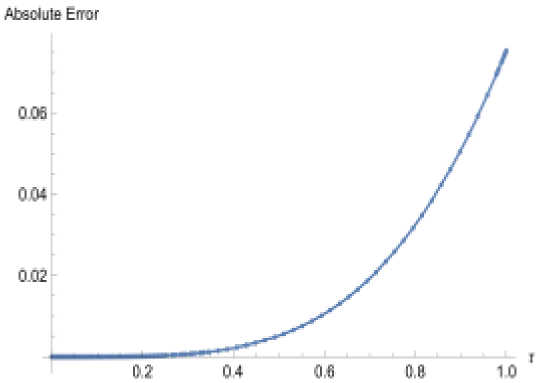

Fig. 2: The absolute error of the exact and approximate solutions of Problem 1.

Problem 2. Consider the following nonlinear VIE of the form

$$\psi(\tau) = \tau + \int_0^\tau \psi^2(u)du. \quad (15)$$

Solution. The exact solution of Equation (15) is $\psi(\tau) = \tan\tau$.

Applying ARA transform to Equation (15), we get

$$\mathcal{G}[\psi(\tau)] = \frac{1}{u} + \frac{1}{u}\mathcal{G}[\psi^2(\tau)]. \quad (16)$$

Thus by similar arguments to Problem 1 one can obtain

$$\psi_0(\tau) = \tau,$$
$$\psi_1(\tau) = \frac{\tau^3}{3},$$
$$\psi_2(\tau) = \frac{2\tau^5}{15},$$
$$\psi_3(\tau) = \frac{17\tau^7}{315}.$$

Thus, the approximate solution can be expressed as

$$\psi(\tau) = \tau + \frac{\tau^3}{3} + \frac{2\tau^5}{15} + \frac{17\tau^7}{315} + \cdots.$$

Table 3 below presents the values of the exact and ARA-DM solutions of Problem 2 and tests the efficiency we compute the absolute error.

Table 3 The exact and ARA-DM solutions of Problem 2, and the absolute error.

	Exact Solution	**ARA-DM Solution**	**Absolute Error**
0.0	0.00000000000	0.0000000000	0.0000000000
0.1	0.1002940335	0.1003346721	0.0000406386
0.2	0.2026262629	0.2027100241	0.0000837612
0.3	0.3092040035	0.3093358029	0.0001317994
0.4	0.4226035289	0.4227870883	0.0001835594
0.5	0.5460413117	0.5462549603	0.0002136486
0.6	0.6837824776	0.6838787656	0.0000962880
0.7	0.8418070516	0.8411871844	0.0006198672
0.8	1.0289756740	1.0256752970	0.0033003770
0.9	1.2592215210	1.2475448490	0.0116766720
1.0	1.5560303730	1.5206349210	0.03539545198

In the following figures below, we sketch the exact and approximate solutions in Figure 3 below, and we sketch the absolute error in Figure 4.

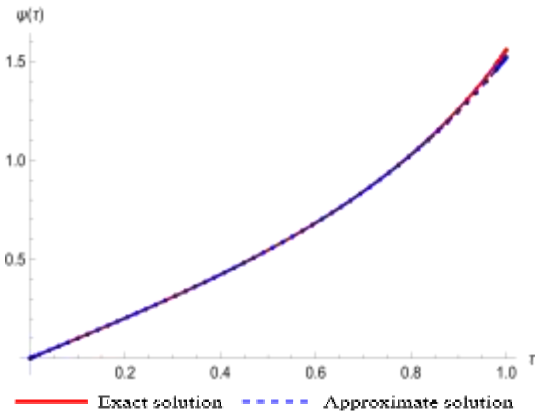

Fig. 3: The exact and approximate solutions of the nonlinear VIE in Problem 2.

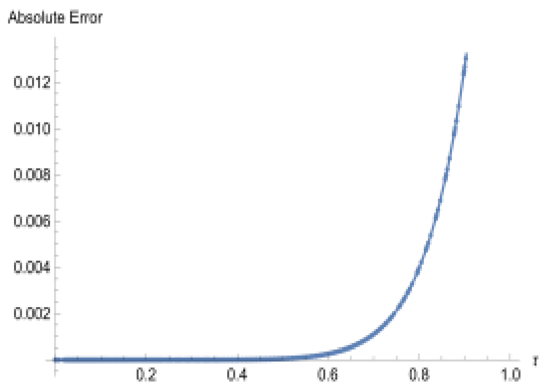

Fig. 4: The absolute error of the exact and approximate solutions of Problem 2.

Problem 3. Consider the following nonlinear VIE of the form

$$\psi'(\tau) = \frac{3}{2}e^{\tau} - \frac{1}{2}e^{3\tau} + \int_0^{\tau} e^{\upsilon-\tau}\psi^3(\tau)d\tau, \qquad (17)$$

$$\psi(0) = 1. \qquad (18)$$

Solution. Applying ARA transform to Equation (17), we get

$$\Psi(u) = 1 + \frac{3}{2(u-1)} - \frac{1}{2(u-3)} + \frac{1}{u^2}\mathcal{G}[e^{\tau}]\mathcal{G}[\psi^3(\tau)] = 1 + \frac{3}{2(u-1)} - \frac{1}{2(u-3)} + \frac{1}{u(u-1)}\mathcal{G}[\psi^3(\tau)].$$

Now, we have

$$\mathcal{G}[\psi_0(\tau)] = 1 + \frac{3}{2(u-1)} - \frac{1}{2(u-3)}, \quad \mathcal{G}[\psi_{n+1}(\tau)] = \frac{1}{u(u-1)}\mathcal{G}[A_n(\tau)], n \geq 0. \qquad (19)$$

The Adomian polynomials $A_n(\tau)$ of $\psi^3(\tau)$, can be determined as

$$A_0 = \psi_0^3,$$
$$A_1 = 3\psi_0^2\psi_1,$$
$$A_2 = 3\psi_0^2\psi_2 + 3\psi_0\psi_1^3,$$
$$A_3 = 3\psi_0^2\psi_3 + 6\psi_0\psi_1\psi_2 + \psi_1^3.$$

Taking the inverse ARA to transform to the functions (19) and using the given recursive relation, one can obtain

$$\psi_0(\tau) = 1 + \tau - \frac{1}{2}\tau^3 - \frac{\tau^4}{2} - \frac{13}{40}\tau^5 + \cdots,$$
$$\psi_1(\tau) = \frac{1}{2}\tau^2 + \frac{2}{3}\tau^3 + \frac{5}{12}\tau^4 + \frac{7}{120}\tau^5 + \cdots,$$
$$\psi_2(\tau) = \frac{1}{8}\tau^4 + \frac{11}{40}\tau^5 + \cdots.$$

Hence, the approximate series solution of Problem 3 is

$$\psi(\tau) = 1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} + \frac{\tau^4}{4!} + \cdots,$$

which converges to the exact solution $\psi(\tau) = e^{\tau}$.

Table 4 below, presents the values of the exact and ARA-DM solutions of Problem 3, and to test the efficiency we compute the absolute error.

Table 4. The exact and ARA-DM solution of Problem 3, and the absolute error.

	Exact Solution	ARA-DM Solution	Absolute Error
0.0	1	1	0
0.1	1.1051709181	1.1051709181	$2.2204460493 \times 10^{-16}$
0.2	1.2214027582	1.2214027582	0
0.3	1.3498588076	1.3498588076	$2.2204460493 \times 10^{-16}$
0.4	1.4918246976	1.4918246976	$2.2204460492 \times 10^{-16}$
0.5	1.6487212707	1.6487212707	$8.8817841970 \times 10^{-16}$
0.6	1.8221188004	1.8221188004	$9.5479180118 \times 10^{-15}$
0.7	2.0137527075	2.0137527075	$8.1268325403 \times 10^{-14}$
0.8	2.2255409285	2.2255409285	$5.3290705182 \times 10^{-13}$
0.9	2.4596031112	2.4596031112	$2.7911006839 \times 10^{-12}$
1.0	2.7182818285	2.7182818284	$1.228617207 \times 10^{-11}$

In the following figure below, we sketch the exact and approximate solutions in Figure 5 below. We also sketch the absolute error of the exact and approximate solutions of Problem 3 in Figure 6.

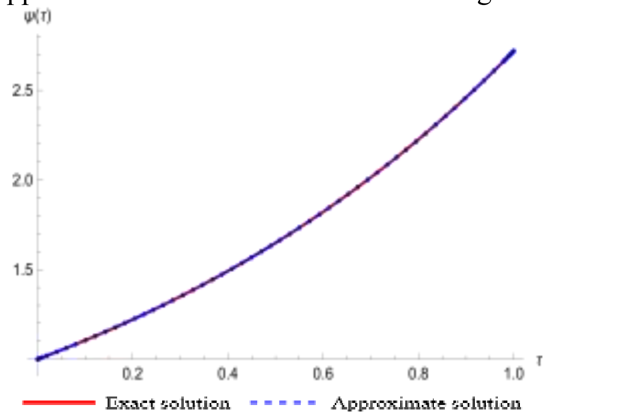

Fig. 5: The exact and approximate solutions of the nonlinear VIE in Problem 3.

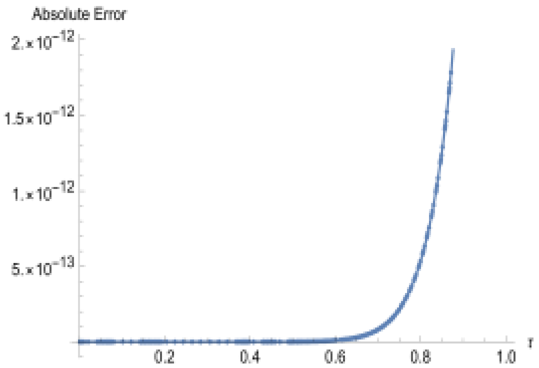

Fig. 6: The absolute error of the exact and approximate solutions of Problem 3.

Problem 4. *Consider the following nonlinear VIE of the form*

$$\psi'(\tau) = -2\sin\tau - \frac{2\tau}{3}\cos\tau + \int_0^\tau \cos(u-\tau)\,\psi^2(\tau)d\tau, \quad (20)$$

$$\psi(0) = 1. \quad (21)$$

Solution. Applying the same procedure from the previous problems, we can obtain

$$\psi_0(\tau) = 1 - \tau - \tau^2 + \frac{1}{2}\tau^3 + \frac{1}{12}\tau^4 - \frac{11}{120}\tau^5 + \cdots,$$

$$\psi_1(\tau) = \frac{1}{2}\tau^2 - \frac{1}{3}\tau^3 - \frac{1}{8}\tau^4 + \frac{1}{6}\tau^5 + \cdots,$$

$$\psi_2(\tau) = \frac{1}{12}\tau^4 - \frac{1}{12}\tau^5 + \cdots.$$

Thus, the approximate solution of (20) and (21) can be expressed as

$$\psi(\tau) = \left(1 - \frac{\tau^2}{2!} + \frac{\tau^4}{4!} + \cdots\right) - \left(\tau - \frac{\tau^3}{3!} - \frac{\tau^5}{5!} + \cdots\right),$$

which converge to the exact solution

$$\psi(\tau) = \cos\tau - \sin\tau.$$

Table 5 below presents the values of the exact and ARA-DM solutions of Problem 4, and to test the efficiency we compute the absolute error.

Table 5. The exact and ARA-DM solutions of Problem 4, and the absolute error.

	Exact Solution	ARA-DM Solution	Absolute Error
0.0	1	1	0
0.1	0.8951707486	0.8951709167	0.0000001680
0.2	0.7813972470	0.7814026667	0.0000054196
0.3	0.6598162825	0.65985775	0.0000414675
0.4	0.5316426517	0.5318186667	0.00017601497
0.5	0.3981570233	0.3986979167	0.0005408933
0.6	0.2606931415	0.262048	0.0013548589
0.7	0.12062450	0.1235714167	0.0029469166
0.8	−0.0206493816	−0.0148693333	0.0057800482
0.9	−0.1617169414	−0.151241750	0.0104751914
1.0	−0.3011686789	−0.2833333333	0.0178353456

In the following figure below, we sketch the exact and approximate solutions in Figure 7 below. Lastly, the absolute error of the exact and approximate solutions of Problem 4 is presented in Figure 8.

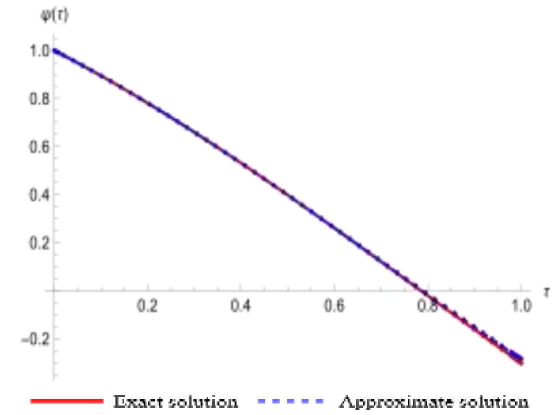

Fig. 7: The exact and approximate solutions of the nonlinear VIE in Problem 4.

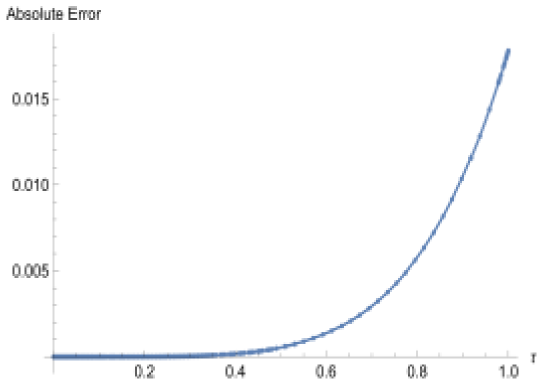

Fig. 8: The absolute error of the exact and approximate solutions of Problem 4.

5 Discussion and Conclusion

The main goal of this research is to develop an effective approach to solving nonlinear VIE. We obtain an approximate series solution of a specific family of nonlinear VIE problems using a new approach, that combines a combination of the ARA transform and the decomposition method, called ARA decomposition approach. The given problems are first simplified using the ARA transform, and then the results are treated by applying the Adomian decomposition method. The solutions to VIE problems are examined and found to best represent the true dynamics of the problem.

To demonstrate the validity of the proposed method, the results are presented graphically and tabulated. The main advantage of the proposed method is the rapid convergence of the series form solutions to the precise ones. It turns out that the presented method for solving nonlinear integro-differential equations is both simple and effective, and thus can be applied to other scientific problems.

The method provides a useful way to develop an analytical treatment for these equations. In future work, we will use the proposed scheme to solve other nonlinear equations and fractional differential equations.

The ARA-DM is used in this research to solve nonlinear integro-differential equations. We solved some numerical examples and sketched the solutions. From the problems discussed, one can see the efficiency of the proposed method. From the previous figure, we can see the agreement between the exact and approximate solution. We also made comparisons and calculated the absolute errors.

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