d-Tribonacci Polynomials and Their Matrix Representations

BAHAR KULOĞLU[1], ENGIN ÖZKAN[2]
[1]Department of Mathematics, Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Yalnızbağ Campus, 24100, Erzincan, TURKEY
[2]Department of Mathematics, Faculty of Arts and Sciences, Erzincan Binali Yıldırım University, Yalnızbağ Campus, 24100, Erzincan, TURKEY

Abstract: - In this study, we define *d*-Tribonacci polynomials. Some combinatorial properties of the *d*-Tribonacci polynomials with matrix representations are obtained with the help of Riordan arrays. In addition, *d*-Tribonacci number sequence, a new generalization of this number sequence, is obtained by considering the Pascal matrix. With the help of the Pascal matrix, two kinds of factors of *d*-Tribonacci polynomials are found. Also, infinite *d*-Tribonacci polynomials matrix and the inverses of these polynomials are found.

Key-Words: - *d*-Tribonacci polynomials, Generating function, Pascal matrix, Riordan matrix.

Received: July 18, 2022. Revised: January 16, 2023. Accepted: February 13, 2023. Published: March 23, 2023.

1 Introduction

The Tribonacci number sequence is inspired by the Fibonacci number sequence and is a number sequence with 3-term recurrence. It is used in many branches, as in the Fibonacci number sequence. Many generalizations of this number sequence such as Padovan, Narayana, Perrin have been put forward and studied [1-8, 10-12].

The term Tribonacci was first used by Feinberg in 1963 [14]. Later, many basic features were studied [15-19].

We know that the Tribonacci numbers T_n are defined by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \qquad n \geq 3$$

with $T_0 = 0, T_1 = 0$ and $T_2 = 1$ [9].

In this study, a new Tribonacci number sequence is obtained with the help of Riordan sequence and Pascal matrix by bringing a new perspective to the existing definitions of traditional number sequences. Additionally, based on Pascal's matrix, we factor two types of d-Tribonacci polynomials.

Also, infinite *d*-Tribonacci polynomial matrices and the inverses of these polynomials are found.

It is thought that if these values are placed in the Riordan array appropriately by working on the initial values, it will allow similar studies to be made on many number sequences where a Riordan array is given as an infinite lower triangular matrix $D = [d_{n,k}]_{n,k\geq 0}$ if its ith column generating function is $g(x)(f(x))^i$ for $i \geq 0$. Note that the first column is indexed by 0 and we accept $d_{0,0} = g_0 = 1$ [13].

Throughout this paper, let $p_i(x)$ and $q_i(x)$ be polynomials with real coefficient for $i = 1, \dots, d+1$.

Definition 1.1 d-Fibonacci polynomials are given as:

$$F_{n+1}(x) = p_1(x)F_n(x) + p_2(x)F_{n-1}(x) + \cdots + p_{d+1}(x)F_{n-d}(x) \tag{1}$$

with $F_n(x) = 0$ for $n \leq 0$ and $F_1(x) = 1$ [12].

Similarly, d-Lucas polynomials are defined by

$$L_{n+1}(x) = F_{n+1}(x) + p_2(x)F_{n-1}(x) + \cdots + p_{d+1}(x)F_{n-d}(x) \tag{2}$$

with $L_n(x) = 0$ for $n < 0$ and $L_0(x) = 2$ and $L_1(x) = p_1(x)$ [12].

The Riordan matrices is given as a set of matrices $M = (m_{ij}), i, j \geq 0$ where (m_{ij}) are complex numbers [13].

The Riordan group is defined as a set of infinite lower-triangular integer matrices where each matrix is defined by pair of formal power series $g(z) = \sum_{n=0}^{\infty} g_n z^n$ and $f(z) = \sum_{n=0}^{\infty} f_n z^n$ with $g_0 \neq 0$ and $f_1 \neq 1$ [13].

In this study, we describe new generalizations of Tribonacci polynomials. Some combinatorial properties of matrix representations of d-Tribonacci polynomials are obtained with the help of Riordan arrays. In addition, d-Tribonacci number sequence is obtained by considering the Pascal matrix. Based on the Pascal matrix, d-Tribonacci polynomials have two types of factors.. Also, infinite d-Tribonacci polynomial matrices and the inverses of these polynomials are given.

2 Generalization of Tribonacci Polynomials

Definition 2.1. $d-$ Tribonacci polynomials are given by

$$T_n(x) = q_1(x)T_{n-1}(x) + q_2(x)T_{n-2}(x) + q_3(x)T_{n-3}(x) + \cdots + q_{d+1}(x)T_{n-d-1}(x) \qquad (3)$$

with $T_0(x) = 0, T_1(x) = 1, T_2(x) = 1$ and $T_n(x) = 0$ for $n < 0$.

A few terms of these polynomials:

$T_0(x) = 0, T_1(x) = 1, T_2(x) = 1, \quad T_3(x) = q_1(x),$
$T_4(x) = q_1^2(x) + q_2(x)$

$T_5(x) = q_1^3(x) + 2q_1(x)q_2(x) + q_3(x)$

From equation (3), its characteristic equation are obtained as

$$s^{d+1} - q_1(x)s^d - q_2(x)s^{d-1} - \cdots - q_{d+1}(x) = 0.$$

Its roots: $\{\delta_1(x), \delta_2(x), \ldots, \delta_{d+1}(x)\}$.

Theorem 2.3. Generating function of d-Tribonacci polynomials $T_n(x)$ is

$$T(x,s) = \sum_{n=0}^{\infty} T_n(x)s^n = \frac{s^2}{(1 - q_1(x)s - q_2(x)s^2 - \cdots - q_{d+1}(x)s^{d+1})}.$$

Proof. We have

$$T(x,s) = T_0(x) + T_1(x)s + T_2(x)s^2 + T_3(x)s^3 + T_4(x)s^4 + \cdots \qquad (4)$$

Multiply Eq. (4) by $q_1(x)s, q_2(x)s^2, \ldots, q_{d+1}(x)s^{d+1}$, respectively. The following equations are obtained.

$$q_1(x)sT(x,s) = q_1(x)sT_0(x) + q_1(x)s^2T_1(x) + \cdots$$

$$\vdots$$

$$q_{d+1}(x)s^{d+1}T(x,s) = q_{d+1}(x)s^{d+1}T_0(x) + q_{d+1}(x)s^{d+2}T_1(x) + \cdots$$

If the necessary calculations are made, we get

$$T(x,s)[1 - q_1(x)s - q_2(x)s^2 - \cdots - q_{d+1}(x)s^{d+1}] = T_0(x) + s\big(T_1(x) - q_1(x)T_0(x)\big) + s^2\big(T_2(x) - q_1(x)T_1(x) - q_2(x)T_0(x)\big) + 0 + \cdots.$$

$$T(x,s) = \frac{s^2}{(1-q_1(x)s-q_2(x)s^2-\cdots-q_{d+1}(x)s^{d+1})}.$$

Its Binet formula has the following form

$$T_n(x) = \sum_{i=1}^{d+1} K_i(x)(\delta_i(x))^n$$

We get the following equation for each value of n.

$$T_0(x) = \sum_{i=1}^{d+1} K_i(x)$$

$$T_1(x) = \sum_{i=1}^{d+1} K_i(x)(\delta_i(x))^1$$

$$\vdots$$

$$T_n(x) = \sum_{i=1}^{d+1} K_i(x)(\delta_i(x))^n$$

$$T_0(x) = \sum_{i=1}^{d+1} K_i(x)$$

$$sT_1(x) = \sum_{i=1}^{d+1} K_i(x)(\delta_i(x))^1 s$$

$$\vdots$$

$$s^n T_n(x) = \sum_{i=1}^{d+1} K_i(x)(\delta_i(x))^n s^n$$

Multiplying both sides of above equations by $s, s^2, \dots, s^n$, respectively, we have:

The sum of the left-hand side of the equations:

$$\frac{s^2}{(1 - q_1(x)s - q_2(x)s^2 - \cdots - q_{d+1}(x)s^{d+1})}.$$

The sum of the right-hand side of the equations:

$$\sum_{i=1}^{d+1} K_i(x)[1 + (\delta_i(x))^1 s + \cdots + (\delta_i(x))^n s^n]$$
$$= \sum_{i=1}^{d+1} K_i(x)\left(\frac{1}{1 - \delta_i(x)s}\right)$$

so, we get

$$\frac{s^2}{(1 - q_1(x)s - q_2(x)s^2 - \cdots - q_{d+1}(x)s^{d+1})} = \sum_{i=1}^{d+1}\left(\frac{K_i(x)}{1 - \delta_i(x)s}\right).$$

Theorem 2.4. We have the following equation for $n \geq 0$.

$$T_n(x) = \left(\sum_{n_1 + 2n_2 + \cdots + (d+1)n_{d+1} = n+2} \binom{n_1 + n_2 + \cdots + n_{d+1}}{n_1, n_2, \dots, n_{d+1}} {q_1}^{n_1}(x){q_2}^{n_2}(x) \dots {q_{d+1}}^{n_{d+1}}(x)\right)s^2.$$

Proof. Generating function for d −Tribonacci polynomials

$$\sum_{n=0}^{\infty} T_n(x)s^n = \frac{s^2}{(1 - q_1(x)s - q_2(x)s^2 - \cdots - q_{d+1}(x)s^{d+1})}$$

$$= \sum_{n=0}^{\infty}\left(q_1(x)s + q_2(x)s^2 + \cdots + q_{d+1}(x)s^{d+1}\right)^{n+2}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{n_1 + n_2 + \cdots + n_{d+1} = n+2}^{\infty}\left[\binom{n+2}{n_1, n_2, \dots, n_{d+1}} {q_1}^{n_1}(x){q_2}^{n_2}(x) \dots {q_{d+1}}^{n_{d+1}}(x)\right] s^{n_1 + 2n_2 + \cdots + (d+1)n_{d+1}}\right)$$

$$\sum_{n=0}^{\infty}\left(\sum_{n_1 + 2n_2 + \cdots + (d+1)n_{d+1} = n+2} \binom{n_1 + n_2 + \cdots + n_{d+1}}{n_1, n_2, \dots, n_{d+1}} {q_1}^{n_1}(x){q_2}^{n_2}(x) \dots {q_{d+1}}^{n_{d+1}}(x)\right)s^{n+2}$$

as desired.

Theorem 2.5. The sum of the d −Tribonacci polynomials:

$$ST_n(x) = \sum_{n=0}^{\infty} T_n(x) = \frac{1}{1 - q_1(x) - q_2(x) - \cdots - q_{d+1}(x)}.$$

Proof. We have

$$ST_n(x) = \sum_{n=0}^{\infty} T_n(x) = T_0(x) + T_1(x) + \cdots + T_n(x) + \cdots$$

Multiplying the last equation by $q_1(x), \dots, q_{d+1}(x)$, respectively then we obtain

$$q_1(x)\, ST_n(x) = q_1(x)T_0(x) + q_1(x)T_1(x) + \cdots + q_1(x)T_n(x) + \cdots$$

$$\vdots$$

$$q_{d+1}(x)ST_n(x) = q_{d+1}(x)T_0(x) + q_{d+1}(x)T_1(x) + \cdots + q_{d+1}(x)T_n(x) + \cdots$$

From here, we have

$$ST_n(x)\big(1 - q_1(x) - q_2(x) - \cdots - q_{d+1}(x)\big) = 1.$$

So, we get

$$ST_n(x) = \sum_{n=0}^{\infty} T_n(x) = \frac{1}{1 - q_1(x) - q_2(x) - \cdots - q_{d+1}(x)}.$$

From [12], the $d -$ Fibonacci polynomials matrix Q_d has the following form

$$Q_d = \begin{pmatrix} q_1(x) & q_2(x) & \cdots & q_{d+1}(x) \\ 1 & 0 & & 0 \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{5}$$

where $detQ_d = (-1)^d q_{d+1}(x)$.

Matrix representation for $T_n(x)$ is given in following theorem.

Theorem 2.7. The representation for $T_n(x)$ is as follows:

$$Q_d{}^n = \begin{pmatrix} T_{n+2}(x) & q_2(x)T_{n+1}(x) + \cdots + q_{d+1}(x)T_{n-d-4}(x) & \cdots & q_{d+1}(x)T_{n+1}(x) \\ T_{n+1}(x) & q_2(x)T_n(x) + \cdots + q_{d+1}(x)T_{n-d-5}(x) & \cdots & q_{d+1}(x)T_{n+1}(x) \\ \vdots & \vdots & & \vdots \\ T_{n-d+2}(x) & q_2(x)T_{n-d+1}(x) + \cdots + q_{d+1}(x)T_{n-2d+2}(x) & \cdots & q_{d+1}(x)T_{n-d+1}(x) \end{pmatrix} \tag{6}$$

Proof. Let's apply the induction over n to prove it.

For $n = 1$,

$$Q_d{}^1 = \begin{pmatrix} T_3(x) & q_2(x)T_2(x) + \cdots + q_{d+1}(x)T_{-d-3}(x) & \cdots & q_{d+1}(x)T_2(x) \\ T_2(x) & q_2(x)T_1(x) + \cdots + q_{d+1}(x)T_{-d-4}(x) & \cdots & q_{d+1}(x)T_1(x) \\ \vdots & \vdots & & \vdots \\ T_{3-d}(x) & q_2(x)T_{2-d}(x) + \cdots + q_{d+1}(x)T_{3-2d}(x) & \cdots & q_{d+1}(x)T_{2-d}(x) \end{pmatrix}$$

$$=\begin{pmatrix} q_1(x) & q_2(x) & \cdots & q_{d+1}(x) \\ 1 & 0 & & 0 \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{7}$$

From the definition of $T_n(x)$, the matrices in (5) and (7) are equal.

Suppose that the result satisfies for n. So, we obtain

$$Q_d{}^n = \begin{pmatrix} T_{n+2}(x) & q_2(x)T_{n+1}(x) + \cdots + q_{d+1}(x)T_{n-d-4}(x) & \cdots & q_{d+1}(x)T_{n+1}(x) \\ T_{n+1}(x) & q_2(x)T_n(x) + \cdots + q_{d+1}(x)T_{n-d-5}(x) & \cdots & q_{d+1}(x)T_{n+1}(x) \\ \vdots & \vdots & & \vdots \\ T_{n-d+2}(x) & q_2(x)T_{n-d+1}(x) + \cdots + q_{d+1}(x)T_{n-2d+2}(x) & \cdots & q_{d+1}(x)T_{n-d+1}(x) \end{pmatrix}$$

Let's prove it for $n+1$. So, we get

$$Q_d^{n+1} = Q_d^n Q_d^1$$

$$= \begin{pmatrix} T_{n+2}(x) & q_2(x)T_{n+1}(x) + \cdots + q_{d+1}(x)T_{n-d-4}(x) & \cdots & q_{d+1}(x)T_{n+1}(x) \\ T_{n+1}(x) & q_2(x)T_n(x) + \cdots + q_{d+1}(x)T_{n-d-5}(x) & \cdots & q_{d+1}(x)T_{n+1}(x) \\ \vdots & \vdots & & \vdots \\ T_{n-d+2}(x) & q_2(x)T_{n-d+1}(x) + \cdots + q_{d+1}(x)T_{n-2d+2}(x) & \cdots & q_{d+1}(x)T_{n-d+1}(x) \end{pmatrix}.$$

$$\begin{pmatrix} q_1(x) & q_2(x) & \cdots & q_{d+1}(x) \\ 1 & 0 & & 0 \\ 0 & \ddots & & \\ & \ddots & & \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} T_{n+3}(x) & q_2(x)T_{n+2}(x) + \cdots + q_{d+1}(x)T_{n-d-3}(x) & \cdots & q_{d+1}(x)T_{n+2}(x) \\ T_{n+2}(x) & q_2(x)T_{n+1}(x) + \cdots + q_{d+1}(x)T_{n-d-4}(x) & \cdots & q_{d+1}(x)T_{n+2}(x) \\ \vdots & \vdots & & \vdots \\ T_{n-d+3}(x) & q_2(x)T_{n-d+2}(x) + \cdots + q_{d+1}(x)T_{n-2d+3}(x) & \cdots & q_{d+1}(x)T_{n-d+2}(x) \end{pmatrix}$$

Corollary 2.8. For $n, m \geq 0$, we have

$$\begin{aligned} T_{n+m}(x) = T_{n+2}(x)T_{m+2}(x) \\ &+ (q_2(x)T_{n+1}(x)T_{n+m}(x)T_{m+1}(x) \\ &+ \cdots \\ &+ q_{d+1}(x)T_{n-d+2}(x)T_{m+1}(x)) \\ &+ \cdots + q_{d+1}(x)T_{n+1}(x)T_{m-d+2}(x) \end{aligned}$$

Proof. We know

$$Q_d^n Q_d^m = Q_d^{n+m}.$$

The first row and column of matrix Q_d^{n+m} is the result.

Lemma 2.9. For $n \geq 1$,

$$T_{\boldsymbol{n}}(x) = F_{\boldsymbol{n-1}}(x)\mathbf{.}$$

Proof. For $n = 2$ equality is true

$$T_{\mathbf{2}}(x) = F_{\mathbf{1}}(x) = 1$$

Let the equality be true for $n = k$. For $n = k + 1$, we show that the equation is true.

$T_{k+1}(x) = q_1(x)T_k(x) + q_2(x)T_{k-1}(x) + \cdots + q_{d+1}(x)T_{n-d}(x)$,

$F_k(x) = q_1(x)F_{k-1}(x) + q_2(x)F_{k-2}(x) + \cdots + q_{d+1}(x)T_{n-d-1}(x)$.

Theorem 2.10. For $d \geq 2, n \geq 0$,

$$\sum_{\substack{n_1,n_2,\ldots,n_{d+1} \\ (d+1)n_1+dn_2+\cdots+n_{d+1}=n+2}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1,n_2,\ldots,n_{d+1}} {q_1}^{n_1}(x){q_2}^{n_2}(x)\ldots{q_{d+1}}^{n_{d+1}}(x)\, T_{n+2-(n_1+n_2+\cdots+n_{d+1})}(x)$$

$$= T_{n(d+1)}(x) \tag{8}$$

Proof. For $n = 1$, we have

$$T_{d+1}(x) = q_1(x)T_d(x) + q_2(x)T_{d-1}(x) + \cdots + q_{d+1}(x)T_0(x).$$

Let us show the right-hand side of (8) by RH.

For $n \geq 0$, we have

RH

$$= \sum_{\substack{n_1,n_2,\ldots,n_{d+1} \\ (d+1)n_1+dn_2+\cdots+n_{d+1}=n+2}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1,n_2,\ldots,n_{d+1}} {q_1}^{n_1}(x)\ldots{q_{d+1}}^{n_{d+1}}(x)\left[\sum_{i=1}^{d+1} K_i(x)(\delta_i(x))^{n+2-(n_1+n_2+\cdots+n_{d+1})}\right]$$

$=$

$$= \sum_{\substack{n_1,n_2,\ldots,n_{d+1} \\ (d+1)n_1+dn_2+\cdots+n_{d+1}=n+2}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1,n_2,\ldots,n_{d+1}} {q_1}^{n_1}(x)\ldots{q_{d+1}}^{n_{d+1}}(x)\left[\sum_{i=1}^{d+1} K_i(x)(\delta_i(x))^{(dn_1+(d-1)n_2+\cdots+n_{d+1})}\right]$$

$$K_1(x) \sum_{\substack{n_1,n_2,\ldots,n_{d+1} \\ (d+1)n_1+dn_2+\cdots+n_{d+1}=n+2}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1,n_2,\ldots,n_{d+1}} \left(\delta_1^d(x)q_1(x)\right)^{n_1}\left(\delta_1^{d-1}(x)q_2(x)\right)^{n_2}\ldots\left(q_{d+1}(x)\right)^{n_{d+1}}$$

$$+ \cdots K_{d+1}(x) \sum_{\substack{n_1,n_2,\ldots,n_{d+1} \\ (d+1)n_1+dn_2+\cdots+n_{d+1}=n+2}} \binom{n_1+n_2+\cdots+n_{d+1}}{n_1,n_2,\ldots,n_{d+1}} \left(\delta_1^d(x)q_1(x)\right)^{n_1}\left(\delta_1^{d-1}(x)q_2(x)\right)^{n_2}\ldots\left(q_{d+1}(x)\right)^{n_{d+1}}$$

$$\begin{aligned} &= K_1(x)\left[\delta_1^d(x)q_1(x) + \delta_1^{d-1}(x)q_2(x) + \cdots + q_{d+1}(x)\right]^n + \cdots \\ &\quad + K_{d+1}(x)\left[\delta_1^d(x)q_1(x) + \delta_1^{d-1}(x)q_2(x) + \cdots + q_{d+1}(x)\right]^n \end{aligned}$$

from characteristic equation, we obtain

$$= \sum_{i=1}^{d+1} K_i(x)(\delta_i(x)^{d+1})^n = T_{n(d+1)}(x).$$

as desired.

Lemma 2.11. For $n \geq 1$,

$$T_n(x) = L_{n-1}(x) - F_n(x) + q_1(x)F_{n-1}(x).$$

Proof. From (2) we get

$$T_n(x) = F_{n-1}(x) = L_{n-1}(x) - q_2(x)F_{n-2}(x) - \cdots - q_{d+1}(x)F_{n-d-1}(x)$$

$$= L_{n-1}(x) - (q_2(x)F_{n-2}(x) + \cdots + q_{d+1}(x)F_{n-d-1}(x))$$

$$= L_{n-1}(x) - F_n(x) + q_1(x)F_{n-1}(x)$$

3 The Infinite Tribonacci Polynomials Matrix

The d −Tribonacci polynomials matrix is showed by

$$\mathcal{T}(x) = [T_{q_1,q_2,\dots,q_{d+1},i,j}(x)]$$

and defined as follows

$$T(x) = \begin{pmatrix} 1 & 0 & . \\ q_1(x) & 1 & . \\ {q_1}^2(x) + q_2(x) & q_1(x) & . \\ s_1(x) & s_2(x) & . \\ \vdots & . & \end{pmatrix}$$

$= (g_{\mathcal{T}(x)}(s), f_{\mathcal{T}(x)}(s))$,

where $s_1(x) = {q_1}^3(x) + 2q_1(x)q_2(x) + q_3(x)$

$s_2(x) = {q_1}^2(x) + q_2(x)$ and $s_3(x) = q_1(x)$

This Tribonacci polynomial matrix can also be written as,

$$\mathcal{T}(x) = \begin{pmatrix} T_2(x) & T_1(x) & T_0(x) & 0 & 0 & \dots \\ T_3(x) & T_2(x) & T_1(x) & T_0(x) & 0 & \dots \\ T_4(x) & T_3(x) & T_2(x) & T_1(x) & T_0(x) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

Note that $\mathcal{T}(x)$ is a Riordan matrix.

Theorem 3.1. The first column of matrix $\mathcal{T}(x)$ is

$$(1, q_1(x), {q_1}^2(x) + q_2(x), \dots)^{\boldsymbol{T}}.$$

From the Riordan group theory, we get the generator function of the first column as follows:

$$g_{\mathcal{T}(x)}(s) = \sum_{n=0}^{\infty} \mathcal{T}_{q_1,q_2,\dots,q_{d+1},i,j}(x)\, s^n = \frac{1}{(1 - q_1(x)s - q_2(x)s^2 - \dots - q_{d+1}(x)s^{d+1})}.$$

Proof. Generating functions of the first column of matrix $\mathcal{T}(x)$ is

$$1 + q_1(x)s + \left({q_1}^2(x) + q_2(x)\right) + \cdots.$$

If we do operations like the proof of Theorem 2.4, then

$$g_{\mathcal{T}(x)}(s) = \frac{1}{(1-q_1(x)s-q_2(x)s^2-\dots-q_{d+1}(x)s^{d+1})}.$$

The desired expression is obtained.

From the Riordan matrix, $f_{\mathcal{T}(x)}(s) = s$.

$$\mathcal{T}(x) = \left(g_{\mathcal{T}(x)}(s), f_{\mathcal{T}(x)}(s)\right) = \left(\frac{1}{(1 - q_1(x)s - q_2(x)s^2 - \dots - q_{d+1}(x)s^{d+1})}, s\right).$$

If the d −Tribonacci polynomials matrix $\mathcal{T}(x)$ is finite, then the matrix is

$$\mathcal{T}_f(x) = \begin{pmatrix} T_2(x) & 0 & 0 & 0 & 0 & \dots \\ T_3(x) & 1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ T_n(x) & T_{n-1}(x) & T_{n-n}(x) & \vdots & \vdots & T_2(x) \end{pmatrix}$$

and

$$det\mathcal{T}_f(x) = \left|\mathcal{T}_f(x)\right| = (1)^n = 1.$$

We give two factorizations of Pascal Matrix with the d −Tribonacci polynomials matrix. Now, we give a matrix $M(x) = (m_{i,j}(x))$,

$$m_{i,j} = \binom{i-1}{j-1} - q_1(x)\binom{i-2}{j-1} - \cdots - q_{d+1}(x)\binom{i-d-2}{j-1}$$

So, we get

$$M(x) = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 1 - q_1(x) & 1 & 0 & \dots \\ 1 - q_1(x) - q_2(x) & 2 - q_1(x) & 1 & \dots \\ 1 - q_1(x) - q_2(x) - q_3(x) & 3 - 2q_1(x) - q_2(x) & 3 - q_1(x) & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

Thus we can introduce the first factorization of the infinite Pascal matrix.

Theorem 3.2. The factorization of the infinite Pascal matrix is

$$P(x) = \mathcal{T}(x)M(x).$$

Proof. The generating function from the first column of matrix $M(x)$ is

$$g_{M(x)}(s) = 1 + \left(1 - q_1(x)\right)s + (1 - q_1(x) - q_2(x))s^2) + \cdots$$

$$= (1 + s + s^2 + \cdots) - q_1(x)(s + s^2 + s^3 + \cdots) - q_2(x)(s^2 + s^3 + \cdots) + \cdots + q_{d+1}(s^{d+1} + s^{d+2} + \cdots)$$

$$= \frac{1}{1-s} - \frac{q_1 s}{1-s} - \frac{q_2 s^2}{1-s} - \cdots - \frac{q_{d+1} s^{d+1}}{1-s}$$

$$= \frac{1 - q_1 s - q_2 s^2 - \cdots - q_{d+1} s^{d+1}}{1-s}.$$

From the Riordan matrix, we get $f_{M(x)}(s)$ as follows

$$f_{M(x)}(s) = s + (2 - q_1(x))s^2 + (3 - 2q_1(x) - q_2(x))s^3 + \cdots$$

$$= (s + 2s^2 + 3s^3 + \cdots) - q_1 s(s + 2s^2 + 3s^3 + \cdots) - \cdots - q_{d+1} s^{d+1}(s + 2s^2 + 3s^3 + \cdots)$$

$$= \frac{s}{1-s}\left(\frac{1 - q_1 s - q_2 s^2 - \cdots - q_{d+1} s^{d+1}}{1-s}\right).$$

From definition of the Riordan array, $i\ th$ column generating function is $g(x)(f(x))^i$

$$f_{M(x)}(s) = \frac{s}{1-s}.$$

Thus, $M(x)$ has the following form

$$M(x) = \left(g_{M(x)}(s), f_{M(x)}(s)\right) = \left(\frac{1 - q_1 s - q_2 s^2 - \cdots - q_{d+1} s^{d+1}}{1-s}, \frac{s}{1-s}\right).$$

From the definitions of infinite Pascal matrix and the infinite $d-$Tribonacci polynomials matrix, the Riordan representations:

$$P = \left(\frac{1}{1-s}, \frac{s}{1-s}\right),$$

$$\mathcal{T}(x) = \left(\frac{1}{(1 - q_1(x)s - q_2(x)s^2 - \cdots - q_{d+1}(x)s^{d+1})}, s\right).$$

From the matrix multiplication, the proof is ok.

Secondly, we introduce other factorization of the Pascal matrix with the $d-$Tribonacci polynomials matrix. Let's give an infinitive $N(x) = (n_{i,j}(x))$ as follows.

$$n_{i,j} = \binom{i-1}{j-1} - q_1(x)\binom{i-1}{j} - q_2(x)\binom{i-1}{j+1} - \cdots - q_{d+1}(x)\binom{i-1}{j+d}.$$

We give the infinite $N(x)$ by

$$N(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 - q_1(x) & 1 & 0 & 0 & \cdots \\ 1 - 2q_1(x) - q_2(x) & 2 - q_1(x) & 1 & 0 & \cdots \\ 1 - 3q_1(x) - 3q_2(x) - q_3(x) & 3 - 2q_1(x) - q_2(x) & 3 - q_1(x) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

Now, we introduce the final factorization of the infinite Pascal matrix.

Theorem 3.3. The factorization of the infinite Pascal matrix:

$$P(x) = \mathcal{T}(x)N(x).$$

Proof. The proof is similar to Theorem 3.2.

Now, we can give the inverse of $d-$Tribonacci polynomials matrix by helping the definition of the reverse element of the Riordan group in [11].

Corollary 3.4 The inverse of $d-$Tribonacci polynomial:

$$\mathcal{T}^{-1}(x) = (1 - q_1 s - q_2 s^2 - \cdots - q_{d+1} s^{d+1}, s).$$

4 Conclusion

In this study, new generalized Tribonacci polynomials have been introduced and studied. Some combinatorial properties of the $d-$Tribonacci polynomials matrix representations are obtained with the help of Riordan arrays. In addition, d-Tribonacci number sequence has been obtained by considering the Pascal matrix. Based on the Pascal matrix, two kinds of factors of d-Tribonacci polynomials were found. Also, infinite $d-$Tribonacci polynomial matrices and the inverses of these polynomials were found.

References:

[1] E. Özkan, B. Kuloğlu, On the new Narayana polynomials, the Gauss Narayana numbers and their polynomials. *Asian-European*

Journal of Mathematics 14(6), 2150100 (2021). https://doi.org/10.1142/S179355712150100X

[2] E. Özkan, B. Kuloğlu and J.F. Peters, k-Narayana sequences self-similarity. Flip graph views of k-Narayana self-similarity, *Chaos Solitons and Fractals*, 153.5 (2021), 111473.

[3] Ö. Deveci, The Pell-Padovan sequences and Jacobsthal-Padovan sequences in finite groups, *Util. Math*., 98, 257-270, (2015).

[4] E. Özkan, 3-Step Fibonacci Sequences in Nilpotent Groups, *Applied Mathematics and Computation*, 144, 517-527, (2003).

[5] E. Özkan, H. Aydın and R. Dikici, Applications of Fibonacci sequences in a finite nilpotent group, *Applied Mathematics and Computation*, 141(2-3), 565-578, (2003) DOI: 10.1016/S0096-3003(02)00276-X.

[6] E. Özkan, M. Uysal, d-Gaussian Pell Polynomials and Their Matrix Representations, *Discrete Mathematics, Algorithms and Applications*, https://doi.org/10.1142/S1793830922501385, 2023.

[7] E. Özkan, M. Uysal, d-Gaussian Jacobsthal, d-Gaussian Jacobsthal-Lucas Polynomials and Their Matrix Representations, *Electronic Journal of Mathematical Analysis and Applications*, 10 (2), 124-140, (2022).

[8] L. Ramirez Jose and F. Sirvent Victor, A note on the *k*-Narayana sequence. *Annales Mathematicae et Informaticae.* 45, 91-105, (2015).

[9] N.J.A. Sloane, The on-line encyclopedia of integer sequences. A000073 - OEIS.

[10] Y. Soykan, İ. Okumuş, On a Generalized Tribonacci Sequence, *Scıtech Research Organısatıon,* 14(3) 2019, 2413-2418.

[11] İ. Durukan, S. Çelik, E. Özkan, New Recurrence Relations For Tribonacci Sequence And Generalized Tribonacci Sequence, *Discrete Mathematical Structures* (RRDMS), 7(3),1-6, (2020).

[12] B. Sadaoui, A. Krelifa, d- Fibonacci and d-Lucas polynomials. *Journal of Mathematical Modeling* 9(3),1-12 (2021). https://doi.org/10.22124/JMM.2021.17837.1538

[13] L. W. Shapiro, S. Getu, W. J. Woan, L. C. Woodson, The riordan group. *Discrete Applied Mathematics* 34 (1-3), 229-239 (1991).

[14] M. Feinberg, Fibonacci-Tribonacci, *The Fibonacci Quarterly,* 1,(3), 71-74, (1963).

[15] I. Bruce, A modified Tribonacci sequence, *The Fibonacci Quarterly*, 22,(3), 244-246,(1984).

[16] A. Scott, T. Delaney, V. Hoggatt Jr., The Tribonacci sequence, *The Fibonacci Quarterly,* 15(3), 193-200, (1977).

[17] A. Shannon, Tribonacci numbers and Pascal's pyramid, *The Fibonacci Quarterly*, 15, (3), 268-275, (1977).

[18] W. Spickerman, Binet's formula for the Tribonacci sequence, *The Fibonacci Quarterly*, 20,118-120, (1981).

[19] C. C. Yalavigi, Properties of Tribonacci numbers, *The Fibonacci Quarterly*, 10, (3), 231-246, (1972).

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by Bahar KULOĞLU, Engin ÖZKAN. The first draft of the manuscript was written by Engin ÖZKAN, and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funds, grants, or other supports were received.

Conflicts of Interest

All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.