On the Diophantine Equation $n^x + 10^y = z^2$

SUTON TADEE
Department of Mathematics,
Faculty of Science and Technology,
Thepsatri Rajabhat University, Lopburi 15000,
THAILAND

Abstract: In this paper, we show that $(n, x, y, z) = (2, 3, 0, 3)$ is the unique non-negative integer solution of the Diophantine equation $n^x + 10^y = z^2$, where n is a positive integer with $n \equiv 2 \pmod{30}$ and x, y, z are non-negative integers. If $n = 5$, then the Diophantine equation has exactly one non-negative integer solution $(x, y, z) = (3, 2, 15)$. We also give some conditions for non-existence of solutions of the Diophantine equation.

Key-Words: Diophantine equation, Mihăilescu's Theorem, congruence, non-negative integer solution

Received: October 29, 2022. Revised: December 24, 2022. Accepted: January 22, 2023. Published: February 23, 2023.

1 Introduction

In 2014, Sroysang, [1], proved that the Diophantine equation $4^x + 10^y = z^2$ has no non-negative integer solution. After that, in 2019, Burshtein, [2], showed that the Diophantine equation $7^x + 10^y = z^2$ has no positive integer solution. In 2020, Orosram and Comemuang, [3], found that the Diophantine equation $8^x + n^y = z^2$, where n is a positive integer with $n \equiv 10 \pmod{15}$, has the unique non-negative integer solution $(x, y, z) = (1, 0, 3)$. In 2021, N. Viriyapong and C. Viriyapong, [4], proved that the Diophantine equation $n^x + 13^y = z^2$, where n is a positive integer with $n \equiv 2 \pmod{39}$ and $n + 1$ is not a square number, has the unique non-negative integer solution $(n, x, y, z) = (2, 3, 0, 3)$. Tangjai and Chubthaisong, [5], studied the Diophantine equation $3^x + p^y = z^2$, where p is prime and $p \equiv 2 \pmod 3$ and found that if $y = 0$, then $(p, x, y, z) = (p, 1, 0, 2)$ is the only one non-negative integer solution and if $4 \nmid y$, then the equation has the unique non-negative integer solution $(p, x, y, z) = (2, 0, 3, 3)$. In 2022, Wannaphan and Tadee, [6], found all non-negative integer solutions of the Diophantine equation $n^{2x} + 2^y = z^2$, where n is an odd positive integer. In the same year, N. Viriyapong and C. Viriyapong, [7], proved that the Diophantine equation $n^x + 19^y = z^2$, where n is a positive integer with $n \equiv 2 \pmod{57}$ has the unique non-negative integer solution $(n, x, y, z) = (2, 3, 0, 3)$. Borah and Dutta, [8], studied the Diophantine equation $n^x + 24^y = z^2$, where n is a positive integer with $n \equiv 5, 7 \pmod 8$.

Inspired by the work mentioned earlier, we study the Diophantine equation $n^x + 10^y = z^2$, where n is a positive integer. We can easily notice that if $n \equiv 1 \pmod 3$, then the Diophantine equation has no non-negative integer solution. Since $n \equiv 1 \pmod 3$, we have $z^2 = n^x + 10^y \equiv 2 \pmod 3$, a contradiction since $z^2 \equiv 0, 1 \pmod 3$. Cases that have not yet been considered, are $n \equiv 0, 2 \pmod 3$. In this research, we will consider the case $n \equiv 2 \pmod 3$ and $n \equiv 2 \pmod{10}$. That is $n \equiv 2 \pmod{30}$. Moreover, we study in case $n = 5$.

2 Preliminaries

In the beginning this section, we present some helpful Theorems.

Theorem 1. If z is an integer, then $z^2 \equiv 0, 1, 4, 5, 6, 9 \pmod{10}$.

Proof. Let z be an integer. Then there exists a non-negative integer r such that $z \equiv r \pmod{10}$, where $0 \le r \le 9$.

Case 1: $r = 0$. Then $z^2 \equiv 0 \pmod{10}$.

Case 2: $r = 1$. Then $z^2 \equiv 1 \pmod{10}$.

Case 3: $r = 2$. Then $z^2 \equiv 4 \pmod{10}$.

Case 4: $r = 3$. Then $z^2 \equiv 9 \pmod{10}$.

Case 5: $r = 4$. Then $z^2 \equiv 16 \equiv 6 \pmod{10}$.

Case 6: $r = 5$. Then $z^2 \equiv 25 \equiv 5 \pmod{10}$.

Case 7: $r = 6$. Then $z^2 \equiv 36 \equiv 6 \pmod{10}$.

Case 8: $r = 7$. Then $z^2 \equiv 49 \equiv 9 \pmod{10}$.

Case 9: $r = 8$. Then $z^2 \equiv 64 \equiv 4 \pmod{10}$.

Case 10: $r = 9$. Then $z^2 \equiv 81 \equiv 1 \pmod{10}$. □

Theorem 2. [9], $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$ are exactly two non-negative integer solutions of the Diophantine equation $2^x + 5^y = z^2$.

Theorem 3. *(Mihăilescu's Theorem)*, [10], The Diophantine equation $a^x - b^y = 1$ has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$, where a, b, x, y are integers and $\min\{a, b, x, y\} > 1$.

Next, we prove two useful Lemmas by using Mihăilescu's Theorem.

Lemma 4. The Diophantine equation

$$1 + 10^y = z^2 \tag{1}$$

has no non-negative integer solution.

Proof. Assume that (y, z) is a non-negative integer solution of (1). Then $z^2 - 10^y = 1$. It is easy to check that $y > 1$ and $z > 1$. Thus $\min\{z, 10, 2, y\} > 1$. By Theorem 3, this is impossible. □

Lemma 5. Let n be a positive integer with $n \equiv 2 \pmod{10}$. Then the Diophantine equation

$$n^x + 1 = z^2 \tag{2}$$

has a unique non-negative integer solution. The solution is $(n, x, z) = (2, 3, 3)$.

Proof. Let (x, z) be a non-negative integer solution of (2). If $n = 1$ or $x = 0$, then $z^2 = 2$, a contradiction. Thus $n > 1$ and $x \geq 1$. If $x = 1$, then $n + 1 = z^2$. Since $n \equiv 2 \pmod{10}$, we have $z^2 \equiv 3 \pmod{10}$. This is impossible by Theorem 1. Then $x > 1$. Next we consider z. If $z = 0$ or $z = 1$, then $n^x = -1$ or $n^x = 0$, respectively, a contradiction. Thus $z > 1$ and so $\min\{z, n, 2, x\} > 1$. By Theorem 3 and (2), we have $(n, x, z) = (2, 3, 3)$. □

3 Main Results

In this section, we give our results.

Theorem 6. The Diophantine equation

$$5^x + 10^y = z^2 \tag{3}$$

has a unique non-negative integer solution. The solution is $(x, y, z) = (3, 2, 15)$.

Proof. Let (x, y, z) be a non-negative integer solution of (3). Suppose that $x \geq y$. From (3), we have

$$5^y(5^{x-y} + 2^y) = z^2. \tag{4}$$

Then y is even and there exists a positive integer z_1 such that

$$5^{x-y} + 2^y = z_1^2. \tag{5}$$

By Theorem 2, we have $y = 2$ and $x - y = 1$. Then $x = 3$, and so $z^2 = 5^3 + 10^2 = 225$. Hence $(x, y, z) = (3, 2, 15)$ is a non-negative integer solution of (3). Now, we consider $x < y$. From (3), it follows that

$$5^x(1 + 2^y \cdot 5^{y-x}) = z^2. \tag{6}$$

Thus, x is even and there exists a positive integer z_2 such that $1 + 2^y \cdot 5^{y-x} = z_2^2$. It implies that

$$(z_2 - 1)(z_2 + 1) = 2^y \cdot 5^{y-x}. \tag{7}$$

Then there exists two non-negative integers u and v such that

$$z_2 - 1 = 2^u \cdot 5^v \tag{8}$$

and

$$z_2 + 1 = 2^{y-u} \cdot 5^{y-x-v}. \tag{9}$$

From (8) and (9), we get

$$2 = 2^{y-u} \cdot 5^{y-x-v} - 2^u \cdot 5^v. \tag{10}$$

Now, we consider three following cases:

Case 1: $y - x - v = 0$. From (10), we obtain that

$$2 = 2^{y-u} - 2^u \cdot 5^v. \tag{11}$$

Subcase 1.1: $y - u \geq u$. From (11), we obtain that $2 = 2^u\left(2^{y-2u} - 5^v\right)$. Then $u = 1$ and $1 = 2^{y-2u} - 5^v$. It is easy to check that $y - 2u > 1$ and $v > 1$. This is impossible by Theorem 3.

Subcase 1.2: $y - u < u$. From (11), we get $2 = 2^{y-u}\left(1 - 2^{2u-y} \cdot 5^v\right)$. Then $y - u = 1$ and $1 = 1 - 2^{2u-y} \cdot 5^v$. Thus $2^{2u-y} \cdot 5^v = 0$, a contradiction.

Case 2: $v = 0$. From (10), we obtain that

$$2 = 2^{y-u} \cdot 5^{y-x} - 2^u. \tag{12}$$

Subcase 2.1: $y - u \geq u$. From (12), we get $2 = 2^u\left(2^{y-2u} \cdot 5^{y-x} - 1\right)$. Then $u = 1$ and $1 = 2^{y-2u} \cdot 5^{y-x} - 1$. Thus $2^{y-2u} \cdot 5^{y-x} = 2$, and so $y - x = 0$. This is impossible since $x < y$.

Subcase 2.2: $y - u < u$. From (12), we get $2 = 2^{y-u}\left(5^{y-x} - 2^{2u-y}\right)$. Then $y - u = 1$ and $5^{y-x} - 2^{2u-y} = 1$. It is easy to check that $y - x > 1$ and $2u - y > 1$. This is impossible by Theorem 3.

Case 3: $y - x - v > 0$ and $v > 0$. From (10), we get $5 \mid 2$, a contradiction. □

Theorem 7. Let n be a positive integer with $n \equiv 2 \pmod{30}$. Then the Diophantine equation

$$n^x + 10^y = z^2 \tag{13}$$

has a unique non-negative integer solution. The solution is $(n, x, y, z) = (2, 3, 0, 3)$.

Proof. Let x, y and z be non-negative integers such that the equation (13) is true.

Case 1: $x = 0$. This is impossible by Lemma 4.

Case 2: $y = 0$. By Lemma 5, it follows that $(n, x, y, z) = (2, 3, 0, 3)$.

Case 3: $x \geq 1$ and $y \geq 1$. Assume that x is even. It follows that $x = 2u$, for some positive integer u. Since $n \equiv 2 \pmod{30}$, we obtain that $n \equiv 2 \pmod 3$, and so $n^x \equiv 2^x \equiv 4^u \equiv 1 \pmod 3$. Then $z^2 = n^x + 10^y \equiv 2 \pmod 3$. This is impossible since $z^2 \equiv 0, 1 \pmod 3$. Thus x is odd. There exists a non-negative integer v such that $x = 2v + 1$. Since $n \equiv 2 \pmod{30}$, we obtain that $n^x = n^{2v+1} \equiv 2^{2v+1} \pmod{30}$.

Subcase 3.1: v is even. Then $v = 2a$, for some non-negative integer a. Since $2^{4a} \equiv 16^a \equiv 1 \pmod 5$, it follows that $n^x \equiv 2^{4a+1} \equiv 2 \pmod{10}$, and so $z^2 = n^x + 10^y \equiv 2 \pmod{10}$. This is impossible by Theorem 1.

Subcase 3.2: v is odd. Then there exists a non-negative integer b such that $v = 2b + 1$. Since $2^{4b} \equiv 16^b \equiv 1 \pmod 5$, it follows that $n^x \equiv 2^{4b+3} \equiv 8 \pmod{10}$, and so $z^2 = n^x + 10^y \equiv 8 \pmod{10}$. This is impossible by Theorem 1. □

By Theorem 7, we have the following examples and the corollary.

Example 8. The Diophantine equation $2^x + 10^y = z^2$ has a unique non-negative integer solution. The solution is $(x, y, z) = (3, 0, 3)$.

Example 9. The Diophantine equation $32^x + 10^y = z^2$ has no non-negative integer solution.

Corollary 10. Let m and n be positive integers with $n \equiv 2 \pmod{30}$. Then the Diophantine equation

$$n^x + 10^y = z^{2m} \tag{14}$$

has a unique non-negative integer solution. The solution is $(n, m, x, y, z) = (2, 1, 3, 0, 3)$.

Proof. Let a, b and c be non-negative integers such that the equation (14) is true. Therefore $(x, y, z) = (a, b, c^m)$ is a solution of the equation (13). By Theorem 7, we get $n = 2, a = 3, b = 0$ and $c^m = 3$. Then $c = 3$ and $m = 1$. Hence $(n, m, x, y, z) = (2, 1, 3, 0, 3)$ is the only one solution of the equation (14). □

Theorem 11. Let n be prime with $n \geq 7$, $n \not\equiv 1 \pmod 4$ and $n \not\equiv 1 \pmod 5$. If y is even, then the Diophantine equation (13) has no non-negative integer solution.

Proof. Let x, y and z be non-negative integers such that the equation (13) is true. Since y is even, we have $y = 2k$, for some non-negative integer k.

Case 1: $k = 0$. Then $y = 0$. From (13), we have

$$z^2 - n^x = 1. \tag{15}$$

It is easy to check that $z > 1$ and $x > 0$. Assume that $x > 1$. Then $\min\{z, n, 2, x\} > 1$. By Theorem 3 and (15), we have $n = 2$, a contradiction. Thus $x = 1$, and so $(z - 1)(z + 1) = n$. Since n is prime, we get $z - 1 = 1$ and $z + 1 = n$. Thus $z = 2$ and $n = 3$, a contradiction.

Case 2: $k > 0$. From (13), it follows that

$$(z - 10^k)(z + 10^k) = n^x. \tag{16}$$

Since n is prime, there exists a non-negative integer h such that

$$z - 10^k = n^h \tag{17}$$

and

$$z + 10^k = n^{x-h}. \tag{18}$$

From (17) and (18), we get $x > 2h$ and

$$2 \cdot 10^k = n^h(n^{x-2h} - 1). \tag{19}$$

Since n is prime with $n \geq 7$, we obtain that $h = 0$ and

$$2 \cdot 10^k = n^x - 1 = (n - 1)(n^{x-1} + n^{x-2} + \cdots + 1).$$

Since $k > 1$ and $n-1 > 2$, it follows that $4 \mid (n-1)$ or $5 \mid (n - 1)$. Thus $n \equiv 1 \pmod 4$ or $n \equiv 1 \pmod 5$, a contradiction. □

Corollary 12. The Diophantine equation

$$7^x + 100^y = z^2 \tag{20}$$

has no non-negative integer solution.

Proof. Assume that (a, b, c) is a non-negative integer solution of (20). Therefore $7^a + 10^{2b} = c^2$. Thus $(n, x, y, z) = (7, a, 2b, c)$ is a non-negative integer solution of (13). This is impossible by Theorem 11. □

Acknowledgment

The author would like to thank reviewers for careful reading of this manuscript and the useful comments. This work was supported by Research and Development Institute and Faculty of Science and Technology, Thepsatri Rajabhat University, Thailand.

References:

[1] B. Sroysang, More on the Diophantine equation $4^x + 10^y = z^2$, *International Journal of Pure and Applied Mathematics*, Vol.91, No.1, 2014, pp. 135-138.

[2] N. Burshtein, On solutions to the Diophantine equation $7^x + 10^y = z^2$ when x, y, z are positive integers, *Annals of Pure and Applied Mathematics*, Vol.20, No.2, 2019, pp. 75-77.

[3] W. Orosram and C. Comemuang, On the Diophantine equation $8^x + n^y = z^2$, *WSEAS Transactions on Mathematics*, Vol.19, 2020, pp. 520-522.

[4] N. Viriyapong and C. Viriyapong, On the Diophantine equation $n^x + 13^y = z^2$ where $n \equiv 2 \pmod{39}$ and $n + 1$ is not a square number, *WSEAS Transactions on Mathematics*, Vol.20, 2021, pp. 442-445.

[5] W. Tangjai and C. Chubthaisong, On the Diophantine equation $3^x + p^y = z^2$ where $p \equiv 2 \pmod 3$, *WSEAS Transactions on Mathematics*, Vol.20, 2021, pp. 283-287.

[6] C. Wannaphan and S. Tadee, On the solutions of two Diophantine equations $n^{2x} + 2^y = z^2$ and $n^{2x} - 2^y = z^2$ (in Thai), *Journal of Science and Science Education*, Vol.5, No. 2, 2022, pp. 236-240.

[7] N. Viriyapong and C. Viriyapong, On the Diophantine equation $n^x + 19^y = z^2$, where $n \equiv 2 \pmod{57}$, *International Journal of Mathematics and Computer Science*, Vol.17, No.4, 2022, pp. 1639-1642.

[8] P.B. Borah and M. Dutta, On two classes of exponential Diophantine equations, *Communications in Mathematics and Applications*, Vol.13, No.1, 2022, pp. 137-145.

[9] D. Acu, On a Diophantine equation $2^x + 5^y = z^2$, *General Mathematics*, Vol.15, No.4, 2007, pp. 145-148.

[10] P. Mihăilescu, Primary cyclotomic units and a proof of Catalan's conjecture, *Journal für die Reine und Angewandte Mathematik*, Vol.572, 2004, pp. 167-195.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This work was supported by Research and Development Institute and Faculty of Science and Technology, Thepsatri Rajabhat University, Thailand.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.