Zero Truncated Poisson - Pareto Distribution: Application and Estimation Methods

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Abstract: This article introduces and discusses a new three-parameter lifespan distribution called Zero-Truncated Poisson Pareto distribution ZTPP. that is built on compounding Pareto distribution as a continuous distribution and Zero-Truncated Poisson distribution as a discrete distribution. Various statistical properties and reliability characteristics of the proposed distribution have been investigated including explicit expressions for the moments, moment generating function, quantile function, and median. With three parameters, the suggested distribution has an advantage over other distributions in that it makes estimating the model parameters simpler. To estimate the unknown parameters of the ZTPP distribution, the maximum likelihood method, and L. Moments method are employed. Moreover, a real data set is used to evaluate the significance and ensure the applicability of the proposed distribution as compared to other probability distributions. The derived model proved to be the best compared to other fitted models, where the criteria values of (AIC), (CAIC), and (BIC) are minimum values by using the ZTPP distribution. The proposed model is hoped to attract a wider application

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1. Introduction

The processes of installing distributions results in new distributions. One such complex probability distribution that is important in practical applications is the Poisson composite distribution [1]. Since the nature of some data or occurrences necessitates the use of composite distributions, which are more flexible than standard distributions, composites distributions are more adaptable to describe some data that cannot be well represented by traditional statistical distributions. For example, failures in electronic devices, the phenomenon of the strength of slime, the phenomenon of rainfall that can occur in certain specific places, and other phenomena in working life. In these cases, the appropriate distribution is one of the composite distributions. Under its name, this distribution involves many distributions. Recently, new distributions have been proposed by integrating continuous distribution with another discrete distribution. For example, we can cite some of them such as the exponential geometric by Adamidis.[2]; Silva proposed the generalized exponential geometric [3]; Barreto-Souza et al. [4] proposed the Weibull geometric; the Poisson exponential by Cancho et al. [5]; the flexible Zero-truncated Poisson by Abouelmagd et al.[6]; the Poisson Burr X Weibull by Abouelmagd et al. [7]; the Zero Truncated Poisson Exponentiated Gamma by Guilherme et al. [1]; the Exponential-Truncated Poisson by Rezaei et al.[8]; the Pareto Poisson Lindley by Asgharzadeh et al. [9]; the Poisson Nadarajah by Muhammad Mansoor et al.[10]; and the Binomial-exponential 2 by Bakouch et al. [11]. The Pareto Geometric by Nassar et al.[12] can also be cited with some distributions related to the Pareto distribution beta modified Weibull by Silva et al.[13]; the Pareto-type distribution by Bourguignon et al.[14]; the bivariate Pareto by Sankaran et al.[15]; and the beta generalized Pareto by Mahmoudi et al.[16].

The shape and scale characteristics of the Generalized Pareto Distribution GPD can be estimated using a variety of methods. Moments-based approaches, maximum likelihood, probability-weighted moments, and others are examples of classical methods. The references [17], [18] provide a thorough analysis of them. Other academics have suggested the following generalizations of the GPD: To estimate Value at Risk, references [17] provided a three-parameter Pareto distribution and used POT; references [19] introduced an extension of the GPD and used parametric estimation. Classical approaches, however, might not be appropriate in all circumstances, as stated in [18]. That is why Zero-Truncated Poisson inference could be advisable.

There aren't many approaches for combining the Pareto and Poisson distributions. We can cite [20], who suggested using conjugate prior distributions; thus in this paper, we derive some structural properties of the (ZTP) and (P) distributions based on a double integrating mechanism to it's with a three-parameter lifetime distribution, installing new distribution called the Zero-Truncated Poisson Pareto distribution or in short ZTPP distribution with three parameters, make ZTPP has an advantage over other distributions in that it makes estimating the model parameters simpler. To estimate the unknown parameters of the ZTPP distribution, the maximum likelihood method and L- method, are employed.

The researchers in this research were interested in installing the Zero-Truncated Poisson distribution with the Pareto distribution, which resulted in a new distribution called the Zero-Truncated Poisson Pareto distribution (ZTPP). Thus the primary goal of this paper is propose a new life distribution consisting of three parameters, which is a direct extension of the Pareto distribution with two parameters. It is obtained by integrating the Pareto distribution as a continuous distribution with the Zero-Truncated Poisson distribution as a discrete distribution and the new distribution is called the Zero-Truncated Poisson Pareto distribution (ZTPP). The probability mass function of the Zero-Truncated Poisson distribution is:

$$P(N=n)=\frac{\lambda^n e^{-\lambda}}{n!\,(1-e^{-\lambda})}\ ;n=1,2,\dots,\infty \quad (1)$$

And, the cumulative distribution function of the Pareto distribution is:

$$F_1(X|\gamma,\beta)=1-\left(\frac{\beta}{X}\right)^{\gamma}\ ,X>\beta \quad \gamma,\beta>0 \quad (2)$$

Let $Y_1,\dots\dots..,Y_N$ be the series of independent symmetric distributions: $U=\min(Y_1,\dots\dots..,Y_N)$then the distribution function for U and its dynasty function are:

$$\mathrm{F}(\mathrm{u}|\theta,\lambda)=\frac{1-\mathrm{e}^{-\lambda \mathrm{F}_1(\mathrm{U}/\theta)}}{1-\mathrm{e}^{-\lambda}} \quad (3)$$

$$f(u|\theta,\lambda)=\frac{\lambda e^{-\lambda F_1(U/\theta)} f_1(v|\theta)}{1-e^{-\lambda}},$$

respectively. The *RV* $V=\max\ (\mathrm{Y}_1,\dots\dots..,\mathrm{Y}_\mathrm{N})$ has a cumulative distribution function (*cdf*) and its probability density function (pdf), given by:

$$F(v|\theta,\lambda)=\frac{e^{-\lambda S_1\left(\frac{v}{\theta}\right)}-e^{-\lambda}}{1-e^{-\lambda}} \quad (4)$$

And

$$f(v|\theta,\lambda)=\frac{\lambda e^{-\lambda s_1\left(\frac{v}{\theta}\right)} f_1(v|\theta)}{1-e^{-\lambda}}\ ,$$

This paper is structured as follows: in Section II is derived the cumulative distribution function, density function, and failure function. We also present in Section III the statistical characteristics of the distribution and in Section IV we present two different methods of estimation: the maximum likelihood method and the L. Moments method. In Section V we present the application on real data and compare the results we obtained for the distribution with other distributions. Section VI is the conclusion.

2. The ZTPP Distribution:

Let the $(Y_1,\dots,Y_n)$ series of identical independent distributions of random variables, the cumulative distribution function, the probability density function, and the failure function, respectively: $(y|\theta)$, $f(y|\theta)$, $S_1(y|\theta)$ where N has ZTP distribution with parameter λ. is given by (1) then a random variable $U=\min(Y_1,\dots\dots..,Y_N)$ is the cumulative distribution function and its probability density function, respectively:

$F(y|\theta)$, $f(y|\theta)$, $S_1(y|\theta)$ where N has ZTP distribution with parameter λ. is given by (1) then a random variable $U=\min(Y_1,\dots\dots..,Y_N)$ is the cumulative distribution function and its probability density function, respectively: by equation (2) and (3) we get cumulative distribution function and its probability density function, respectively with ZTPP:

$$F(X|\gamma,\beta,\alpha)=\frac{1-e^{-\alpha\left[1-\left(\frac{\beta}{X}\right)^{\gamma}\right]}}{1-e^{-\alpha}}\ , \qquad (5)$$

where: $0<\alpha=\lambda\ if\ X=min\ (Y_1,\dots,Y_N)$

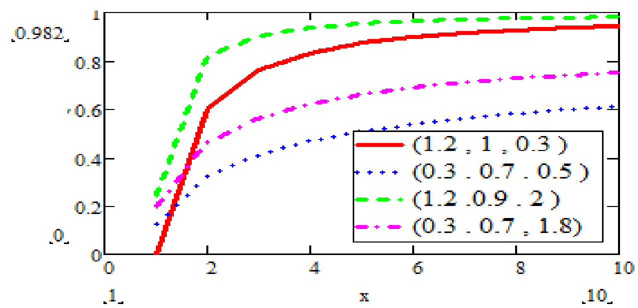

Figure 1. Plots of the ZTPP CDF for some parameter values

$$f(X|\gamma,\beta,\alpha)=\frac{\alpha\gamma\beta^{\gamma}X^{-\gamma-1}e^{-\alpha\left[1-\left(\frac{\beta}{X}\right)^{\gamma}\right]}}{1-e^{-\alpha}}$$
$$,X>\beta \quad \gamma,\beta,\alpha>0(6)$$

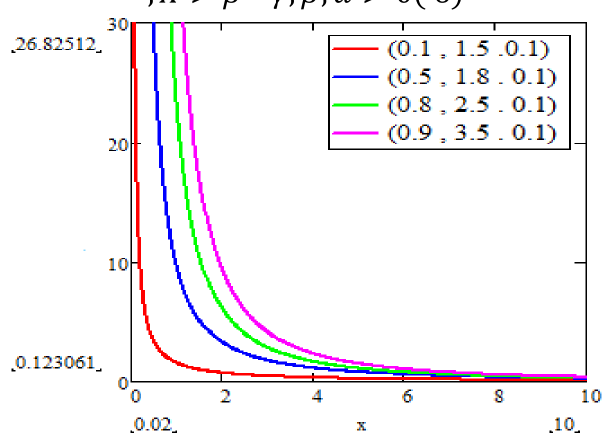

Figure 2. Plots of the ZTPP PDF for some parameter values

2.1 Survival and Hazard Rate Functions:

The survival function for the ZTPP distribution and hazard rate function of *X* the random variable with, $f(x)$ probability density function, $F(x)$ cumulative distribution function and survival function, $s(x)$ is assumed respectively, to be given by:

$$S(x) = 1 - F(x) = \frac{e^{-\alpha\left[1-\left(\frac{\beta}{X}\right)^{\gamma}\right]} - e^{-\alpha}}{1-e^{-\alpha}}$$ and

$$h(x) = \frac{f(x)}{S(x)} = \frac{\alpha\gamma B^{\gamma} e^{-\alpha\left[1-\left(\frac{\beta}{X}\right)^{\gamma}\right]}}{x^{\gamma+1}\left[e^{-\alpha\left[1-\left(\frac{\beta}{X}\right)^{\gamma}\right]} - e^{-\alpha}\right]}$$

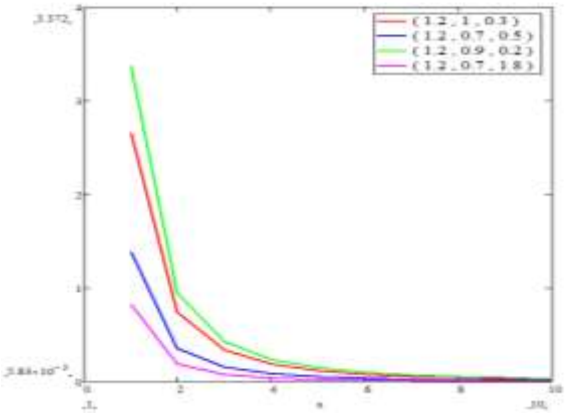

Figure 3. Plots of the ZTPP HF for some parameter values

3. Mathematical Properties

The following is a derivation of some statistical properties of the ZTPP distribution, which include moments and the quantile function and median.

3.1 General Properties

The other ordinary moment of X is given by $\mu_r^{/} = E[X^r] = \int_{-\infty}^{\infty} x^r f(x)dx$

Using (6), we obtain:

$$\bar{\mu_r} = E[X^r] = (1-e^{\alpha})^{-1}\beta^{\gamma} \left(\frac{1}{1-\beta^{\gamma}}\right)^{\frac{-r-1}{\gamma}} \sum_{j=0}^{\infty} \frac{(-\alpha)^{j+1}}{j!}\left(\frac{\gamma}{-r-1+\gamma j+\gamma}\right)$$

$$\mu = E[X] = (1-e^{\alpha})^{-1}\beta^{\gamma} \left(\frac{1}{1-\beta^{\gamma}}\right)^{\frac{-2}{\gamma}} \sum_{j=0}^{\infty} \frac{(-\alpha)^{j+1}}{j!}\left(\frac{\gamma}{-r-1+\gamma j+\gamma}\right)$$

$$(E[X^2] = (1-e^{\alpha})^{-1}\beta^{\gamma} \left(\frac{1}{1-\beta^{\gamma}}\right)^{\frac{-3}{\gamma}} \sum_{j=0}^{\infty} \frac{(-\alpha)^{j+1}}{j!}\left(\frac{\gamma}{-r-1+\gamma j+\gamma}\right)$$

$$\sigma^2 = E[X^2] - [E[X]]^2 = (1-e^{\alpha})^{-1} * \beta^{\gamma} \left(\frac{1}{1-\beta^{\gamma}}\right)^{\frac{-3}{\gamma}} \sum_{j=0}^{\infty} \frac{(-\alpha)^{j+1}}{j!}\left(\frac{\gamma}{-r-1+\gamma j+\gamma}\right) - \left[(1-e^{\alpha})^{-1}\beta^{\gamma} \left(\frac{1}{1-\beta^{\gamma}}\right)^{\frac{-2}{\gamma}} \sum_{j=0}^{\infty} \frac{(-\alpha)^{j+1}}{j!}\left(\frac{\gamma}{-r-1+\gamma j+\gamma}\right)\right]^2$$

3.2 Quantile Function:

We obtained a quantile function, which is the inverse function of equation (5), by solving the following equation $F(X) = q$ for $0 \le q \le 1$:

$$F(x) = \frac{1-e^{-\alpha\left[1-\left(\frac{\beta}{x}\right)^{\gamma}\right]}}{1-e^{-\alpha}} = q$$

$$q[1-e^{-\alpha}] - 1 = -e^{-\alpha\left[1-\left(\frac{\beta}{x}\right)^{\gamma}\right]}$$

By entering the natural logarithm on both sides of the previous equation and solving it, we get:

$$x = \beta\{\alpha^{-1}([\alpha - \ln(q(1-e^{-\alpha})] - 1)\}^{-\frac{1}{\gamma}}$$

By Substituting $q = u$ in the previous equation for, where u follows the uniform distribution [0,1], we get:

$$x = \beta\{\alpha^{-1}([\alpha - \ln(U(1-e^{-\alpha})] - 1)\}^{-\frac{1}{\gamma}} \quad (7)$$

We can use the previous inverse function to generate random numbers to simulate the random variable of the ZTTP distribution.

Median:

The median for the distribution is obtained by substituting in the previous inverse function for $(U = \frac{1}{2})$ as follows:

$$Median = \beta\left[\alpha^{-1}\left[\alpha - ln\left(\frac{1}{2}(1-e^{-\alpha}) - 1\right)\right]\right]^{-\frac{1}{\gamma}}$$

4. Estimation Methods

In this section, we offer two methods for estimating distribution parameters in (a): Maximum Likelihood Method and L. Moments in (b).

4.1 Maximum Likelihood Method:

The Maximum Likelihood method is one of the traditional and widely used methods for estimating the parameters of the model ZTPP that makes the logarithm of the Likelihood function at its end and is easy to use analytically or numerically with parameter estimators for large samples.

$$L\left(\alpha,\beta,\frac{\gamma}{x}\right) = \alpha^n\gamma^n\beta^{n\gamma}(1-e^{-\alpha})^{-n}e^{-n\alpha} + \alpha\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\gamma}\prod_{i=1}^{n}X_i^{-\gamma-1}$$

Let $\ell = \ln L(\alpha,\beta,\gamma/x)$ then:

$$\ell = n\ln\alpha + n\ln\gamma + n\gamma\ln\beta - n\ln(1-e^{-\alpha}) - n\alpha$$

$$+\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\gamma}-(\gamma+1)\sum_{i=1}^{n}\ln x_i$$

By deriving the function L for the parameters (α,β,γ) is given by:

$$\frac{\partial\ell}{\partial\beta}=\frac{n\gamma}{\beta}+\alpha\sum_{i=1}^{n}\frac{1}{x_i{}^{\gamma}}(\gamma\beta^{\gamma-1})=0$$

$$\frac{\partial^2\ell}{\partial\alpha^2}=\frac{-n}{\alpha^2}+\frac{ne^{-\alpha}}{(1-e^{-\alpha})^2}$$

The second partial derivation of L with respect to the parameters of the distribution is given by:

$$\frac{\partial^2\ell}{\partial\beta^2}=\frac{-n\gamma}{\beta^2}+\alpha\gamma(\gamma-1)\beta^{\gamma-2}\sum_{i=1}^{n}x_i^{-\gamma}$$

$$\frac{\partial^2\ell}{\partial\gamma^2}=\frac{-n}{\gamma^2}+\alpha\sum_{I=1}^{n}\left(\frac{\beta}{x_i}\right)^{\gamma}\left(\ln\left(\frac{\beta}{x_i}\right)\right)^2$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\gamma}=\sum_{I=1}^{n}\left(\frac{\beta}{x_i}\right)^{\gamma}\left(\ln\left(\frac{\beta}{x_i}\right)\right),\ \frac{\partial^2L}{\partial\alpha\partial\beta}=\sum_{i=1}^{n}\gamma\beta^{-1}\left(\frac{\beta}{x_i}\right)^{\gamma}$$

$$\frac{\partial^2\ell}{\partial\beta\partial\gamma}=\frac{\partial^2\ell}{\partial\gamma\partial\beta}=\frac{n}{\beta}+\alpha\beta^{\gamma-1}\sum_{i=1}^{n}x_i^{-\gamma}\left[1+\gamma\ln\left(\frac{\beta}{x_i}\right)\right]$$

Covariance Matrix:

Is a square and symmetric matrix that contains approximate covariances of the Maximum Likelihood estimators for the parameters of the ZTPP model of that matrix, representing the covariance between each pair of estimators and its main diagonal contains variances. It can be used to estimate model parameters with confidence intervals.

$$I(\alpha,\beta,\gamma)=-E\begin{bmatrix}\frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\beta} & \frac{\partial^2\ell}{\partial\alpha\partial\gamma}\\ \frac{\partial^2\ell}{\partial\beta\partial\alpha} & \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\partial\gamma}\\ \frac{\partial^2\ell}{\partial\gamma\partial\alpha} & \frac{\partial^2\ell}{\partial\gamma\partial\beta} & \frac{\partial^2\ell}{\partial\gamma^2}\end{bmatrix}$$

4.2 L. Moments (LM) Statistics

The characteristics of a one-variable distribution can be described using moments such as mean, variance, skewness, and kurtosis. Hosking [21] introduced another method called L-Moments and this method can be defined as a linear set of ordinal statistics in a similar way. Therefore, the mean vector and the variance and variance matrix include various elements of the covariance and its properties, which are usually used to summarize the features of multivariate distributions. To overcome this drawback, Serfling and Xiao [22] proposed the multivariate L-moments method, and its components are the central moments, but this method does not assume the central moments of the second and higher than the second must be specified. Suppose x is a continuous random variable, the cumulative distribution function $F(X)$ and the quantitative function $XF(X)$, and that:

$$LM_{r+1}=\sum_{i=0}^{r}M_{1,r,0}(-1)^{r-i}\binom{r}{i}\binom{r+i}{i},\ r=1,2,\dots,n$$

For any distribution, the first four moments can easily be calculated from the weighted moments as follows:

$$LM_1=M_{100}=\int_0^1 X(F)dF$$

$$LM_2=2M_{110}-M_{100}=\int_0^1 X(F)(2F-1)dF$$

$$LM_3=6M_{120}-6M_{110}+M_{100}=\int_0^1 X(F)(6F^2-6F-1)dF$$

$$LM_4=20M_{130}-30M_{120}+12M_{110}-M_{100}=\int_0^1 X(F)(20F^3-30F^2+12F-1)dF$$

Hosking (1990) proposed an unbiased estimator of L-moments as the following: Considering $X_1,X_2,\dots,X_n$ as the complete Lifetimes from the (ZPP) distribution with the three parameters, which is defined in (a) the probability-weighted moment is based on the following steps.

$$L\ddot{M}_{r+1}=\sum_{i=0}^{r}\widetilde{M}_{1,r,0}(-1)^{r-i}\binom{r}{i}\binom{r+i}{i}$$

Step (1): Obtain the inverse distribution $XF(X)$ of the distribution, which is given by: $XF(X)$ is the inverse function of the function F we obtained when U=F in the equation (7).

$$XF(X)=\beta\alpha^{\frac{1}{\gamma}}[([(\alpha-1)-\ln(F(1-e^{-\alpha})])]^{-\frac{1}{\gamma}}$$

Step (2): Obtain the theoretical probability-weighted moments of the $M_{1,r,0}$,where :

$M_{1,r,0}=\int_0^1 X(F)F^r dF$, $r=0,1,2\ldots$ then

$$M_{1,r,0}=\int_0^1\beta\alpha^{\frac{1}{\gamma}}[([\alpha-\ln(F(1-e^{-\alpha})]-1)]^{-\frac{1}{\gamma}}F^r dF,\ r=0,1,2\ldots$$

Letting $=(F(1-e^{-\alpha}))$, we have

$$M_{1,r,0}=\frac{-\beta\alpha^{\frac{1}{\gamma}}}{(1-e^{-\alpha})^{r+1}}\int_1^{e^{-\alpha}}(\alpha-\ln Z)^{-\frac{1}{\gamma}}\sum_{j=0}^{r}\binom{r}{j}Z^j dz$$

Letting $U=\ln Z$ we get:

$$M_{1,r,0}=\frac{\beta\alpha^{\frac{1}{\gamma}}}{(1-e^{-\alpha})^{r+1}}\sum_{j=0}^{r}\binom{r}{j}\sum_{k=0}^{\infty}\frac{(j+1)^k}{k!}$$

$$\int_0^{\alpha} (\alpha - U)^{-\frac{1}{\gamma}}\ U^K du$$

Where:

$$(\alpha - U)^{-\frac{1}{\gamma}} = \alpha^{-\frac{1}{\gamma}} \sum_{v=0}^{\infty} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1} \left(\frac{U}{\alpha}\right)^v$$

then: $M_{1,r,0} = \beta$

$$\sum_{v=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{r} \binom{r}{j} \frac{(j+1)^k}{k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1} \left(\frac{1}{k+v+1}\right)$$

$$\alpha^{-v} M_{1,0,0} = \beta \sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1}$$

$M_{1,1,0} = \beta$

$$\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1} \left[\binom{1}{0}(1)^k + \binom{1}{1}(2)^k\right] =$$

$$\beta \sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1} [(1 + 2^k)]$$

$M_{1,2,0} = \beta$

$$\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1} [(1 + 2^{k+1} + 3^k)]$$

From $M_{1,2,0}$, $M_{1,1,0}$, $M_{1,0,0}$ We get

$$\beta = \frac{M_{1,0,0}}{\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1}} \qquad (8)$$

$$\beta = \frac{M_{1,1,0}}{\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1} [(1 + 2^k)]} \qquad (9)$$

$$\beta = \frac{M_{1,2,0}}{\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1} [(1 + 2^{k+1} + 3^k)]} \qquad (10)$$

From the previous three equations, and by substitution $\frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1} = w$

we find that:

$$\frac{M_{1,0,0}}{\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} w} = \frac{M_{1,1,0}}{\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} w[(1 + 2^k)]} = \frac{M_{1,2,0}}{\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} w[(1 + 2^{k+1} + 3^k)]}$$

$$M_{1,0,0} \sum_{v=0}^{\infty}\sum_{k=0}^{\infty} w[(1 + 2^k)] = M_{1,1,0} \sum_{v=0}^{\infty}\sum_{k=0}^{\infty} w \qquad (11)$$

$$M_{1,1,0} \sum_{v=0}^{\infty}\sum_{k=0}^{\infty} w[(1 + 2^{k+1} + 3^k)] = M_{1,2,0} \sum_{v=0}^{\infty}\sum_{k=0}^{\infty} w\,[(1 + 2^k)] \qquad (12)$$

$$M_{1,0,0} \sum_{v=0}^{\infty}\sum_{k=0}^{\infty} w\left[(1 + 2^{k+1} + 3^k)\right] = M_{1,2,0} \sum_{v=0}^{\infty}\sum_{k=0}^{\infty} w \qquad (13)$$

Step(3): Replace the theoretical probability weighted moment $M_{1,0,0}$, $M_{1,1,0}$ and $M_{1,2,0}$ by their sample estimator, since the sample estimators are:

$$\widehat{M}_{1,r,0} = \frac{n^{-1} \sum_{j=1}^{n} X_J \binom{j-1}{r}}{\binom{n-1}{r}}$$

Where X_J is the order statistic, the first estimation of $M_{1,0,0}$, $M_{1,1,0}$ and $M_{1,2,0}$ is given by:

$$\hat{\beta} = \frac{\widehat{M}_{1,0,0}}{\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1}} \ ; \ \frac{1}{\gamma} \geq 1 \qquad (14)$$

$$\widehat{M}_{1,0,0} = \frac{\widehat{M}_{1,1,0} \sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1}}{\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1} [(1 + 2^k)]} \qquad (15)$$

$$\widehat{M}_{1,0,0} = \frac{\widehat{M}_{1,2,0} \sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1}}{\sum_{v=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{-v}}{(k+v+1)k!} \binom{v + \frac{1}{\gamma} - 1}{\frac{1}{\gamma} - 1} [(1 + 2^{k+1} + 3^k)]} \qquad (16)$$

Equation (15) and (16) can be solved for an unknown $\hat{\alpha}$, $\hat{\gamma}$ numerical and $\hat{\beta}$ becomes easy from equation (14).

5. Application

Here, we compare the ZTPP distribution to the BP beta distribution, the exponentiated pareto EP distribution, the Pareto distribution P, and the BEP beta exponentiated pareto distribution using an actual data sample. The models' fit is compared with the outcomes. We take into consideration an unedited data set that represents to remission times in months includes 128 bladder cancer patients chosen at random. Lee et. al [23] and Lemonte et.al [24] already examined these data. In the case of bladder cancer, abnormal cells in the bladder become out of control. Transitional cell carcinoma, the most prevalent form of bladder cancer, mimics the typical urothelial histology. The statistics data are as follows:

0.08, 0.402.02, 2.02,2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.690.20, 0.50, 0.90, 1.05, 0.51, 0.81, 1.35, 1.40, 1.19, 1.26, 1.76, , 2.75, 2.831.46, 3.02, 3.25, , 3.36, 3.36, 3.48, , 3.31, 3.52, 3.57, 3.70, 3.82, 3.88, 4.18, 3.64, 4.23, 4.26, 4.33, , 4.40, 4.50, 4.51, 4.34, 4.87, 4.98, 5.06, 5.17, 5.32, 5.32, 5.09, 5.34, 5.41, 5.49, 5.62, 5.71, 5.41, 5.85, 6.25, 6.54, 6.93, 6.94, 6.97, 6.76, 7.09, 7.26, 7.32, 7.39, 7.59, 7.28, 7.62, 7.63, 7.66, 7.87, 8.26, 8.37, 7.93, 8.53, 8.65, 9.02, 9.22, 9.47, 8.66,9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.63, 13.11, 12.07, 13.29, 13.80, 14.76, 14.24, 14.77, 14.83, 16.62, 17.12, 15.96, 17.14, 17.36, 19.13, 20.28, 18.10, 21.73, 22.69, 25.74, 25.82, 23.63, 26.31, 32.15, 36.66, 43.01, 46.12, 34.26, 79.05.

Table 1. ML Estimates and Information Criteria

Model	MLE Estimates					Statistic		
	$\hat{a}$	$\hat{b}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	AIC	BIC	CAIC
ZTPP			0.099	13.84	0.75	603.4	617.3	603.5
Pareto			0.1519	0.0800	-	1189.3	1192.1	1189.3
BEP	0.348	15983	0.0508	0.0800	8.6121	874.8	886.2	875.1
BP	4.805	100.5	0.0109	0.0800	-	970.7	979.2	970.9
EP			0.4722	0.0800	4.1518	992.2	997.9	992.3

Comparisons of models entailed the consideration of various criteria such as maximized likelihood -2ℓ, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian information criterion (BIC). The minimum values rule of AIC, BIC, CAIC is taken into consideration for selecting the best model to fit. These statistics are given by $AIC = -2\hat{\ell} + 2K, BIC = -2\hat{\ell} + KLog(n),\ , CAIC = -2\hat{\ell} + {}^{2Kn}/_{(n-k-1)}$ where n is a sample size, ℓ is log-likelihood and k is the number of parameters. Results show that our model satisfied the minimum rule, hence it is the best one.

6. Discussion and Conclusion

In this study, we introduce a new distribution of life called Zero Truncated Poisson Pareto distribution (ZTPP). Despite the multiplicity of research in the field of compound distributions, but the authors do not discuss properties for distributions (ZTP) and (P) based on a double integrating mechanism to the Pareto (P) distribution as a continuous distribution with the Zero-Truncated Poisson (ZTP) distribution as a discrete distribution, with three parameters, thus in this paper, we derive some structural properties of the (ZTP) and (P) distributions based on a double integrating mechanism to it's with three-parameter lifetime which resulted in a new distribution called the Zero-Truncated Poisson Pareto distribution (ZTPP) or in short ZTPP distribution. with three parameters, the suggested distribution has an advantage over other distributions in that it makes estimating the model parameters simpler. To estimate the unknown parameters of the ZTPP distribution, the maximum likelihood method and L, method are employed. The distribution was applied to real data to ensure the possibility of applying it to life data and through the application, the distribution ZTPP was compared with other distributions, as Pareto, BEP, BP, and EP, by the comparison proving that the new distribution is better than the distributions that had been compared with its in economics, and other fields. Overall the result indicated the ZTPP is better than the other distributions where the criteria values of (AIC), (CAIC), and (BIC) are minimum values by using the ZTPP distribution as shown in table 1.

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Conflicts of Interest:

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Conflict of Interest

The authors declare that they have no conflicts of interest