Comparison of Several Topologies Generated by the Convergence

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Abstract: This paper studies the construction of three functional topologies in Y^X production spaces. Their topological bases have been found and compared between them. An important place is a comparison with well-known topologies such as uniform, point, and open compact convergence. The convergences used are uniform local convergence, strongly uniformed local convergence, and the convergence known as α-convergence.

Keywords. Generation of new topologies, locally uniform convergence, convergence locally uniformly strongly, α-convergence, closure operator.

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1 Introduction

Let's briefly present some of the concepts we used to carry out this study.

Definitions 1.1: [1] Let (X, d), (Y, p) be two metric spaces x∈X, and $f_n, f : X \to Y$. The function $f(x)$ is a δ-limit of the sequence $(f_n)_{n\in N}$ if for every ε>0, there exists $n_0(\varepsilon, x) \in \mathbb{N}$ and $\delta > 0$ that for every $n \geq n_0$, and $y \in S(x,\delta)$, we have

$p(f_n(y), f(y)) < \varepsilon$.

Thus, we can say that $\left(f_n(x)\right)_{n\in N}$ converges locally uniformly to f(x).

(2). Let (X, d), (Y, p) be two metric spaces, x∈X, and $f_n, f : X \to Y$. We say that the sequence $(f_n)_{n\in N}$ is locally uniformly strongly convergent (or short

δ_a -convergent) to f(x) if for every ε>0 and x∈X, there exists $n_0(\varepsilon, x) \in \mathbb{N}$ and δ>0, such that for $n \geq n_0(\varepsilon, x)$ and $y \in S(x,\delta)$ we have

$p(f_n(y), f(x)) < \varepsilon$

(3) It is said that the sequence of functions $f_n(x_n) : (X, d) \to (Y, \rho)$ is α-convergent in X if: $x_n \to x$ then $f_n(x_n) \to f(x)$. This means that when $x_n \overset{d}{\to} x$ then $f_n(x_n) \overset{\rho}{\to} f(x)$, [3].

Thus we can say that $(f_n)_{n\in N}$ *converges locally uniformly to* f(x), [1].

Definition 1.2. The closure operator is called the operator that enjoys the following properties:

$(CO1)$ $\overline{\emptyset} = \emptyset$

$(CO2)$ $A \subset \bar{A}$

$(CO3)$ $\overline{A \cup B} = \bar{A} \cup \bar{B}$

$(CO4)$ $\overline{(\bar{A})} = \bar{A}$.

Let X be any set, by generating a topology on X we mean the selection of the family τ of subsets of X that satisfies the known conditions of open sets. It is often more useful not to describe the family τ of open sets directly. The other methods consist first of all in defining a family that serves as the basis of the topology, or of the adjacency system, of the closure operator or the interiority operator. In this paper, we are considering the closure operator.

Suppose that given a set X and closure operator, each A⊂X defines a set $\bar{A}$⊂X such that conditions (C01) -(C04) are satisfied. The family τ= {X|A: A=$\bar{A}$} satisfies the conditions of open sets and for each A⊂X the set $\bar{A}$ is the closure of A in the topological space (X, τ). The topology τ is called the topology generated by the closure operator.

2 Main Results

Let us carry out in this section the comparison of some uniform Cartesian production topologies.

Let X and Y be two topological spaces corresponding to the product Y^X of all continuous functions leading X to Y. We denote by F a family of functions contained in Y^X and we raise the question of whether there are topologies in Y^X belonging to the family F. Let's examine the mechanism of construction of these topologies related to the family of functions.

Let A⊂Y^X and f∈Y^X where X, Y be two metric spaces. We define these convergences:

(1) $f \in \bar{A}$ if $f = \lim f_i$, where $f_i \in A$, for i=1,2, …,

With the equation $f = \lim f_i$, we understand that the sequence (f_i) is uniformly convergent to f.

(2) Let x ∈X and V be a neighborhood of f(x). Denote $B = f^{-1}(V)$) and $C = V^B$, then $f \in \bar{C}$ if $f = \lim f_i$, where $f_i \in C$, for i=1, 2, … and with $f = \lim f_i$ we understand uniform convergence to f for every $\bar{x} \in B$.

(3) Let $x \in X$ and $V_{f(x)}$ be the neighborhood of $f(x)$. We denote $= f^{-1}(V_f)$, where $E = \left(V_{f(x)}\right)^D$, $f \in \bar{E}$ if $\lim f_i = f, f_i \in E$, and with $\lim f_i = f$, we understand that $(f_i(\bar{x}), f(x)) \in V_{f(x)}$ for every $\bar{x} \in D$.

(4) Let $x \in X$ and $V_{f(x)}$ be the neighborhood of $f(x)$. We denote $= f^{-1}(V_f)$, where $G = \left(V_{f(x)}\right)^F$ $f \in \bar{G}$ if $\lim f_i = f, f_i \in G$, and with $\lim f_i = f$ for i=1, 2, …, we understand that

$(f_i(x_i), f(x)) \in V_{f(x)}$ for every $x_i \in F$.

Proposition 2.1. The closure operator defined on Y^X by each of the formulas (1), (2), (3), (4) satisfies the conditions (C1), (C2), (C3), (C4) and defines the respective topology.

Proof. We do the proof only for (2) since they are similar for other operators (for (1) see, [2]).

Condition (C1) is fulfilled. Since when $f_i = f$, i=1, 2, … it follows that $\lim f_i = f$ it turns out that the condition (C2) is also fulfilled.

It immediately follows from (2) that

(ii) when $A \subset B$ then $\bar{A} \subset \bar{B}$,

since, to prove condition (C3), it suffices to show that

(iii) $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

Taking $f \in \overline{A \cup B}$ it follows that there exists the sequence (f_i) of functions belonging to A∪B and that $f = \lim f_i$ which means for $f \in A$ or in B and every $C = V^D$, where V is a neighborhood of f, there is δ>0, that for x ∈D and $n > n(x,\delta)$, $f_n(x) \in V$ or that $f_n \in V^D \subset A$. This shows that (C3) is satisfied.

It follows from statement (ii) that $\bar{A} \subset \overline{(\bar{A})}$.

In order to prove the condition (C4), it is sufficient to prove that

(iv) $\overline{(\bar{A})} \subset \bar{A}$.

Taking $f \in \overline{(\bar{A})}$ the sequence (f_i) will belong to $\bar{A}$ and satisfy the equation $f = \lim f_i$ which means that if $f \in C \subset Y^X$ for every $V_{f(x)}$, there is $\delta > 0$, and $i(x,\delta)$, such that for

(v) $\bar{x} \in B$ and $i(k) > i(x,\delta)$, $f_{i(k)}(\bar{x}) \in V_{f(x)}$,

from which it follows that $f_{i(k)} \in \bar{C}$. But for this function the sequence g_i^k, such that $\lim g_j^k(x) = f_{i(k)}(x)$, where $g_i^k \subset C$ and $C = V^D$ for j=1, 2, ... g and here it will be found $j(k)$, such that

(vi) $g_{j(k)}(x) \in V_{f_{i(k)}}$ $V_{f_{i(k)}} \subset V_{f(x)}$,$\forall x \in D$.

Since the functions g_j^k belong to C for k=1, 2, 3… from (v) and (vi) it follows that $f \in \bar{C}$ which completes the proof.

(I) Following, [2], the topology generated by operator (1) is called the uniform convergence topology on Y^X. It can be easily verified that for a ∈Y^X , the family $\{U_i(f)\}_{i=1}^{\infty}$, where

$U_i(f) = \left\{g \in Y^X\text{: exists } \alpha < \frac{1}{i}\text{, such that } d\big(f(x), g(x)\big) < \alpha \text{ for every } x \in X\right\}$

is a basis at f of the space Y^X with the topology of uniform convergence.

For the topologies introduced in this paper we will have:

II) The topology generated by the operator (2) is called locally uniform convergence topology in Y^X.

For each $f \in Y^X$ family $\{V_j(f)\}_{i=1}^{\infty}$, where

$$V_f = B(f, \alpha),$$
$$V_i(f) = \{g \in Y^X\text{: where for every } \alpha < \frac{1}{j}\text{, there exist U} = f^{-1}(V_f)\text{ that for } x \in f^{-1}(V_f)\text{ we have that } g(x) \in V_{f(x)}\}$$

is a basis at f for the space Y^X endowed with the locally uniform topology.

(III) The topology generated by operator (3) is called the topology of δ_a-convergence in Y^X. Here too for each $f \in Y^X$, family $\{W_j(f)\}_{i=1}^{\infty}$, where

$$W_i(f) = \left\{g \in Y^X\text{: where for every } V_f = B(f, \alpha), \alpha < \frac{1}{j}\text{, there exist U} = f^{-1}(V_f)\text{ that for } \bar{x} \in f^{-1}(V_f)\text{we have that } g(\bar{x}) \in V_{f(x)}\right\}$$

The topology generated by operator (4), is called the topology of α-convergence in Y^X. Here too for each f$\in Y^X$, family $\{W_j(f)\}_{i=1}^{\infty}$, where

$$W_i(f) = \left\{g \in Y^X\text{: where for every } V_f = B(f, \alpha), \alpha < \frac{1}{j}\text{, exist U} = f^{-1}(V_f)\text{ that for } x_i \in f^{-1}(V_f)\text{, we have that } g(x_i) \in V_{f(x)}\right\}$$

This family is a basis at f for the space Y^X equipped with the δ_a-topology. Considering the topology in a subspace, we conclude e.g., that the space of uniform convergence in $\mathbb{R}^X$ induces the uniform topology of the subspace in I^X, where I⊂P is the unit interval.

Regarding this subspace, the statement has been proved:

Proposition 2.2. For any topological space X, the set I^X is closed in the space with uniform convergence topology, [2].

Let X and Y be two arbitrary topological spaces for which A⊂X and B⊂Y.

We determine

(vii) $\mu(A, B) = \{f \in Y^X : f(A) \subset B\}$.

Denote $\mathcal{F}$ the family of finite subsets of Y^X and denote τ the topology on Y. The family β of all sets $\bigcap_{i=1}^{k} M(A_i, U_i)$, where $A_i \in \mathcal{F}$ and $U_i \in \tau$ for i=1, 2, ..., k generates, as is known, [2], [3] a topology in Y^X which is called the topology of pointwise convergence in Y^X.

Due to the proof below, we bring to attention, together with the argument, a well-known statement in topology following in this case [2].

Proposition 2.3. The topology of pointwise convergence in Y^X coincides with the topology of the Cartesian product subspace $\prod_{x \in X} Y_x$ where $Y_x = Y$ for every $x \in X$.

Proof. As is known, any open space in Y^X, equipped with the topology of the Cartesian product subspace $\prod_x Y_x$ is a union of sets in the form

(viii) $Y^X \cap p_{x_1}^{-1}(U_1) \cap p_{x_2}^{-1}(U_2) \cap \cdots \cap p_{x_k}^{-1}(U_k)$

where $x_i \in X$ and $U_i \in \tau$ for i=1, 2, …, k.

But considering that

(ix) $Y^X \cap p_{x_1}^{-1}(U) = \mu(\{x\}, U)$

it turns out that the sets of the form (viii) and all the sets that are open concerning the topology of the Cartesian product subspace are open concerning the pointwise convergence topology.

Conversely, from equation (ix) it follows that for $A = \{x_1, x_2, \dots, x_n\} \in \mathcal{F}$ and $U \in \tau$ we will have $\mu(A, U) = Y^X \cap p_{x_1}^{-1}(U) \cap p_{x_2}^{-1}(U) \cap \cdots \cap p_{x_k}^{-1}(U)$ which means that all sets that are open concerning the pointwise convergence topology are also open concerning the topology of a subspace of the Cartesian product, [2].

Corollary 2.4. A net $\{f_\sigma(x), \sigma \in \Sigma\}$ in the space Y^X with pointwise convergence topology converges to $f \in Y^X$ if and only if the net $\{f_\sigma(x), \sigma \in \Sigma\}$ } converges to f(x) for every x ∈X, [1].

If we compare the topologies constructed from definitions (1), (2), (3), and (4) it is not difficult to establish a ranking. Since for each C (of convergences (2) and (3)), there exists an A such that $\bar{A} \subset \bar{C}$ then the topology of uniform convergence is richer than the topology of locally uniform convergence. Likewise, since for every C, there is a D such that $\bar{D} \subset \bar{C}$, it turns out that the topology of δ_a-convergence is richer than the topology of locally uniform convergence by one more type, $\overline{F} \subset \overline{D}$, so, we have proved the assertion.

Theorem 2.5.: $\tau_{l.u} \subset \tau_{\delta_a} \subset \tau_\alpha \subset \tau_u$,
where $\tau_{l,u}$ is the locally uniform convergence topology, τ_{δ_a} is the convergence's topology, τ_α is the α-convergence topology, and τ_u is the uniform convergence topology.
The above inclusions can be given in the metric spaces also using the corresponding spheres:
(1) $[B,\varepsilon, \delta]^{l.u.} = \{(f, g): \forall\varepsilon,\exists\delta; p(f(y), g(y)) < \varepsilon, y\in S(x,\delta)\}$
(2) $[B,\varepsilon, \delta]^{\delta a} = \{(f, g): \forall\varepsilon,\exists\delta; p(f(x), g(y)) < \varepsilon, y\in S(x,\delta)\}$
(3) $[B,\varepsilon, \delta]^{\alpha} = \{(f, g): \forall\varepsilon,\exists\delta; p(f(x), g(x_i)) < \varepsilon, x_i\in S(x,\delta)\}$
(4) $[B,\varepsilon]^{u} = \{(f. g): p(f(x), g(x)) < \varepsilon, \forall x \in X\}$
Knowing that the topology of uniform convergence is richer than the topology of pointwise convergence, the question arises: If the other two convergences shown above are between these two topologies?

Proposition 2.6. For any topological space X, the locally uniform convergence topology on $\mathbb{R}^X$ is richer than the pointwise convergence topology.
Proof: The equivalence of conditions (i) and (v) in proposition 1.4.1, [2], it is shown that it suffices to prove that when $f \in \mathbb{R}^X$ is in the closure of the set $C \subset \mathbb{R}^X$, concerning the locally uniform convergence topology then f is in the closure of A concerning the pointwise convergence topology.
Let
$U = \mathbb{R}^X \cap \bigcap_{i=1}^{k} p_{x_i}^{-1}(U_i)$
be a neighborhood of f in the pointwise convergence topology of Proposition 2.3. As long as U_i are opened in $\mathbb{R}$, there is an $\varepsilon > 0$, such that $V =]f(x_i) - \varepsilon; f(x_i) + \varepsilon[\subset U_i$, for $i = 1, 2, .., k$. As long as f is continuous $f^{-1}(V)$ is opened and $x_i \in B = f^{-1}(V)$. Since, $f = \lim f_i$, it follows that $x \in B = f^{-1}(V) \Rightarrow \left|f(x) - f_j(x)\right| < \varepsilon$ for j=1, 2,.., n and every $\varepsilon > 0$. This shows that $f_j \in B^V = C$.It follows that $f \in \bar{C}$.
In many manuals of topology and functional analysis such as, [2], [3], related to topologies in the space of continuous functions Y^X, where X and Y are topological spaces, a prominent place is the study of topological spaces of pointwise convergence and also of compact open topology. Following [4], it is proved that the compact-open topology is richer than the topology of pointwise convergence. Let us compare below an independent way of how the topology of locally uniform convergence is related to the compact open-convergence. The compact-open topology in Y^X is a topology generated from the basis consisting of the sets $\bigcap_{i=1}^{k} M(C_i, U_i)$, where K_i is a compact set in X and U_i is an open set in Y for i=1, 2, ..., k. In general, the compact-open topology in R^X differs from the topology of uniform convergence, however, in the case of R^N, it is observed that this topology coincides with the topology of pointwise convergence.

Proposition 2.7. For any topological space X, the locally uniform convergence topology on $\mathbb{R}^X$ is richer than the compact open topology.
Proof: Even here it suffices to prove that when $f \in \mathbb{R}^X$ is in the closure of the set C considered in (2), where $C \subset \mathbb{R}^X$ concerning the locally uniform topology then f is in the closure A in relation to compact open topology.
As we noted, the basis of the open compact topology has the form

$$B = \bigcap_{i=1}^{k} M(K_i, U_i) ,$$

where, K_i are compact sets in X, whereas U_i are opened sets in P and
$M (K_i, U_i) = \{f: f(K_i) \subset U_i\}$,

Let $U = \mathbb{R}^X \cap \bigcap_{i=1}^{k} M(C_i, U_i)$, be a neighbourhood of f in the compact open topology, $f(x_i) \in U_i$ and U_i opened in P, then there exists $\varepsilon > 0$, such that $]f(x_i) - \varepsilon; f(x_i) + \varepsilon[\subset U_i$ for i=1, 2,…, k. Given the definition of locally uniform convergence $f(x) \overset{\delta}{=} \lim_{j\to\infty} f_j(x)$, for every ε > 0 and for x∈X, there exists δ_x and n (x, ε), such that for $x_i \in S(x,\delta)$, it follows that $|f_j(x)-f(x)| < \varepsilon$.

Since $f(x_i) \in U_i$ it follows that $x_i \in K_i$. From the fact that K_i is a compact set from any covering $\{S(x, \delta_x)\}$, any covering of it will yield a finite sub-covering.

If x_i will take part in one of them, e.g., $x_i \in S(x_j, \delta_{x_j})$ then $|f_j(x_i)-f(x_i)| < \varepsilon$. This means that $f_j(x_i) \in U_i$ from which it follows that $U \cap A \neq \varnothing$.

3 Conclusion

It was proved that the α topology is richer than the δ_a topology, richer than the locally uniform topology, and that the locally uniform topology is richer than the compact-open topology.

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