A new algorithm for the split feasibility problem with multiple output sets in Hilbert spaces

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Abstract: - In this paper, we study the split feasibility problem with multiple output sets in Hilbert spaces. We propose a new self-adaptive algorithm combing with ball-relaxation and inertial acceleration, and prove its strong convergence. Numerical simulations are provided to illustrate the effectiveness of the proposed algorithm.

Key-Words: - split feasibility problem with multiple output sets, self-adaptive step size, ball-relaxation, inertial acceleration, strong convergence.

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1 Introduction

The multiple-sets split feasibility problem (MSSFP) is to find $x^* \in H_1$ such that

$$x^* \in \bigcap_{i=1}^{t} C_i, \quad Ax^* \in \bigcap_{j=1}^{r} Q_j, \tag{1}$$

where $C_i, i = 1, 2, \cdots, t \subset H_1, Q_j, j = 1, 2, \cdots, r \subset H_2$ are nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator.

It is obviously that if $r = t = 1$, the MSSFP is reduced to the split feasibility problem (SFP).

The SFP and the MSSFP were first proposed by Censor and Elfving in [1] and [2] for modeling certain inverse problems, which have been widely used in many application fields, such as, medical image reconstruction, [1, 3, 4], intensity-modulated radiation therapy (IMRT), [5, 6], and gene regulatory network inference, [7], etc. Many authors have also made a continuation of the study on the MSSFP and its variant form, for instance, see, [8–16].

Recently, Reich et al. proposed the split feasibility problem with multiple output sets in [14]. Let $H, H_j, j = 1, 2, \cdots, r$, be real Hilbert spaces and let $A_j : H \to H_j, j = 1, 2, \cdots, r$, be bounded linear operators. Let C and Q_j be nonempty, closed and convex subsets of H and $H_j, j = 1, 2, \cdots, r$, respectively. Find an element x^*, such that

$$x^* \in S = C \cap (\bigcap_{j=1}^{r} A_j^{-1}(Q_j)). \tag{2}$$

They also provided algorithms for solving this problem.

In this paper, we study a slightly generalized multiple-sets split feasibility problem with multiple output sets: Let H, $H_j, j = 1, 2, \cdots, r$, be real Hilbert spaces and let $A_j : H \to H_j, j = 1, 2, \cdots, r$, be bounded linear operators. Let C_i and Q_j be nonempty, closed and convex subsets of H and $H_j, j = 1, 2, \cdots, r$, respectively. Find an element x^*, such that

$$x^* \in S = \bigcap_{i=1}^{t} C_i \cap (\bigcap_{j=1}^{r} A_j^{-1}(Q_j)). \tag{3}$$

In other words, the aim is to find an $x^* \in C_i$ such that $A_j x^* \in Q_j$ for all $i = 1, 2, \cdots, t, j = 1, 2 \cdots, r$.

If $t = 1$, the problem (3) reduces to the problem (2). If $A_j \equiv A, H_j \equiv H_1$, the problem (3) reduces to the MSSFP (1).

Many iterative methods have been proposed for solving the SFP. One of the well-known algorithms is the CQ method proposed by Byrne, [3], which is formulated as follows

$$x_{n+1} = P_C(x_n - \alpha_n A^*(I - P_Q)Ax_n), \tag{4}$$

where the step size $\alpha_n \in (0, \frac{2}{\|A\|^2})$, and P_C and P_Q stand for the metric projection onto C and Q, respectively.

Since the projections onto a general nonempty closed convex subset is hard to be implemented, Yang [15] proposed the half-space relaxation projection CQ algorithm. Yu et al. [16] introduced the ball-relaxed projection CQ algorithms.

Since the norm estimation of $||A_j||$ for step size is hard to get, Several choice of the self-adaptive step size have been presented, see for instance, Yang [17], López et al. [18], Gibali et a.l. [19], etc.

To achieve a faster convergence of the algorithms, many references have investigated the inertial technique, see for example, Suantai et al. [20], etc.

In this paper, we adopt the ball-relaxation, a new self-adaptive step size and inertial acceleration technique to the algorithm solving the problem (3). Since the orthogonal projections onto balls and the self-adaptive step size can be directly calculated, the proposed algorithm is easy to implement.

The rest is outlined as follows. Some useful concepts and lemmas for our analysis are reviewed in the next section. In section 3, we present our algorithm and prove its strong convergence. Finally, in section 4, we exhibit a numerical example in order to illustrate our results and observe the performance of our algorithm.

2 Preliminaries

In this section, we introduce some definitions and basic lemmas that will be used in the sequel. Let H be a real Hilbert space, and its inner product and norm be expressed by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Besides, we use the symbol $x_n \to x$ ($x_n \rightharpoonup x$) to express that the sequence $\{x_n\}$ converges strongly (weakly) to x.

Definition 2.1 Let C be a nonempty closed convex subset of H. Then the mapping $T : C \to H$ is said to be:
(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{5}$$

(2) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C, \tag{6}$$

or equivalently if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall x, y \in C, \tag{7}$$

where I is the identity operator.

Definition 2.2 Let C be a nonempty, closed and convex subset of H. The metric projection $P_C : H \to C$ defined by

$$P_C(x) = \arg\min_{y \in C} \|x - y\|^2, \quad x \in C. \tag{8}$$

Definition 2.3 Let $f : H \to (-\infty, +\infty]$ be a proper function. Then f is said to be weakly lower semicontinuous at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n \to \infty} f(x_n). \tag{9}$$

f is lower semicontinuous on H if it is lower semicontinuous at every point $x \in H$ and f is weakly lower semicontinuous on H if it is weakly lower semicontinuous at every point $x \in H$.

Lemma 2.1 [21] Let C be a nonempty closed and convex subset of H. Then for all $x, y \in H$ and $z \in C$, we have the following statements:
(1) $\langle x - P_C(x), y - P_C(x) \rangle \leq 0$;
(2) P_C and $I - P_C$ are both firmly nonexpansive;
(3) $\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2$;
(4) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$.

Lemma 2.2 [21] Let $f : H \to (-\infty, +\infty]$ be a strongly convex function with constant λ. Then for all $x, y \in H$,

$$f(y) \geq f(x) + \langle \xi, y - x \rangle + \frac{\lambda}{2}\|y - x\|^2, \quad \xi \in \partial f(x) \tag{10}$$

Lemma 2.3 [4] Let H_1 and H_2 be real Hilbert spaces and $f : H_1 \to \mathbb{R}$ is given by $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ where Q is closed convex subset of H_2 and $A : H_1 \to H_2$ be a bounded linear operator. Then
(1) the function f is convex and weakly lower semicontinuous on H_1;
(2) $\nabla f(x) = A^*(I - P_Q)Ax$, for $x \in H_1$;
(3) ∇f is $\|A\|^2$-Lipschitzian continuous, i.e., $\|\nabla f(x) - \nabla f(y)\| \leq \|A\|^2\|x - y\|, \forall x, y \in H_1$.

Lemma 2.4 [22] Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\delta_n, \quad n \geq 1, \tag{11}$$

$$s_{n+1} \leq s_n - \eta_n + \gamma_n, \quad n \geq 1, \tag{12}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers, $\{\delta_n\}$ and $\{\gamma_n\}$ are two sequences in $\mathbb{R}$ such that
(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(2) $\lim_{n \to \infty} \gamma_n = 0$;
(3) $\lim_{k \to \infty} \eta_{n_k} = 0$ implies $\limsup_{k \to \infty} \delta_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$.
Then $\lim_{n \to \infty} s_n = 0$.

3 Algorithm and its convergence

In this section, we introduce ball-relaxed algorithm with a new self-adaptive step size and inertial acceleration for solving the problem (3) and prove its strong convergence.

Set

$$C_i = \{x \in H : c_i(x) \leq 0\}, \tag{13}$$

$$Q_j = \{y \in H_j : q_j(y) \leq 0\}, \tag{14}$$

where $c_i(x), i = 1, 2, \cdots, t$ and $q_j(y), j = 1, 2, \cdots, r$ are convex, weakly lower semi-continuous functions, respectively.

If $c_i(x), i = 1, 2, \cdots, t$ and $q_j(y), j = 1, 2, \cdots, r$ are λ_i- and γ_j- strongly convex, define a series of sets $C^b_{i,n}$ and $Q^b_{j,n}$, $n \geq 1$, by

$$C^b_{i,n} = \{x \in H : c_i(x_n) + \langle \xi^n_i, x - x_n\rangle + \frac{\lambda_i}{2}\|x - x_n\|^2 \leq 0\}, \quad (15)$$

$$Q^b_{j,n} = \{y \in H_j : q_j(A_j x_n) + \langle \zeta^n_j, y - A_j x_n\rangle + \frac{\gamma_j}{2}\|y - A_j x_n\|^2 \leq 0\}, \quad (16)$$

where $\xi^n_i \in \partial c_i(x_n)$, $i = 1, 2, \cdots, t$ and $\zeta^n_j \in \partial q_j(A_j x_n)$, $j = 1, 2, \cdots r$.

It is easy to verify that $C^b_{i,n}, i = 1, 2, \cdots, t$ and $Q^b_{j,n}, j = 1, 2, \cdots, r$ are closed balls containning C and Q, respectively, see, [23].

Define that

$$\begin{aligned} d_n &= \max_{i=1,\cdots,t} \|x - P_{C_i}(x)\|, \\ v_n &= \max_{j=1,\cdots,r} \|A_j x - P_{Q_j}(A_j x)\|, \end{aligned} \quad (17)$$

Then the problem (3) is equivalent to the following minimization problem:

$$\min f(x) = \frac{1}{2}\sum_{i=1}^{t} l_i \|x - P_{C_i}(x)\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j \|A_j x - P_{Q_j}(A_j x)\|^2. \quad (18)$$

where $l_i, i = 1, \cdots, t$ and $\lambda_j, j = 1, \cdots, r$ are all positive constants such that $\sum_{i=1}^{t} l_i + \sum_{j=1}^{r} \lambda_j = 1$.

Using (17), the problem (18) is equivalent to the following minimization problem:

$$\min f(x) = \frac{1}{2}\Phi^2_n. \quad (19)$$

where $\Phi_n = \max\{d_n, v_n\}$.

In the sequel, we assume that the following three assumptions hold.

(A1) The solution set S of (3) is nonempty.

(A2) The functions $c_i : H \longrightarrow \mathbb{R}$ and $q_j : H_j \longrightarrow \mathbb{R}$ defined in (13) and (14) are λ_i- and γ_j- strongly convex lower semicontinuous functions.

(A3) For any $x \in H$ and $y_j \in H_j$, at least one subgradient $\xi_i \in \partial c_i(x)$ and $\zeta_j \in \partial q_j(y_j)$ can be calculated. The subdifferentials ∂c_i and ∂q_j are bounded on the bounded sets.

Algorithm 3.1 For any initial point x_0, $x_1 \in H$, the sequence $\{x_n\}$ be defined as follows:

1. Compute y_n

$$y_n = x_n + \beta_n(x_n - x_{n-1}). \quad (20)$$

2. Compute d_n and define L_n

$$d_n = \max_{i=1,\cdots,t} \|y_n - P_{C^b_{i,n}}(y_n)\|, \quad (21)$$

$$L_n = \{i \in \{1, 2, \cdots, t\} : \|y_n - P_{C^b_{i,n}}(y_n)\| = d_n\}. \quad (22)$$

3. Compute v_n and define H_n

$$v_n = \max_{j=1,\cdots,r} \|A_j y_n - P_{Q^b_{j,n}}(A_j y_n)\|, \quad (23)$$

$$H_n = \{j \in \{1, 2, \cdots, r\} : \|A_j y_n - P_{Q^b_{j,n}}(A_j y_n)\| = v_n\}. \quad (24)$$

4. If $d_n \geq v_n$, then choose $i_n \in L_n$ and let $\Delta = I$, $\mu_n = P^b_{C_{i_n,n}} y_n$, $f_n(y_n) = \frac{1}{2}\|(I - P_{C^b_{i_n,n}})y_n\|^2$; else choose $j_n \in H_n$ and let $\Delta = A_{j_n}$, $\mu_n = P^b_{Q_{j_n,n}} A_{j_n} y_n$, $f_n(y_n) = \frac{1}{2}\|(I - P_{Q^b_{j_n,n}})A_{j_n} y_n\|^2$.
5. Compute x_{n+1}

$$x_{n+1} = \alpha_n g(y_n) + (1 - \alpha_n)(y_n - \tau_n \nabla f_n(y_n)), \quad (25)$$

where

$$\tau_n = \rho_n \frac{\|\Delta y_n - \mu_n\|^2}{\|\Delta^*(\Delta y_n - \mu_n)\|^2 + \theta_n}, \quad (26)$$

$\alpha_n \in (0, 1)$, $\rho_n \in (0, 2)$ and $\{\theta_n\}$ is a bounded sequence of positive real numbers. $g : H \longrightarrow H$ is a strict contraction mapping H into itself with the contraction coefficient $c \in [0, 1)$.

Now we establish the strong convergence for Algorithm 3.1.

Theorem 3.1 Let H and H_j be real Hilbert spaces, $C_i, i = 1, 2, \cdots, t$ and $Q_j, j = 1, 2, \cdots, r$ be nonempty, closed and convex subsets of H and H_j, respectively. Let $A_j : H \longrightarrow H_j, j = 1, 2, \cdots, r$ be bounded linear operators with their adjoint denoted by A^*_j. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and ρ_n satisfy the following conditions:
(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\inf_{n\in\mathbb{N}} \rho_n(2 - \rho_n) > 0$;
(C3) $\{\beta_n\} \subset [0, \beta]$, where $\beta \in [0, 1)$ and $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$;
(C4) $0 < \inf\{\theta_n\} \leq \sup\{\theta_n\} < +\infty$.
Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $z \in S$, and z is the unique solution to the variational inequality:

$$\langle (I - g)(z), y - z\rangle \geq 0, \quad \forall y \in S, \quad (27)$$

Proof Note that $g : H \to H$ is contractive, so $P_S g$ is also contractive, thus $P_S g$ has a unique fixed point z, which by Lemma 2.1(1) is the unique solution of (27).

Notice that

$$\begin{aligned} &\|x_{n+1} - z\| \\ &= \|\alpha_n g(y_n) + (1-\alpha_n)(y_n - \tau_n \nabla f_n(y_n)) - z\| \\ &\le \alpha_n \|g(y_n) - z\| \\ &\quad + (1-\alpha_n)\|y_n - \tau_n \nabla f_n(y_n) - z\|. \end{aligned} \tag{28}$$

Next, we consider the following two cases.

Case A: $d_n \ge v_n$.

In this case, $\tau_n = \frac{\rho_n \|(I-P_{C^b_{i_n,n}})y_n\|^2}{\|(I-P_{C^b_{i_n,n}})y_n\|^2+\theta_n}$, $f_n(y_n) = \frac{1}{2}\|(I - P_{C^b_{i_n,n}})y_n\|^2$ and $\nabla f_n(y_n) = (I - P_{C^b_{i_n,n}})y_n$. Applying Lemma 2.1 (2-3) and the nonexpansivity of $P_{C^b_{i_n,n}}$ that

$$\begin{aligned} &\|y_n - \tau_n \nabla f_n(y_n) - z\|^2 \\ &= \|y_n - z\|^2 + \tau_n^2 \|\nabla f_n(y_n)\|^2 \\ &\quad - 2\tau_n \langle \nabla f_n(y_n), y_n - z\rangle \\ &= \|y_n - z\|^2 + \tau_n^2 \|(I - P_{C^b_{i_n,n}})y_n\|^2 \\ &\qquad - 2\tau_n \langle (I - P_{C^b_{i_n,n}})y_n, y_n - z\rangle \\ &= \|y_n - z\|^2 + \tau_n^2 \|(I - P_{C^b_{i_n,n}})y_n\|^2 \\ &\qquad - 2\tau_n \langle (I - P_{C^b_{i_n,n}})z - (I - P_{C^b_{i_n,n}})y_n, y_n - z\rangle \\ &\le \|y_n - z\|^2 + \tau_n^2 \|(I - P_{C^b_{i_n,n}})y_n\|^2 \\ &\qquad - 2\tau_n \|I - P_{C^b_{i_n,n}})z - (I - P_{C^b_{i_n,n}})y_n\|^2 \\ &= \|y_n - z\|^2 + \tau_n^2 \|(I - P_{C^b_{i_n,n}})y_n\|^2 \\ &\qquad - 2\tau_n \|(I - P_{C^b_{i_n,n}})y_n\|^2 \\ &\le \|y_n - z\|^2 + \rho_n^2 \frac{\|(I - P_{C^b_{i_n,n}})y_n\|^4}{(\|(I - P_{C^b_{i_n,n}})y_n\|^2 + \theta_n)^2} \\ &\quad (\|(I - P_{C^b_{i_n,n}})y_n\|^2 + \theta_n) \\ &\qquad - 2\rho_n \frac{\|(I - P_{C^b_{i_n,n}})y_n\|^4}{\|(I - P_{C^b_{i_n,n}})y_n\|^2 + \theta_n} \\ &= \|y_n - z\|^2 - \rho_n(2-\rho_n)\frac{\|(I - P_{C^b_{i_n,n}})y_n\|^4}{\|(I - P_{C^b_{i_n,n}})y_n\|^2 + \theta_n}. \end{aligned} \tag{29}$$

Case B: $d_n < v_n$.

In this case, $\tau_n = \frac{\rho_n \|(I-P_{Q^b_{j_n,n}})A_{j_n} y_n\|^2}{\|A^*_{j_n}(I-P_{Q^b_{j_n,n}})A_{j_n} y_n\|^2+\theta_n}$, $f_n(y_n) = \frac{1}{2}\|(I - P_{Q^b_{j_n,n}})A_{j_n} y_n\|^2$ and $\nabla f_n(y_n) = A^*_{j_n}(I - P_{Q^b_{j_n,n}})A_{j_n} y_n$. Similar with the deduction of Case A, we obtain that

$$\begin{aligned} &\|y_n - \tau_n \nabla f_n(y_n) - z\|^2 \\ \le\ &\|y_n - z\|^2 - \rho_n(2-\rho_n)\frac{\|(I-P_{Q^b_{j_n,n}})A_{j_n} y_n\|^4}{\|A^*_{j_n}(I-P_{Q^b_{j_n,n}})A_{j_n} y_n\|^2+\theta_n}. \end{aligned} \tag{30}$$

From (C2), (29) and (30) we have that

$$\|y_n - \tau_n \nabla f_n(y_n) - z\| \le \|y_n - z\|. \tag{31}$$

By (20), we also have

$$\begin{aligned} \|y_n - z\| &= \|x_n + \beta_n(x_n - x_{n-1}) - z\| \\ &\le \|x_n - z\| + \beta_n \|x_n - x_{n-1}\|. \end{aligned} \tag{32}$$

Combining (28), (31) and (32)

$$\begin{aligned} \|x_{n+1} - z\| &\le \alpha_n \|g(y_n) - g(z)\| + \alpha_n \|g(z) - z\| \\ &\quad + (1-\alpha_n)\|y_n - \tau_n \nabla f_n(y_n) - z\| \\ &\le \alpha_n c\|y_n - z\| + \alpha_n \|g(z) - z\| \\ &\quad + (1-\alpha_n)\|y_n - z\| \\ &\le [1 - \alpha_n(1-c)]\|x_n - z\| \\ &\quad + [1 - \alpha_n(1-c)]\beta_n \|x_n - x_{n-1}\| \\ &\quad + \alpha_n \|g(z) - z\| \\ &= [1 - \alpha_n(1-c)]\|x_n - z\| + \alpha_n(1-c) \\ &\quad \frac{\|g(z) - z\| + \frac{1-\alpha_n(1-c)}{\alpha_n}\beta_n \|x_n - x_{n-1}\|}{1-c}. \end{aligned} \tag{33}$$

According to (C3), we see that $t_n = \frac{[1-\alpha_n(1-c)]\beta_n \|x_n - x_{n-1}\|}{\alpha_n} \to 0$. Hence the sequence t_n is bounded. There exists some $M > 0$ such that

$$\begin{aligned} \|x_{n+1} - z\| &\le [1 - \alpha_n(1-c)]\|x_n - z\| + \alpha_n(1-c)M \\ &\le \max\{\|x_n - z\|, M\} \\ &\quad \vdots \\ &\le \max\{\|x_0 - z\|, M\}. \end{aligned} \tag{34}$$

We conclude that $\{x_n\}$ is bounded and hence $\{y_n\}$ is bounded.

According to the definition of y_n, it holds that

$$\begin{aligned} \|y_n - z\|^2 &= \|x_n + \beta_n(x_n - x_{n-1}) - z\|^2 \\ &= \|x_n - z\|^2 + 2\beta_n \langle x_n - x_{n-1}, x_n - z\rangle \\ &\quad + \beta_n^2 \|x_n - x_{n-1}\|^2. \end{aligned} \tag{35}$$

By Lemma 2.1(3), the following equation holds.

$$\begin{aligned} \langle x_n - x_{n-1}, x_n - z\rangle = -\frac{1}{2}\|x_{n-1} - z\|^2 + \frac{1}{2}\|x_n - z\|^2 \\ + \frac{1}{2}\|x_n - x_{n-1}\|^2. \end{aligned} \tag{36}$$

Substituting the equality (36) into (35), we obtain

$$\begin{aligned}\|y_n - z\|^2 &= \|x_n - z\|^2 + \beta_n(-\|x_{n-1} - z\|^2 \\ &\quad + \|x_n - z\|^2 + \|x_n - x_{n-1}\|^2) \\ &\quad + \beta_n^2\|x_n - x_{n-1}\|^2 \\ &\le \|x_n - z\|^2 + \beta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\ &\quad + 2\beta_n\|x_n - x_{n-1}\|^2 \\ &\le \|x_n - z\|^2 + \beta_n\|x_n - x_{n-1}\|(\|x_n - z\| \\ &\quad - \|x_{n-1} - z\|) + 2\beta_n\|x_n - x_{n-1}\|^2 \\ &= \|x_n - z\|^2 + E_n,\end{aligned} \tag{37}$$

where $E_n = \beta_n\|x_n - x_{n-1}\|(\|x_n - z\| - \|x_{n-1} - z\|) + 2\beta_n\|x_n - x_{n-1}\|^2$.

According to the boundedness of $\{\theta_n\}$ and $\{y_n\}$, put

$$\begin{aligned}L = \max\{&\sup_n\{\|(I - P_{C^b_{i_n,n}})y_n\|^2 + \theta_n\}, \\ &\sup_n\{\|A^*_{j_n}(I - P_{Q^b_{j_n,n}})A_{j_n}y_n\|^2 + \theta_n\}\}.\end{aligned} \tag{38}$$

It follows from (29) and (30) that

$$\Phi_n^4 \le \frac{L}{\rho_n(1-\rho_n)}(\|y_n - z\|^2 - \|y_n - \tau_n\nabla f_n(y_n) - z\|^2), \tag{39}$$

where $\Phi_n = \max\{d_n, v_n\}$.

Using (31) and (39), we have

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &= \langle\alpha_n g(y_n) + (1-\alpha_n)(y_n - \tau_n\nabla f_n(y_n)) - z, x_{n+1} - z\rangle \\ &= (1-\alpha_n)\langle y_n - \tau_n\nabla f_n(y_n) - z, x_{n+1} - z\rangle \\ &\quad + \alpha_n\langle g(y_n) - z, x_{n+1} - z\rangle \\ &\le \frac{1-\alpha_n}{2}(\|x_{n+1} - z\|^2 + \|y_n - \tau_n\nabla f_n(y_n) - z\|^2) \\ &\quad + \alpha_n\langle g(y_n) - g(z), x_{n+1} - z\rangle \\ &\quad + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &\le \frac{1-\alpha_n}{2}(\|x_{n+1} - z\|^2 + \|y_n - \tau_n\nabla f_n(y_n) - z\|^2) \\ &\quad + \frac{\alpha_n}{2}(c^2\|y_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &\quad + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &\le \frac{1-\alpha_n}{2}(\|x_{n+1} - z\|^2 + \|y_n - z\|^2 - \rho_n(1-\rho_n)\frac{\Phi_n^4}{L}) \\ &\quad + \frac{\alpha_n}{2}(c\|y_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &\quad + \alpha_n\langle g(z) - z, x_{n+1} - z\rangle,\end{aligned} \tag{40}$$

which is rearranged to obtain

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &\le (1-\alpha_n)\|y_n - z\|^2 \\ &\quad - (1-\alpha_n)\rho_n(1-\rho_n)\frac{\Phi_n^4}{L} + \alpha_n c\|y_n - z\|^2 \\ &\quad + 2\alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &= [1 - \alpha_n(1-c)]\|y_n - z\|^2 \\ &\quad - (1-\alpha_n)\rho_n(1-\rho_n)\frac{\Phi_n^4}{L} \\ &\quad + 2\alpha_n\langle g(z) - z, x_{n+1} - z\rangle.\end{aligned} \tag{41}$$

Substituting (37) into (41) yields that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le [1 - \alpha_n(1-c)](\|x_n - z\|^2 + E_n) \\ &\quad - (1-\alpha_n)\rho_n(1-\rho_n)\frac{\Phi_n^4}{L} \\ &\quad + 2\alpha_n\langle g(z) - z, x_{n+1} - z\rangle \\ &= [1 - \alpha_n(1-c)]\|x_n - z\|^2 \\ &\quad + [1 - \alpha_n(1-c)]E_n \\ &\quad - (1-\alpha_n)\rho_n(1-\rho_n)\frac{\Phi_n^4}{L} \\ &\quad + 2\alpha_n\langle g(z) - z, x_{n+1} - z\rangle.\end{aligned} \tag{42}$$

Set

$$\begin{aligned}&s_n = \|x_n - z\|^2; \\ &\gamma_n = [1 - \alpha_n(1-c)]E_n + 2\alpha_n\langle g(z) - z, x_{n+1} - z\rangle; \\ &\delta_n = [1 - \alpha_n(1-c)]\frac{E_n}{\alpha_n(1-c)} \\ &\qquad + \frac{2}{1-c}\langle g(z) - z, x_{n+1} - z\rangle; \\ &\eta_n = (1-\alpha_n)\rho_n(1-\rho_n)\frac{\Phi_n^4}{L}.\end{aligned} \tag{43}$$

From (42) and (43), we derive that

$$s_{n+1} \le [1 - \alpha_n(1-c)]s_n + \alpha_n(1-c)\delta_n, \quad n \ge 1, \tag{44}$$

$$s_{n+1} \le s_n - \eta_n + \gamma_n, \quad n \ge 1. \tag{45}$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ such that

$$\limsup_{k\to\infty}\eta_{n_k} \le 0. \tag{46}$$

That is

$$\limsup_{k\to\infty}(1-\alpha_{n_k})\rho_{n_k}(1-\rho_{n_k})\frac{\Phi_{n_k}^4}{L} \le 0, \tag{47}$$

which by conditions (C1) and (C2) implies

$$\lim_{k\to\infty}\Phi_{n_k} = 0. \tag{48}$$

By the definition of Φ_n, it indicates that

$$\lim_{k\to\infty} \|(I - P_{C^b_{i_{n_k},n_k}})y_{n_k}\| = 0, \quad (49)$$
$$\lim_{k\to\infty} \|(I - P_{Q^b_{j_{n_k},n_k}})A_{j_n} y_{n_k}\| = 0. \quad (50)$$

The definitions of i_{n_k} and j_{n_k} ensure that

$$\lim_{k\to\infty} \|(I - P_{C^b_{i,n_k}})y_{n_k}\| = 0, \quad i = 1, 2, \cdots, t, \quad (51)$$
$$\lim_{k\to\infty} \|(I - P_{Q^b_{j,n_k}})A_j y_{n_k}\| = 0, \quad j = 1, 2, \cdots, r. \quad (52)$$

Since $\partial q_j, j = 1, 2, \cdots, r$ are bounded on bounded sets , there exists a constant $\mu > 0$ such that $\|\zeta_j^{n_k}\| \le \mu, j = 1, 2, \cdots, r, k \in \mathbb{N}$, where $\zeta_j^{n_k} \in \partial q_j(y_{n_k})$. Note that $P_{Q^b_{j,n_k}} A_j y_{n_k} \in Q^b_{j,n_k}$ and from (52) we obtain

$$\begin{aligned} q_j(A_j y_{n_k}) &\le \langle \zeta_j^{n_k}, A_j y_{n_k} - P_{Q^b_{j,n_k}} A_j y_{n_k}\rangle \\ &\quad - \frac{\gamma_j}{2}\|A_j y_{n_k} - P_{Q^b_{j,n_k}} A_j y_{n_k}\|^2 \\ &\le \|\zeta_j^{n_k}\|\|A_j y_{n_k} - P_{Q^b_{j,n_k}} A_j y_{n_k}\| \\ &\le \mu\|(I - P_{Q^b_{j,n_k}})A_j y_{n_k}\| \to 0, \quad k \to \infty. \end{aligned} \quad (53)$$

Since $\{y_{n_k}\}$ is bounded, there exists a subsequence $\{y_{n_{k_m}}\} \subset \{y_{n_k}\}$ such that $y_{n_{k_m}} \rightharpoonup x^*$ and

$$\limsup_{k\to\infty}\langle g(z) - z, y_{n_k} - z\rangle = \lim_{m\to\infty}\langle g(z) - z, y_{n_{k_m}} - z\rangle. \quad (54)$$

Since $q_j(\cdot)$ is convex and weakly lower semicontinuous and that $A_j y_{n_{k_m}} \rightharpoonup A_j x^*$, by (53) we have

$$q_j(A_j x^*) \le \liminf_{m\to\infty} q_j(A_j y_{n_{k_m}}) \le 0. \quad (55)$$

Hence $A_j x^* \in Q_j$.

By the definition of C^b_{i,n_k}, the assumption (A3) and (51), there exists a constant $\varepsilon > 0$ such that

$$\begin{aligned} c_i(y_{n_k}) &\le \langle \xi_i^{n_k}, y_{n_k} - P_{C^b_{i,n_k}}(y_{n_k})\rangle \\ &\quad - \frac{\lambda_i}{2}\|y_{n_k} - P_{C^b_{i,n_k}}(y_{n_k})\|^2 \\ &\le \varepsilon\|y_{n_k} - P_{C^b_{i,n_k}}(y_{n_k})\| \to 0, \quad k \to \infty. \end{aligned} \quad (56)$$

By the weakly lower semi-continuity of c_i and $y_{n_{k_m}} \rightharpoonup x^*$, we have

$$c_i(x^*) \le \liminf_{m\to\infty} c_i(y_{n_{k_m}}) \le 0. \quad (57)$$

Therefore $x^* \in C_i (i = 1, ..., t)$. So $x^* \in S$.

From Lemma 2.1(1) and (54) we obtain:

$$\begin{aligned} &\limsup_{k\to\infty}\langle g(z) - z, y_{n_k} - z\rangle \\ &= \lim_{m\to\infty}\langle g(z) - z, y_{n_{k_m}} - z\rangle \\ &= \langle g(z) - z, x^* - z\rangle \le 0. \end{aligned} \quad (58)$$

On the other hand, by (C3), we have

$$\|y_n - x_n\| = \beta_n\|x_n - x_{n-1}\| \to 0, \quad n \to \infty. \quad (59)$$

So we have

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \|\alpha_n(g(y_n) - x_n) + (1 - \alpha_n)(y_n - \tau_n\nabla f_n(y_n) - x_n)\| \\ &\le \alpha_n\|g(y_n) - x_n\| + (1 - \alpha_n)\|y_n - x_n\| \\ &\quad + (1 - \alpha_n)\tau_n\|\nabla f_n(y_n)\|. \end{aligned} \quad (60)$$

Note that

$$\begin{aligned} \tau_n &= \rho_n\frac{\Phi_n^2}{||\nabla f_n(y_n)||^2 + \theta_n} \\ &\le \frac{2}{\inf\{\theta_n\}}\Phi_n^2 \to 0, \quad n \to \infty. \end{aligned} \quad (61)$$

Thus

$$\|x_{n+1} - x_n\| \to 0, \quad n \to \infty. \quad (62)$$

From (58), (59) and (62), we derive that

$$\limsup_{k\to\infty}\langle g(z) - z, x_{n_k+1} - z\rangle \le 0. \quad (63)$$

Then (C3) and (63) implies that

$$\limsup_{k\to\infty}\delta_{n_k} \le 0. \quad (64)$$

Using Lemma 2.4 , we conclude that $x_n \to z$. The proof is complete. □

4 Numerical experiments

In this section, we provide some numerical experiments to show the efficiency of Algorithm 3.1 and compare the convergence rates of our algorithm and other algorithms. The codes are written in Matlab R2018b and run on Inter(R) Core(TM) i9-12900H CPU @ 2.50 GHz , RAM 16.00 GB.

The inertial coefficient β_n is given by

$$\beta_n = \begin{cases} \dfrac{\varepsilon_n}{||x_n - x_{n-1}||}, & ||x_n - x_{n-1}|| > 1 \\ \varepsilon_n, & ||x_n - x_{n-1}|| \le 1. \end{cases} \quad (65)$$

It is easy to check that $\beta_n \le \varepsilon_n$ and that $\beta_n||x_n - x_{n-1}|| \le \varepsilon_n$. It also can be proved, see [24], that if $\varepsilon_n \ge 0$ and $\sum_{n=0}^{\infty}\varepsilon_n < +\infty$, the corresponding algorithms are strongly convergent.

Example 4.1 Consider the following problem: find an element $x^* \in \mathbb{R}^N$ such that

$$x^* \in S = \bigcap_{i=1}^{t} C_i \cap (\bigcap_{j=1}^{r} A_j^{-1}(Q_j)), \tag{66}$$

where the closed convex subsets C_i and Q_j are given by

$$C_i = \{x \in \mathbb{R}^N : \sum_{k=i}^{N} 10^{\frac{k-1}{N-1}} x_k^2 - 1 \leq 0\}, \tag{67}$$

$$Q_j = \{y \in \mathbb{R}^{N(j+1)} : \sum_{k=j}^{N(j+1)} 10^{\frac{k-1}{N(j+1)-1}} y_k^2 - 1 \leq 0\}. \tag{68}$$

It is obvious that C_i and Q_j are ellipsoids (see [25]), and $c_i(x) = \sum_{k=i}^{N} 10^{\frac{k-1}{N-1}} x_k^2 - 1$ and $q_j(y) = \sum_{k=j}^{N(j+1)} 10^{\frac{k-1}{N(j+1)-1}} y_k^2 - 1$ are both 2-strongly convex functions, see [16]).

Set $N = 5$, $t = 10$, $r = 20$, $\rho_n = 0.1$, $\alpha_n = \frac{1}{n}$, $\varepsilon_n = \frac{1}{n^{1.2}}$, $\theta_n = 0.1$, $g(x) = 0.8x$, and e denotes the vector of corresponding dimension of which the coordinates are all 1. Let $A_j : \mathbb{R}^N \to \mathbb{R}^{N(j+1)}$ be bounded linear operators, of which the elements are randomly generated in the closed interval $[0, 10]$. Set

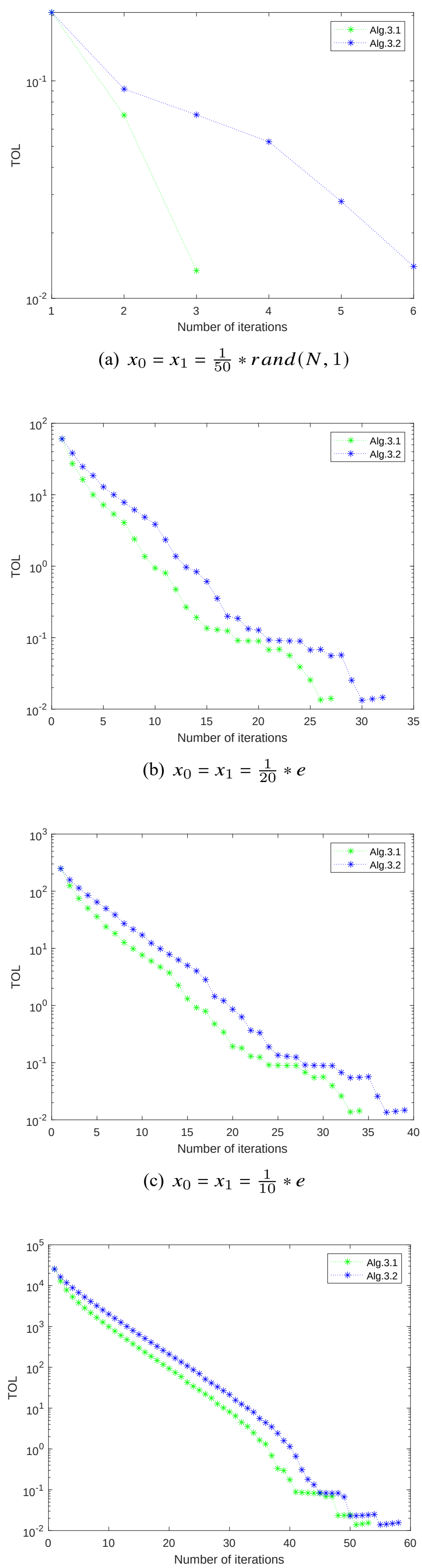
$$\mathrm{TOL} = \frac{1}{t+r}(\sum_{i=1}^{t} \|x_n - P_{C_i}x_n\|^2 + \sum_{j=1}^{r} \|A_j x_n - P_{Q_j} A_j x_n\|^2) \tag{69}$$

for all $n \geq 1$. Note that if at the nth step, $\mathrm{TOL} = 0$, then $x_n \in S$, that is, x_n is a solution to this problem. We use $\mathrm{TOL} < 10^{-3}$ as a stopping criteria.

First, we consider the impact of inertial term on the convergence rate under different initial values x_0 and x_1. We use Alg.3.1 to refer to Algorithm 3.1 and Alg.3.2 to refer to Algorithm 3.1 without inertial term. The results of numerical experiments are reported in Table 1 and Fig.1.

From Table 1 and Fig.1, we can see that Alg.3.1 has advantage over Alg.3.2 in both the iteration number and the CPU time. This shows that the inertial perturbation can improve the convergence of the algorithms.

Next, we consider the impact of ρ_n in the self-adaptive step size on the convergence rate. For $\rho_n = 0.1$, $\rho_n = 0.3$, $\rho_n = 0.6$, $\rho_n = 0.9$, $\rho = 1.2$, $\rho = 1.5$, and $\rho = 1.9$, with other parameters retaining the same values as above, we examine the convergence of the sequence $\{x_n\}$ which is generated by Algorithm 3.1. The results as shown in Fig.2(e).



(a) $x_0 = x_1 = \frac{1}{50} * rand(N, 1)$

(b) $x_0 = x_1 = \frac{1}{20} * e$

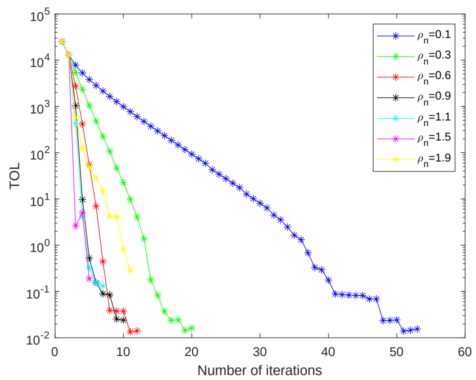
(c) $x_0 = x_1 = \frac{1}{10} * e$

(d) $x_0 = x_1 = e$

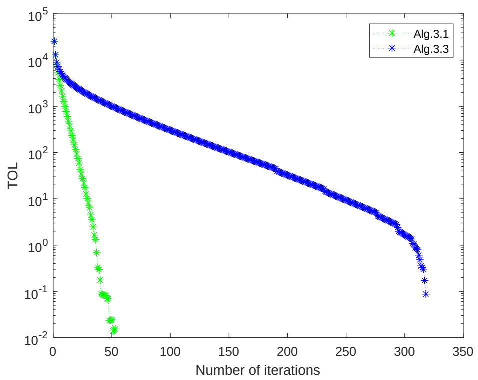
Fig.1: Comparison of Alg.3.1 and Alg.3.2 under different choices of initial values.

Table 1: The numerical results for Alg.3.1 and Alg.3.2

	Alg.3.1		Alg.3.2	
Initial Point	n	Time(s)	n	Time(s)
$x_0 = x_1 = \frac{1}{50} * rand(N,1)$	4	0.0223	7	0.0330
$x_0 = x_1 = \frac{1}{20} * e$	28	0.0823	33	0.0941
$x_0 = x_1 = \frac{1}{10} * e$	35	0.0986	40	0.1218
$x_0 = x_1 = e$	54	0.1500	59	0.1495



(e) Different choices of ρ_n



(f) Different choices of step sizes

Fig.2: Comparison of different choices of ρ_n and step sizes.

It seems that Algorithm 3.1 converges faster if ρ_n takes values around the midpoint of the interval $(0, 2)$.

Finally, we consider the convergence rate under different choices of self-adaptive step sizes. We denote by Alg.3.3 the algorithm the same as ours except that the the step size is the ordinary self-adaptive one:

$$\tau_n = \rho_n \frac{a_1}{a_2}, \tag{70}$$

where

$$a_1 = \sum_{i=1}^{t} ||(I - P_{C_{i,n}^b})y_n||^2 + \sum_{j=1}^{r} ||(I - P_{Q_{j,n}^b})A_j y_n||^2, \tag{71}$$

$$a_2 = ||\sum_{i=1}^{t}(I - P_{C_{i,n}^b})y_n + \sum_{j=1}^{r} A_j^*(I - P_{Q_{j,n}^b})A_j y_n||^2, \tag{72}$$

with other parameters retaining the same values as above, we provide the results in Fig.2(f).

From Fig.2(f), we see that our algorithm with step size defined as in (26) is more effective in that it used fewer iterates in the experiment. The reason may be that we applied the largest distance among $\|A_j y_n - P_{Q_{j,n}^b}(A_j y_n)\|, i = 1, 2, \cdots, t$ and $\|A_j y_n - P_{Q_{j,n}^b}(A_j y_n)\|, j = 1, 2, \cdots, r$ while Alg.3.3 use the sum of them.

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Contribution of individual authors

Yaxuan Zhang proposed the algorithm and checked the correctness of the manuscript.
Yuming Guan proved the convergence theorem of the algorithm and carried out the numerical simulation.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.