On The Local Multiset Dimension of some Families of Graphs

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Abstract: Let G be a connected graph. G is said to be unicyclic if it contains exactly one cycle, and bicyclic if the number of edges equals the number of vertices plus one. For a k-ordered set $W = \{s_1, s_2, \dots, s_k\} \subset V(G)$, the multiset representation of a vertex x in G with respect to W is given as $r_m(x|W) = \{d(x, s_1), d(x, s_2), \dots, d(x, s_k)\}$, where $d(x, s_i)$ is the distance between x and the ordered subset s_i of W together with their multiplicities. The set W is called a local m-resolving set of G if for every $uv \in E(G)$, $r_m(u|W) \neq r_m(v|W)$. The local m-resolving set with minimum cardinality is called the local multiset basis and its cardinality is called the local multiset dimension of G, denoted by $md_l(G)$. If G has no local m-resolving set, we write $md_t(G) = \infty$ and say that G has an infinite local multiset dimension. In this paper, we determine the local multiset dimension of the unicyclic and bicyclic graphs.

Key-Words: Networks infrastructure, local m-resolving set, local multiset dimension, unicyclic graphs, bicyclic graphs.

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1 Introduction

In this paper, we consider graphs G that are simple, connected and finite. The vertex set and edge set are denoted, respectively, as $V(G)$ and $E(G)$. The distance between u and v, denoted as $d_G(u, v)$ or simply as $d(u, v)$, is the length of the shortest path between u and v in G. A vertex x is said to *resolve* a pair $u, v \in V(G)$ if $d(x, u) \neq d(x, v)$. A subset $S \subseteq V(G)$ is then said to be a *resolving set* of G if any pair of adjacent vertices of G can be resolved by some vertex in S. The set S with minimum cardinality in G is called a metric basis and its cardinality is the *metric dimension* of G, denoted as $dim(G)$ by [2].

A new variant of the resolving set problems introduced by Okamoto *et al.*, [9], is the local resolving set problems. For a k-ordered set, $W = \{s_1, s_2, \dots, s_k\} \subset V(G)$, the vertex representations of vertex x to the set W is an ordered k-tuple, $r(x|W) = (d(x, s_1), d(x, s_2), \dots, d(x, ss_k))$. The set W is called a local resolving set if $\forall\, xy \in E(G), r(x|W) \neq r(y|W)$. The minimum cardinality of a local resolving set is called *local basis* and its cardinality is called the *local metric dimension* of G, denoted by $ldim(G)$.

Both the revolving set problems and the local set resolving problems are topics in distances in graphs that have increased in popularity over the past few decades due to their applications in many real-life problems, especially in network infrastructures, chemical structures, computer connectivity and navigation robots optimization problems. More detail about these applications can be seen in Khuller *et al*., [1].

A generalized form of the local resolving set problem is the multiset dimension of a graph first defined by Simanjuntak *et al*., [3]. The multiset dimension of a graph G is defined as the set $W = \{s_1, s_2, \dots, s_k\} \subset V(G)$, such that the vertex representations of a vertex $x \in V(G)$ to the set W is the multiset, $r_m(x|W) = \{d(x, s_1), d(x, s_2), \dots, d(x, s_k)\}$ where $d(x, s_i)$ is the length of the shortest path between x and a vertex in W together with their multiplicities. The set W is called an *m-resolving set* if $\forall\, xy \in E(G)$, $r_m(x|W) \neq r_m(y|W)$. If G has an *m-resolving* set, then the minimum resolving set W is a *multiset basis* of G and its cardinality is called the *multiset dimension* of G, denoted as $md(G)$. Otherwise, we say that G has an infinite multiset dimension and we write $md(G) = \infty$.

Several results on multiset dimensions can be found in the literature. For example, for multiset dimensions at least 3 were proven by Bong *et al*., [12], for almost hypercube graphs by Alfarisi *et al*., [5], trees by Hafidh *et al*., [11], starphene and zigzag-edge coronoid by Liu *et al*., [13]. A new notion based on the multiset dimension of G was defined by Alfarisi *et al*., [4], as the *local multiset dimension*.

As in previous definitions, for a given set $W = \{s_1, s_2, \dots, s_k\} \subset V(G)$, the vertex representations of a vertex $x \in V(G)$ to the set W is $r_m(x|W) = \{d(x, s_1), d(x, s_2), \dots, d(x, s_k)\}$. The set W is called a *local m-resolving set* of G if $r_m(v|W) \neq r_m(u|W)$ for $uv \in E(G)$. The minimum cardinality of a local m-resolving set is called the *local multiset basis* and its cardinality is called the *local multiset dimension*, denoted by $md_l(G)$: otherwise, we say that G has an infinite local multiset dimension and we write $md_l(G) = \infty$.

We illustrate this concept in Fig. 1. In this case, we have the resolving set $W = \{v_2, v_6\}$, shown in Fig. 1(a) whose metric dimension is $dim(G) = 2$. Furthermore, the m-resolving set is $W = \{v_2, v_3, v_6\}$, shown in Fig. 1(b) with multiset dimension, $md(G) = 3$. The representations of $v \in V(G)$ with respect to W are all distinct. We only need to make sure the adjacent vertices have distinct representations for the local multiset dimension. Thus, we have the local m-resolving $W = \{v_1\}$, shown in Fig. 1(c) that the local multiset dimension is $md_l(G) = 1$. The vertex representation v with respect to basis can be seen in Fig.1. In Fig. 1(a) and (b), every vertex in G has distinct representations. For Fig. 1(c), every two adjacent vertices have a different representation.

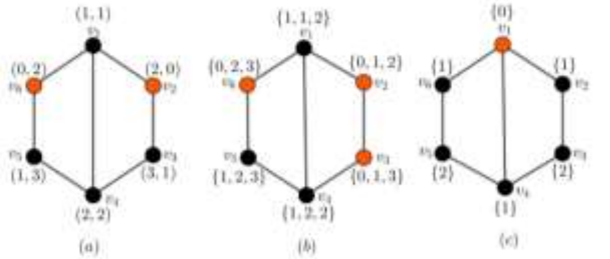

Fig. 1: (a) A graph with metric dimension 2; (b) A graph with multiset dimension 3; (c) A graph with local multiset dimension 1.

The natural question what the local multiset dimension of some special graphs namely paths, stars, trees and cycles and also the local multiset dimension of graph operations namely, the cartesian product was answered, for example, in a classical paper by Alfarisi *et al*., [4]. Other known results are, for example, the m-shadow graph by Adawiyah *et al*., [8], and the local multiset dimension of unicyclic graphs was also studied by Adawiyah *et al*., [7].

This paper aims to provide similar results for the unicyclic and bicyclic graphs. In particular, we show that if G is a unicyclic graph of order $p \geq 3$, then $md_l(G)$ is 1 for p even and 2 for p odd. Similarly, if G is bicyclic with cycles of order $p_1, p_2 \geq 3$, then $md_l(G)$ is 1 if both p_1 and p_2 are even, and 2 otherwise.

The following known definitions and results will be useful in the proof of our main results.

Proposition 1.1 [10] A graph is bipartite if and only if it contains no odd cycle.

Proposition 1.2 [4] Let K_n be a complete graph with $n \geq 3$. Then $md_l(K_n) = \infty$.

Proposition 1.3 [6] Let G be a connected graph. Then $md_l(G) = 1$ if and only if G is a bipartite graph.

Definition 1.1 The unicyclic graph is a graph that has only one cycle.

Definition 1.2 The bicyclic graph is a graph that has only two cycles.

By Definition 1.1, it is obvious that the unicyclic graph can be obtained from a tree by connecting any two vertices by an edge. An example of a unicyclic graph (left) and a bicyclic graph (right) can be seen in Fig. 2.

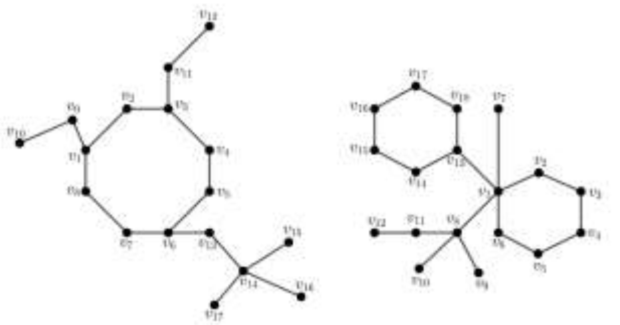

Fig. 2: Unicyclic (left) and bicyclic (right) graphs

2 Results and Discussion

In this section, we investigate the local multiset dimension of a unicyclic and bicyclic graph.

2.1 For Unicyclic Graphs

Let G be a unicyclic graph obtained from a tree T by adding an edge $e = uv$ between two non-adjacent vertices $u, v \in V(T)$. We state our first results for both even and odd cycle subgraphs of G.

Proposition 2.1 *Let G be a unicyclic graph that contains an even cycle. Then $md_l(G) = 1$.*

Proof. Let C_p be the cycle subgraph of G, where p is the number of vertices in C_p. If p is even, then G is bipartite by Proposition 1.1 and $md_l(G) = 1$ by Proposition 1.3

Lemma 2.1 *Let G be a unicyclic graph that contains an odd cycle C_p. For every $v \in V(G)$, there exists exactly one pair of adjacent vertices x and y satisfying $d(x, v) = d(y, v)$ where x and y are in a cycle C_p.*

Proof. Let G be a unicyclic graph containing an odd cycle C_p where $p \geq 3$. Suppose $V(C_p) = \{v_1, v_2, \dots, v_p\}$ and $E(C_p) = \{v_1 v_p, v_i v_{\{i+1\}}; 1 \leq i \leq p-1\}$ with $p = 2k+1$ for $k \geq 1$. Let $G' = G - E(C_p)$. Then G' is a disconnected graph that contains p components where each component is a tree. For $i \in \{1,2,3, \dots, p\}$, we define T_i as the component of G' containing vertex $v_i \in V(C_p)$.

Suppose v is a vertex in G, then there are $i \in \{1,2,3, \dots, p\}$ such that $v \in V(T_i)$. Suppose x and y are two adjacent vertices in G and there are $j \in \{1,2,3, \dots, p\}$ such that $x, y \in V(T_j)$, then $d(v, z) = d(v, v_i) + d(v_i, v_j) + d(v_j, z)$ for $z \in \{x, y\}$. Since T_j are a tree and every two distinct vertices in the tree have a unique path between them, we get that either $d(v_j, x) < d(v_j, y)$ or $d(v_j, x) > d(v_j, y)$, which implies $d(v, x) \neq d(v, y)$. Therefore, x and y must be two different components of G', implying that x and y must be in C_p.

Now let x and y be two adjacent vertices at C_p and $v \in V(T_i)$. Suppose $d(v_i, x) < diam(C_p) = k$, then $d(v_i, x) < d(v_i, y)$ or $d(v_i, x) > d(v_i, y)$, which implies $d(v, x) \neq d(v, y)$. So, $d(v_i, x) = k = d(v_i, y)$. When p is odd, two adjacent vertices are obtained, namely $x = v_{\{i+k\}}$ and $y = v_{\{i+k+1\}}$ where both indices are at modulo p.

Suppose there are two distinct pairs of adjacent vertices x_1, y_1 and x_2, y_2 of G such that for any vertices $v \in V(G), d(x_1, v) = d(y_1, v)$ and $d(x_2, v) = d(y_2, v)$. With the same arguments above, we get that x_1, y_1 and x_2, y_2 come from different cycles in G. Therefore, G contains at least two cycles which is a contradiction.

Theorem 2.1 *Let G be a unicyclic graph of order at least 3. If G contains a circle of order p, then*

$$md_l(G) = \begin{cases} 1, & for\ p\ is\ even \\ 2, & for\ p\ is\ odd \end{cases}$$

Proof. Let C_p be a cycle of G. We consider two cases for p as follows.

Case 1. For p even

Since p is even, G is a bipartite graph by Proposition 1.3 and $md_l(G) = 1$ by Proposition 2.1.

Case 2. For p odd.

If p is odd, then G is not bipartite based on Proposition 1.1 implying that $md_l(G) \leq 2$ by Proposition 1.3. Furthermore, we will prove the upper bound of the local multiset dimension of G that $md_l(G) \leq 2$. Let C_p be a cycle contained in G. By Lemma 2.1, there is exactly one pair of x and y adjacent vertices in C_p such that $d(x, v) = d(y, v)$ for every $v \in V(G)$. Now, define $W = \{v, x | v \in V(T_i)$ and $x \in V(C_p)\}$. Since no two adjacent vertices have the same representation of W, W is a local multiset resolving set of G and thus, $md_l(G) = 2$.

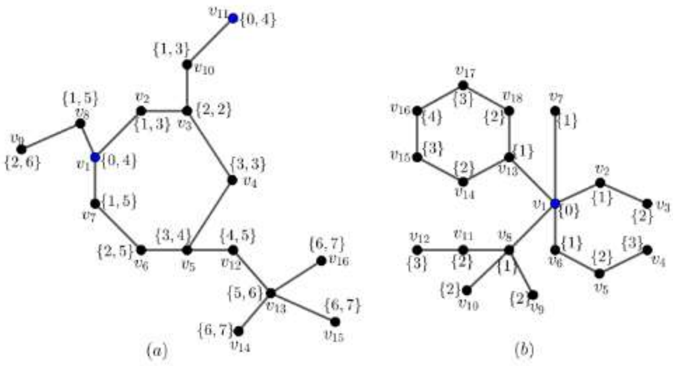

Fig. 3: (a) $md_l(G) = 2$ with p is odd and (b) $md_l(G) = 1$ with p is even

2.2 For Bicyclic Graphs

Let H be a bicyclic graph. Recall that a bicyclic graph is obtained from a tree T by adding two edges say, $e_1 = u_1 v_1$ and $e_2 = u_2 v_2$ with two non-adjacent vertices $u_1, v_1 \in V(T)$ and $u_2, v_2 \in V(T)$, respectively. In other words, a bicyclic graph is a graph that contains only two cycle subgraphs. Suppose p_1 is the number of vertices in the cycle subgraph C_1 and p_2 is the number of vertices in the cycle subgraph C_2.

It is important to note the following conditions for the type of bicycle graph with C_1 and C_2 being considered.

1. the bicyclic graph contains two disjoint cycles.
2. C_1 and C_2 are disjoint with the property that $V(C_1) \cap V(C_2) = \emptyset$. The example can be seen in Fig. 4.

For conditions 1 and 2, if p_1 and p_2 are even, then H is a bipartite graph, and so by Proposition 1.3, $md_l(H) = 1$. Our next lemma shows the condition if at least one of C_1 and C_2 is odd.

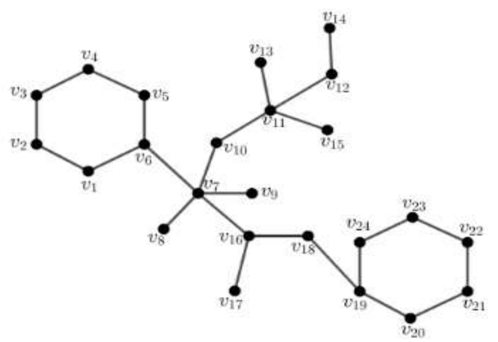

Fig. 4: Bicyclic graph with cycle C_1 and C_2 is disjoint with properties $V(C_1) \cap V(C_2) = \emptyset$

Lemma 2.2. *Let G be a bicyclic graph that contains an odd cycle C_1 or C_2. For every $v \in V(H)$, there exists exactly one pair of adjacent vertices x and y that satisfy $d(x, v) = d(y, v)$ where x and y are in an odd cycle C_1 or C_2.*

Proof. Let H be a unicyclic graph containing an odd cycle C_1 or C_2 where $p_1, p_2 \geq 3$. Suppose $V(C_1) = \{v_1, v_2, \dots, v_{p_1}\}$, $V(C_2) = \{v_1, v_2, \dots, v_{p_1}\}$ $E(C_1) = \{v_1 v_{p_1}, v_i v_{i+1}; 1 \leq i \leq p_1 - 1\}$ and $E(C_2) = \{v_1 v_{p_2}, v_i v_{i+1}; 1 \leq i \leq p_2 - 1\}$ with $p_1 = 2k + 1$ for $k \geq 1$ and $p_2 = 2l + 1$ for $l \geq 1$. Let H_1' be a subgraph of H such that $H_1' = G \setminus E(C_1)$. So, H_1' is a disconnected graph that contains $p_1 - 1$ components where each component is a tree. For $i \in \{1,2,3, \dots, p_1 - 1\}$, we define T_{1_i} as the component of H_1' containing vertex $v_i \in V(C_1)$.

Suppose v is a vertex in H, then there are $i \in \{1,2,3, \dots, p_1 - 1\}$ such that $v \in V(T_{1_i})$. Suppose x and y are two adjacent vertices in H and there are $j \in \{1,2,3, \dots, p_1 - 1\}$ such that $x, y \in V(T_{1_j})$, then $d(v, z) = d(v, v_i) + d(v_i, v_j) + d(v_j, z)$ for $z \in \{x, y\}$. Since T_{1_j} is a tree and every two distinct vertices in the tree have a unique path between them, we get that either $d(v_j, x) < d(v_j, y)$ or $d(v_j, x) > d(v_j, y)$, which implies $d(v, x) \neq d(v, y)$. Therefore, x and y must be two different components of H_1'. Hence, x and y must be in C_{p_1}.

Let x and y be two adjacent vertices at C_{p_1} and $v \in V(T_{1_i})$. Suppose $d(v_i, x) < diam(C_{p_1}) = k$, then $d(v_i, x) < d(v_i, y)$ or $d(v_i, x) > d(v_i, y)$, which implies $d(v, x) \neq d(v, y)$. So, $d(v_i, x) = k = d(v_i, y)$. When p is odd, two adjacent vertices are obtained, namely $x = v_{i+k}$ and $y = v_{i+k+1}$ where both indices are at modulo p.

Furthermore, we have the properties of two odd cycles C_1 or C_2 in Observation 2.1 as follows.

Observation 2.1 *Let H be a bicyclic graph that contains an odd cycle C_1 or C_2. For p_1 is odd and p_2 is even (or p_1 is even and p_2 is odd). We have some properties as follows.*

1. *We define two vertices $u, v \in V(H)$ with $u \in V(C_1)$ and $v \in V(C_2)$ where there is a path $u - v$. This path is called a bridge, denoted by B_r.*
2. *If p_2 is even, then $d(v_i, v) \neq d(v_j, v)$ for two adjacent vertices $v_i, v_j \in V(C_2)$.*
3. *For $x_2, y_2 \in V(T_{2_j})$, since T_{2_j} is a tree and every two distinct vertices in the tree have a unique path between them, we get that either $d(v_j, x_2) < d(v_j, y_2)$ or $d(v_j, x_2) > d(v_j, y_2)$, which implies $d(v, x_2) \neq d(v, y_2)$.*

Theorem 2.2 *Let H be a bicyclic graph of order at least 3. If H contains cycles of order p_1 and p_2, then*

$$md_l(H) = \begin{cases} 1, & if\ p_1\ and\ p_2\ are\ even, \\ 2, & otherwise \end{cases}$$

Proof. Let C_1 and C_2 be a cycle graph of H. There are three cases for p_1 and p_2 as follows.

Case 1. If p_1 and p_2 are even

Because p_1 and p_2 are even, then H is a bipartite graph, based on Proposition 1.3 that $md_l(H) = 1$.

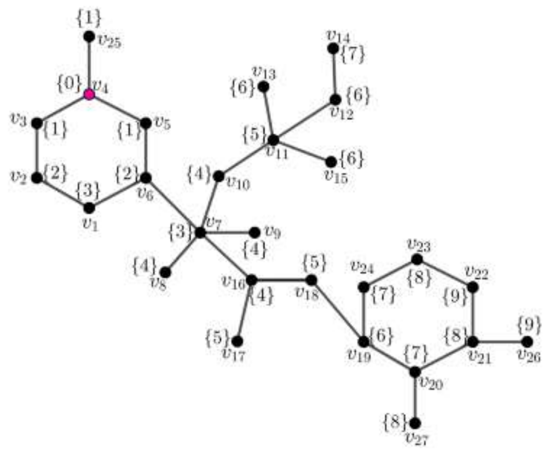

Fig. 5: $md_l(H) = 1$ with p_1 and p_2 are even

Case 2. If p_1 is odd and p_2 is even.

If p_1 is odd, then H is not a bipartite graph, resulting in $md_l(H) \geq 2$. Furthermore, we will prove the upper bound of the local multiset dimension of H that $md_l(H) \leq 2$. Let C_1 and C_2 be cycles contained in H. By Lemma 2.2, there is exactly one pair of x and y adjacent vertices in C_1 or C_2 such that $d(x, v) = d(y, v)$ for every $v \in V(H)$. Now, define $W = \{u_l, x | x \in V(T_{1_i})\}$ and $u_l \in V(C_{n_1})$. We show that vertex representation for two adjacent vertices is distinct as follows.

1. For two adjacent vertices $u_i, u_j \in V(C_2)$, $d(u_i, u_l) = d(u_j, u_l)$ and $d(u_i, x) \neq d(u_j, x)$. Thus, $r_m(u_i W) \neq r_m(u_j|W)$.
2. For two adjacent vertices $x_1, y_1 \in V(T_{1_k})$, $d(x_1, u_k) \neq d(y_1, u_k)$ by Observation 2.1 (3), implying that $d(x_1, u_l) \neq d(y_1, u_l)$ and \$$d(x_1, x) \neq d(y_1, x)$. Thus, $r_m(x_1|W \neq r_m(y_1|W)$.
3. For two adjacent vertices in a bridge, say, $w_i, w_j \in V(B_r)$, $d(w_i, u) \neq d(w_j, u)$ implying that $d(w_i, u_l) \neq d(w_i, u_l)$ and $d(w_j, x) \neq d(w_j, x)$. Thus, $r_m(w_i|W) \neq r_m(w_j|W)$.
4. For two adjacent vertices in C_2, i.e. $v_i, v_j \in V(C_2)$, $d(v_i, v) \neq d(v_j, v)$ by Observation 2.1 (2) implying that $d(v_i, u_l) \neq d(v_i, u_l)$ and $d(v_j, x) \neq d(v_j, x)$. Thus, $r_m(v_i|W) \neq r_m(v_j|W)$.
5. For two adjacent vertices in C_2 and $x_2, y_2 \in V(T_{2k})$, $d(x_2, v_k) \neq d(y_2, v_k)$ and $d(v_k, v) \neq d(v_k, v)$ by Observation 2.1 (3) implying that $d(x_2, u_l) \neq d(x_2, u_l)$ and $d(y_2, x) \neq d(y_2, x)$, respectively. Thus, $r_m(x_2|W) \neq r_m(y_2|W)$.

It is easy to see that by Equations 1. to 5. above, W is a local m-resolving set of H. Hence, $md_l(H) = 2$.

*Case 3***.** If p_1 and p_2 are odd.

If p_1 and p_2 are odd, then H is not a bipartite graph, implying that $md_l(H) \geq 2$. Furthermore, we will prove the upper bound of the local multiset dimension of H that $md_l(H) \leq 2$. Let C_1 and C_2 be cycles contained in H. By Lemma 2.2, there is exactly one pair of x and y adjacent vertices in C_1 or C_2 such that $d(x, v) = d(y, v)$ for every $v \in V(H)$. Now, define $W = \{u_l, v_l | u_l \in V(C_1)$ and $v_l \in V(C_2)\}$ with $d(u_l, u) \neq d(v_l, v)$. We show that vertex representation for two adjacent vertices is distinct as follows.

1. For two adjacent vertices $u_i, u_j \in V(C_1)$, $d(u_i, u_l) = d(u_j, u_l)$ and $d(u_i, u) = d(u_j, u)$ and so $d(u_i, v_l) \neq d(u_j, v_l)$. Thus, $r_m(u_i|W) \neq r_m(u_j|W)$.
2. For two adjacent vertices $x_1, y_1 \in V(T_{1k})$, based on Observation 2.1 (3) that $d(x_1, u_k) \neq d(y_1, u_k)$ then $d(x_1, u_l) \neq d(x_1, u_l)$. Since $(x_1, u) = d(y_1, u)$, then $d(x_1, v_l) \neq d(y_1, v_l)$. Thus, $r_m(x_1|W) \neq r_m(y_1|W)$.
3. For two adjacent vertices in bridges $w_i, w_j \in V(B_r)$, $d(w_i, u) \neq d(w_j, u)$, then $d(w_i, u_l) \neq d(w_i, u_l)$ and $d(w_i, v) \neq d(w_j, v)$ then $d(w_i, v_l) \neq d(w_i, v_l)$. Thus, $r_m(w_i|W) \neq r_m(w_j|W)$.
4. For two adjacent vertices $v_i, v_j \in V(C_2)$, $d(v_i, v_l) = d(v_j, v_l)$ and $d(v_i, v) = d(v_j, v)$ then $d(v_i, u_l) \neq d(v_j, u_l)$. Thus, $r_m(v_i|W) \neq r_m(v_j|W)$.
5. For two adjacent vertices in C_2 and $x_2, y_2 \in V(T_{2k})$, based on Observation 2.1 (3) that $d(x_2, v_k) \neq d(y_2, v_k)$ then $d(x_2, v_l) \neq d(y_2, v_l)$. Since $d(x_2, v) = d(y_2, v)$ then $d(x_2, u_l) \neq d(y_2, u_l)$. Thus, $r_m(x_2|W) \neq r_m(y_2|W)$.

Based on Equations 1 to 5 above show that W is a local \$m\$-resolving set of H. Therefore, $md_l(H) = 2$.

3 Conclusion

In this paper, the local multiset dimensions of unicyclic graphs and bicyclic graphs in which the cycles have no common vertex have been studied. Based on the results of this research, we propose the following open problem.

Open Problem 3.1 *Characterise the local multiset dimension of bicyclic graphs in which the two cycles have at least one vertex in common.*

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-Liliek Susilowati carried out the methodology, supervision, funding acquisition, writing review and editing.

-Dafik carried out the supervision and writing review and editing.

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