Reliability Evaluation Based on Uncertain Bayesian rule

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Abstract: This paper focuses on the reliability evaluation of a one-unit system based on uncertain Bayesian rule, in which the unit's lifetime is assumed to be an uncertain variable. Considering two types of the posterior uncertainty distribution of the lifetime, the Bayesian estimation method of uncertainty parameter is first proposed. Then reliability evaluation is carried out by calculating uncertainty reliability $R(T)$ with a specific time T and mean time between failure $MTBF$. Finally, some numerical examples are conducted to illustrate the application of the new method.

Key-Words: Uncertainty Bayesian rule, Parameter estimation, Reliability evaluation, $MTBF$

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1 Introduction

Reliability evaluation is an essential aspect of reliability research, which assesses the ability of electronic equipment or systems to realize their functions under specific conditions based on the life distribution function. Its method mainly uses mathematical statistics theory to analyze the specific distribution of the lifetime and evaluates the reliability of systems by estimating the distribution parameters. Commonly, reliability is characterized by reliability function, mean time between failures ($MTBF$), and other indicators, [1], [2], [3] and [4].

Bayes method has become an important means of reliability evaluation, which can improve the accuracy of parameter estimation by introducing the prior engineering knowledge of systems into reliability evaluation in the form of the prior distribution, [5], [6]. First, the prior distribution of the assumed parameters and the likelihood function associated with the observed data are obtained. Second, the posterior distribution of the parameter is deduced. Finally, the analytical solution or approximate solution of the parameter is determined.

In the Bayesian framework, the lifetime of the system is regarded as an unknown parameter, which is generally estimated using the expectation algorithm. In [7] Breipohl et al. indicated the application of Bayesian theory in making typical reliability decisions via decision theory. Tillman et al. in [8] reviewed some reliability problems using Bayesian inference. In [9] Sharma et al. analyzed various engineering systems to their reliability characteristics and studied the Bayesian analysis of system availability. Ando, T. in [10] proposed a Bayesian prediction information criterion to estimate the posterior mean of the expected log-likelihood of the prediction distribution. In [11] Guo et al. investigated the Bayesian melding method (BMM) for system reliability analysis by effectively integrating various available sources of expert knowledge and data at both subsystem and system levels. In [12] Lu and L. proposed a Bayesian approach for evaluating the system structure based on estimating the multiplicative or additive discrepancy between the system and component test data under the assumed structure while quantifying the uncertainty. The real systems are mostly uncertain random systems that are affected by both aleatory and epistemic uncertainties. It is of great significance to study effective reliability evaluation methods in various fields. Song et al. in [13] proposed a system reliability evaluation method based on Bayesian theory and multi-source information fusion. In [14] Alharbi et al. proposed a fuzzy Bayesian procedure to estimate the unknown parameters and fuzzy reliability function and applied it to compare estimators of cancer data set.

As mentioned in previous literature, the lifetime of the system was usually assumed as a stochastic variable. In practical cases, it is known that the estimated distribution is not close enough to the frequency in the world of eternal change. Therefore, the classical method is not available and it should be treated as an uncertainty distribution, [15],[16]. To deal with this kind of problem, uncertainty theory was founded by Liu in [17] and refined by Liu in [18]. It has become a branch of axiomatic mathematics for modeling human uncertainty.

At present, the uncertainty theory has been further developed and popularized. It has become a mathematical branch of modeling epistemic uncertainties under small data sizes or no data and has been introduced to the field of reliability. Zhang et al. in [19],[20] developed some system belief reliability formulas for different systems configurations. Zhang et al. in [21] considered the structure component's failure time as an uncertain variable because of the absence of historical data. In recent years, based on uncertainty theory, how to use limited failure time data to obtain reliability distribution has become the focus of scholars. For example, Z. et al. in [22] developed a new method called the graduation formula to construct belief reliability distribution with limited observations. In [23] Kang presented the lifetime model and reliability evaluations based on uncertainty theory. In [24] Lio and Kang gave a method to update a prior uncertainty distribution to a posterior uncertainty distribution based on the likelihood function and observation data in the sense of uncertainty theory.

In practical engineering, since system reliability testing is widely costly and with high reliability, the sample size of the system is normally very small, and the problem of non-failure frequently occurs. In addition, they have few historical operating data of their lifetime. The above issues can lead to a lack of knowledge in evaluating system reliability. Especially under the small sample size, the probability theory based on large samples is not appropriate anymore. However, using the system reliability in uncertain Bayesian rule has not been investigated in the literature. Therefore, this paper will propose a new method to present the lifetime distribution and reliability evaluations based on uncertain Bayesian rule.

The remainder of this paper is organized as follows: Section 2 is a preliminary basic knowledge about uncertainty theory. In Section 3 and Section 4, the definitions and theorems for calculating parameter values and evaluating the reliability from two special uncertainty distributions are provided. Some numerical examples with reliability evaluation are conducted to illustrate the application of the new method in Section 5. Finally, a concise conclusion is made in Section 6.

2 Preliminary

This section introduces some fundamental definitions and theorems of uncertainty theory.

Definition 1. *(Liu, [17]) An uncertain variable is a measurable function ξ from the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B of real numbers.*

Definition 2. *(Liu, [17]) The uncertainty distribution Φ of an uncertain variable ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Definition 3. *(Liu, [18]) Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.*

An uncertain variable ξ is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0 & , if\, x \leq a \\ \frac{x-a}{b-a} & , if\, a < x \leq b \\ 1 & , if\, x > b \end{cases}$$

denoted by $L(a, b)$ where a and b are real numbers with $a < b$ and the inverse uncertainty distribution of linear uncertain variable $L(a, b)$ is

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b.$$

An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = (1 + exp(\frac{\pi(e - x)}{\sqrt{3}\sigma}))^{-1}, x \in \Re$$

denoted by $N(e, \sigma)$ where e and σ are real numbers with $\sigma > 0$ and the inverse uncertainty distribution of normal uncertain variable $N(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = e + \frac{\sqrt{3}\sigma}{\pi} ln \frac{\alpha}{1 - \alpha}.$$

Expected value is the average value of uncertain variable in the sense of uncertain measure. It is an important feature of distribution and reflects the average value of the uncertain variable.

Theorem 1. *(Liu, [17]) Let ξ be an uncertain variable with uncertainty distribution Φ. Then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx$$

.

Remark 1. *(Liu, [18]) Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha$$

Theorem 2. *(Lio and Liu, [26], Likelihood Function) Suppose η_1,η_2, ...,η_n are iid uncertain variables with uncertainty distribution $F(y|\theta)$ where θ is an unknown parameter, and have observed values*

y_1,y_2,. . .,y_n, *respectively. If* $F(y|\theta)$ *is differentiable at* y_1,y_2,. . .,y_n, *then the likelihood function associated with* y_1,y_2,. . .,y_n *is*

$$L(s|y_1, y_2, \ldots, y_n) = \bigwedge_{i=1}^{m} F'(y_i|s). \qquad (1)$$

Definition 4. *(Lio, [24])Suppose* ξ *is an uncertain variable with prior uncertainty distribution* $\Phi(x)$*, and* η_1,η_2, . . .,η_n *are iid uncertain variables from a population with uncertainty distribution* $F(y|\xi)$*. Suppose* $\Phi'(x)$ *and* $F'(y|\xi)$ *can be obtained, and* η_1,η_2, . . .,η_n *have observed values* y_1,y_2,. . .,y_n*, respectively. Then the posterior uncertainty distribution is defined by*

$$\begin{aligned}
&\Psi(x|y_1, y_2, \ldots, y_n)\\
&= \frac{\int_{-\infty}^{x} L(s|y_1, y_2, \ldots, y_n) \wedge \Phi'(x)ds}{\int_{-\infty}^{+\infty} L(s|y_1, y_2, \ldots, y_n) \wedge \Phi'(x)ds}\\
&= \frac{\int_{-\infty}^{x} \bigwedge_{i=1}^{m} F'(y_i|s) \wedge \Phi'(x)ds}{\int_{-\infty}^{+\infty} \bigwedge_{i=1}^{m} F'(y_i|s) \wedge \Phi'(x)ds}.
\end{aligned} \qquad (2)$$

It is clear that if

$$\int_{-\infty}^{+\infty} \bigwedge_{i=1}^{m} F'(y_i|s) \bigwedge \Phi'(x)ds \neq 0$$

then the posterior uncertainty distribution defined by Eq. (2) is a continuous monotone increasing function satisfying

$$\begin{gathered}
0 \leq \Psi(x|y_1, y_2, \ldots, y_n) \leq 1,\\
\Psi(x|y_1, y_2, \ldots, y_n) \neq 0,\\
\Psi(x|y_1, y_2, \ldots, y_n) \neq 1.
\end{gathered}$$

It was proved by Peng and Iwamura, [25], and Liu and Lio, [26], that Eq.(2) is indeed an uncertainty distribution.

3 Uncertain Bayesian parameter estimation

In this section, we first present an estimation method of uncertainty parameters based on the posterior uncertainty distribution.

Definition 5. *(Uncertain Posterior Expected Estimation) Suppose* ξ *is an uncertain variable with the posterior uncertainty distribution* $\Psi(x|y_1, y_2, \ldots, y_n)$ *and* $y_1, y_2, \ldots, y_m$ *are observed values, respectively. If the inverse uncertainty distribution* $\Psi^{-1}(\alpha)$ *exists, then the posterior expected estimation of* ξ *is*

$$\begin{aligned}
\hat{\xi}_E &= E(\xi|y_1, y_2, \ldots, y_n)\\
&= \int_0^{+\infty} (1 - \Psi(x|y_1, y_2, \ldots, y_n))dx\\
&= \int_{\Psi(0)}^{1} \Psi^{-1}(\alpha)d\alpha.
\end{aligned} \qquad (3)$$

To present the lifetime distribution of systems in the uncertainty theory, we consider two simple and commonly use situations.

Lemma 1. *(Lio, [24]) Suppose* ξ *is an uncertain variable with linear prior uncertainty distribution* $L(a, b)$*, and* $\eta_1, \eta_2, \ldots, \eta_n$ *are iid uncertain variables from a population with linear uncertainty distribution* $L(\xi - c, \xi + d)$*, c,d≥0 and observed values* $y_1, y_2, \ldots, y_m$*, respectively. If it is assumed that*

$$\bigvee_{i=1}^{n} (y_i - d) \vee a \leq \bigwedge_{i=1}^{n} (y_i + c) \wedge b,$$

then the posterior uncertainty distribution is

$$L(\bigvee_{i=1}^{n} (y_i - d) \vee a, \bigwedge_{i=1}^{n} (y_i + c) \wedge b). \qquad (4)$$

Theorem 3. *Suppose an uncertain variable* ξ *has the posterior uncertainty distribution* $\Psi(x|y_1, y_2, \ldots, y_n)$ *and* $y_1, y_2, \ldots, y_m$ *are observed values, when the prior distribution* $L(a, b)$ *is assumed to be linear and the data are observed from a population whose distribution function* $L(\xi - c, \xi + d)$*, c,d≥0 is also linear. If the inverse uncertainty distribution* $\Psi^{-1}(\alpha)$ *exists, then the posterior expected estimation is*

$$\hat{\xi}_E = \frac{(\bigvee_{i=1}^{n} (y_i - d) \vee a) + (\bigwedge_{i=1}^{n} (y_i + c) \wedge b)}{2}. \qquad (5)$$

Proof. It follows from Definition 5 and Lemma 1 that the posterior expected estimation is

$$\begin{aligned}
\hat{\xi}_E &= \int_0^1 \Psi^{-1}(\alpha)d\alpha\\
&= \int_0^1 ((\bigvee_{i=1}^{n} (y_i - d) \vee a) - (\bigwedge_{i=1}^{n} (y_i + c) \wedge b))\alpha\\
&+ (\bigwedge_{i=1}^{n} (y_i + c) \wedge b)d\alpha\\
&= \frac{(\bigvee_{i=1}^{n} (y_i - d) \vee a) + (\bigwedge_{i=1}^{n} (y_i + c) \wedge b)}{2}
\end{aligned}$$

The Theorem is proved. □

According to Eq.(5), then the uncertainty distribution F of the unit can be obtained.

$$F(y|\hat{\xi}_E) = \begin{cases} 0 & , if\, y \le \hat{\xi}_E - c \\ \frac{y-(\hat{\xi}_E - c)}{d+c} & , if\, \hat{\xi}_E - c < y \le \hat{\xi}_E + d \\ 1 & , if\, y > \hat{\xi}_E + d \end{cases} \tag{6}$$

Lemma 2. *(Lio, [24]) Suppose ξ is an uncertain variable with normal prior uncertainty distribution $N(e,\sigma)$, and $\eta_1, \eta_2, \dots, \eta_n$ are iid uncertain variables from a population with normal uncertainty distribution $N(\xi,\sigma)$ and observed values $y_1, y_2, \dots, y_m$, respectively. Then the posterior uncertainty distribution is*

$$\Psi(x|y_1, y_2, \dots, y_n) = \begin{cases} \frac{\Phi_M(x)}{\Phi_M((m+M)/2)+1-\Phi_m((m+M)/2)} & , if\, x \le \frac{m+M}{2} \\ \frac{\Phi_M((m+M)/2)+\Phi_m(x)-\Phi_m((m+M)/2)}{\Phi_M((m+M)/2)+1-\Phi_m((m+M)/2)} & , if\, x > \frac{m+M}{2} \end{cases} \tag{7}$$

where Φ_M and Φ_m are the uncertainty distributions $N(M,\sigma)$ and $N(m,\sigma)$, respectively, and

$$M = \bigvee_{i=1}^{n} y_i \wedge e, m = \bigwedge_{i=1}^{n} y_i \wedge e.$$

Theorem 4. *Suppose an uncertain variable ξ has the posterior uncertainty distribution $\Psi(x|y_1, y_2, \dots, y_n)$ and $y_1, y_2, \dots, y_m$ are observed values, when the prior distribution $N(e,\sigma)$ is assumed to be normal and the data are observed from a population whose distribution function $N(\xi,\sigma)$ is also normal. If the inverse uncertainty distribution $\Psi^{-1}(\alpha)$ exists, then the posterior expected estimation is*

$$\hat{\xi}_E = \frac{m+M}{2} \tag{8}$$

where

$$M = \bigvee_{i=1}^{n} y_i \wedge e, m = \bigwedge_{i=1}^{n} y_i \wedge e.$$

Proof. It follows from Definition 5 and Lemma 2 that the posterior expected estimation is

$$\begin{aligned} \hat{\xi}_E &= \int_0^1 \Psi^{-1}(\alpha) d\alpha \\ &= \int_0^1 (\frac{m+M}{2} - \frac{\sqrt{3}\sigma}{\pi} ln \frac{\alpha}{1-\alpha}) d\alpha \\ &= \frac{m+M}{2} \end{aligned}$$

The Theorem is verified. □

According to Eq.(8), then the uncertainty distribution F of the unit can be obtained.

$$F(y|\hat{\xi}_E) = (1 + exp(\frac{\pi(\hat{\xi}_E - y)}{\sqrt{3}\sigma}))^{-1} \tag{9}$$

4 Uncertain Bayesian reliability evaluation

In this section, we consider uncertainty reliability evaluation, which is defined as the measure that it will perform a required function under stated conditions for a stated period. To assess the system reliability, two indicators the $MTBF$ and $R(T)$ for uncertainty assessment are first defined.

Definition 6. *(Mean Time Between Failure)Suppose the failure free time η ($\eta \ge 0$) is a nonnegative uncertain variable, and the uncertainty distribution of the unit is $F(y) \equiv \mathcal{M}\{\eta \leqslant y\}$. The MTBF of the unit is defined as*

$$\begin{aligned} MTBF &= \int_0^{+\infty} \mathcal{M}\{\eta > y\} dy \\ &= \int_0^{+\infty} (1 - F(y)) dy \\ &= \int_{F(0)}^1 F^{-1}(\alpha) d\alpha \end{aligned} \tag{10}$$

where F^{-1} is the inverse uncertainty distributions of F.

In practical application, the uncertainty measure of failure free time over a certain value T is also an important index, which represent the unceratinty reliability $R(T)$ that a system will perform a required function at the specific time T under stated operating conditions using the uncertainty theory, expressed as,

$$R(T) = \mathcal{M}\{y > T\} = 1 - F(T). \tag{11}$$

Theorem 5. *Suppose the failure free time η is a nonnegative uncertain variable, and the uncertainty distribution of the unit $F(y|\hat{\xi}_E)$ is $L(\hat{\xi}_E - c, \hat{\xi}_E + d)$. If the inverse uncertainty distribution $F^{-1}(\alpha)$ exists, the $MTBF$ of the unit and uncertainty reliability R are*

$$MTBF = \int_0^1 (1-\alpha)(\hat{\xi}_E - c) + \alpha(\hat{\xi}_E + d) d\alpha \tag{12}$$

$$R(T) = \begin{cases} 1 & , if\, y \le \hat{\xi}_E - c \\ \frac{(\hat{\xi}_E + d) - T}{d+c} & , if\, \hat{\xi}_E - c < y \le \hat{\xi}_E + d \\ 0 & , if\, y > \hat{\xi}_E + d \end{cases} . \tag{13}$$

Proof. It follows from Definition 6 and

$$F(y|\hat{\xi}_E) = \begin{cases} 0 & , if\, y \le \hat{\xi}_E - c \\ \frac{y-(\hat{\xi}_E - c)}{d+c} & , if\, \hat{\xi}_E - c < y \le \hat{\xi}_E + d \\ 1 & , if\, y > \hat{\xi}_E + d \end{cases}$$

that the $MTBF$ of the unit is

$$\begin{aligned} MTBF &= \int_0^{+\infty} (1 - F(y|\hat{\xi}_E))dy \\ &= \int_0^1 F^{-1}(\alpha)d\alpha \\ &= \int_0^1 (1-\alpha)(\hat{\xi}_E - c) + \alpha(\hat{\xi}_E + d)d\alpha \end{aligned}$$

where F^{-1} is the inverse uncertainty distributions of F, and uncertainty reliability R is

$$\begin{aligned} R(T) &= \mathcal{M}\{y > T\} \\ &= 1 - F(T|\hat{\xi}_E) \\ &= \begin{cases} 1 & , if\, y \le \hat{\xi}_E - c \\ \frac{(\hat{\xi}_E + d) - T}{d+c} & , if\, \hat{\xi}_E - c < y \le \hat{\xi}_E + d \\ 0 & , if\, y > \hat{\xi}_E + d \end{cases} \end{aligned}$$

The theorem is proved. □

Theorem 6. *Suppose the failure free time η is a nonnegative uncertain variable, and the uncertainty distribution of the unit $F(y|\hat{\xi}_E)$ is $N(\hat{\xi}_E, \sigma)$. If the inverse uncertainty distribution $F^{-1}(\alpha)$ exists, the $MTBF$ of the unit and uncertainty reliability R are*

$$MTBF = \int_{(1+exp(\frac{\pi\hat{\xi}_E}{\sqrt{3}\sigma}))^{-1}}^1 (\hat{\xi}_E + \frac{\sqrt{3}\sigma}{\pi}\ln\frac{\alpha}{1-\alpha})d\alpha \tag{14}$$

$$R(T) = 1 - (1 + exp(\frac{\pi(\hat{\xi}_E - T)}{\sqrt{3}\sigma}))^{-1}. \tag{15}$$

Proof. It follows from Definition 6 and

$$F(y|\hat{\xi}_E) = 1 + exp(\frac{\pi(\hat{\xi}_E - y)}{\sqrt{3}\sigma})^{-1}$$

that the $MTBF$ of the unit is

$$\begin{aligned} MTBF &= \int_0^{+\infty} (1 - F(y|\hat{\xi}_E))dy \\ &= \int_{(1+exp(\frac{\pi\hat{\xi}_E}{\sqrt{3}\sigma}))^{-1}}^1 F^{-1}(\alpha)d\alpha \\ &= \int_{(1+exp(\frac{\pi\hat{\xi}_E}{\sqrt{3}\sigma}))^{-1}}^1 (\hat{\xi}_E + \frac{\sqrt{3}\sigma}{\pi}\ln\frac{\alpha}{1-\alpha})d\alpha \end{aligned}$$

where F^{-1} is the inverse uncertainty distributions of F, and uncertainty reliability R is

$$\begin{aligned} R(T) &= \mathcal{M}\{y > T\} \\ &= 1 - F(T|\hat{\xi}_E) \\ &= 1 - (1 + exp(\frac{\pi(\hat{\xi}_E - T)}{\sqrt{3}\sigma}))^{-1} \end{aligned}$$

The theorem is verified. □

5 Numerical examples

This section will provided some examples to illustrate the application of the new method.

Example 1. *Suppose ξ is an uncertain variable with linear prior uncertainty distribution $L(1510, 1550)$, and η_1, η_2, η_3 are iid uncertain variables from a population with linear uncertainty distribution $L(\xi - 10, \xi + 20)$ and observed values $y_1 = 1520, y_2 = 1530, y_3 = 1540$, respectively. Then it follows from Lemma 1 that the posterior uncertainty distribution is $L(1520, 1530)$, i.e.*

$$\Psi(x|1520, 1530, 1540) = \begin{cases} 0 & , if\, x \le 1520 \\ \frac{x-1520}{10} & , if\, 1520 < x \le 1530 \\ 1 & , if\, x > 1530 \end{cases} .$$

Then, using the observation y_1, y_2, y_3, the unknown parameter can be estimated.

$$\hat{\xi}_E = E(\xi) = 1525$$

According to Eq.(6), the uncertainty distribution of the unit and its derivative F' can be obtained(see Figure 1).

$$F(y|\hat{\xi}_E) = \begin{cases} 0 & , if\, y \le 1515 \\ \frac{y-1515}{30} & , if\, 1515 < y \le 1545 \\ 1 & , if\, y > 1545 \end{cases}$$

$$F'(y|\hat{\xi}_E) = \begin{cases} \frac{y}{30} & , if\, 1515 < y \le 1545 \\ 0 & , otherwise \end{cases}$$

By substituting the estimation results into Eq.(12) *and Eq.*(13) *, the $MTBF$ and $R(1520)$ can be obtained. As shown in Figure 2, it can be seen that the uncertainty reliability of the unit decreases with the increase in the expected time of failure.*

$$MTBF = \int_0^1 1515 + 30\alpha d\alpha = 1530$$
$$R(1520) = 1 - F(1520) \approx 83.3\%$$

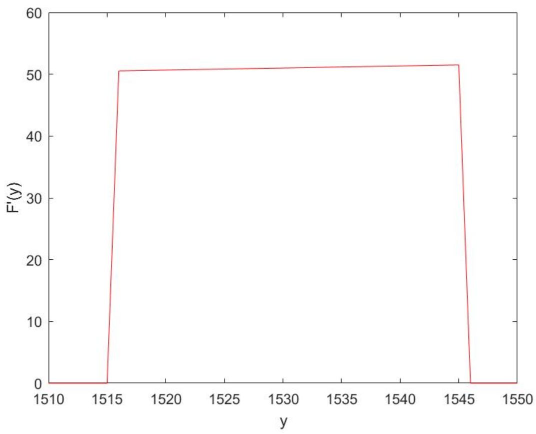

Figure 1: Function F' in Example 1

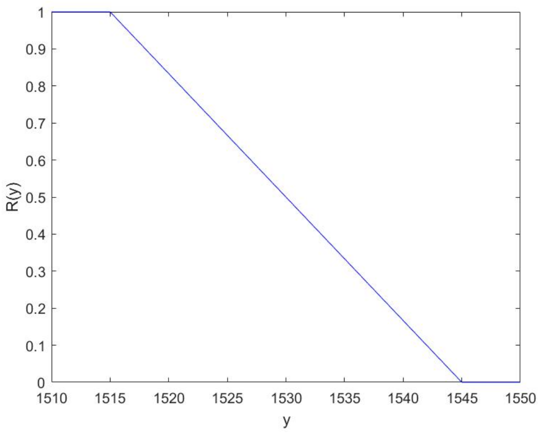

Figure 2: Uncertainty reliability R in Example 1

Example 2. *Suppose ξ is an uncertain variable with normal prior uncertainty distribution $N(1540, 3)$, and η_1, η_2 are iid uncertain variables from a population with normal uncertainty distribution $N(\xi, 3)$ and observed values $y_1 = 1510, y_2 = 1550$, respectively. Then it follows from Lemma 2 that the posterior uncertainty distribution is*

$$\Psi(x|1510, 1550) = \begin{cases} \frac{\Phi_2(x)}{\Phi_2(1530)+1-\Phi_m(1530)}, & if\, x \le 1530 \\ \frac{\Phi_2(1530)+\Phi_2(x)-\Phi_1(1530)}{\Phi_2(1530)+1-\Phi_1(1530)}, & if\, x > 1530 \end{cases}$$

where Φ_2 and Φ_1 are the uncertainty distributions $N(1550, 3)$ and $N(1510, 3)$. Then, using the observation y_1, y_2, the unknown parameter can be estimated.

$$\hat{\xi}_E = E(\xi) = 1530$$

According to Eq.(9)*, the uncertainty distribution of the unit and its derivative F' can be obtained (see Figure 3).*

$$F(y|\hat{\xi}_E) = (1 + exp(\frac{\pi(1530 - y)}{3\sqrt{3}}))^{-1}$$
$$F'(y|\hat{\xi}_E) = \frac{\frac{\pi}{3\sqrt{3}} exp(\frac{\pi(1530-y)}{3\sqrt{3}})}{(1 + exp(\frac{\pi(1530-y)}{3\sqrt{3}}))^2}$$

By substituting the estimation results into Eq.(14) *and Eq.*(15)*, the $MTBF$ and $R(1520)$ can be obtained. As shown in Figure 4, it can be seen that the uncertainty reliability of the unit decreases with the increase in the expected time of failure.*

$$MTBF = \int_{(1+exp(\frac{1530\pi}{3\sqrt{3}}))^{-1}}^1 (1530 + \frac{3\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}) d\alpha$$
$$= 1530$$
$$R(1520) = 1 - (1 + exp(\frac{\pi(1530 - 1520)}{3\sqrt{3}}))^{-1}$$
$$\approx 99.76\%$$

Example 3. *Suppose ξ is an uncertain variable with normal prior uncertainty distribution $N(1540, 3)$, and η_1, η_2 are iid uncertain variables from a population with linear uncertainty distribution $L(\xi - 20, \xi + 10)$ and observed values $y_1 = 1530, y_2 = 1540$, respectively. Then it follows from Definition 4 that the posterior uncertainty distribution is*

$$\Psi(x|1530, 1540) = \begin{cases} \frac{\Phi(x)-0.0024}{0.3886}, & if\, 1530 < x \le 1538.5 \\ \frac{(x-1538.5)/30}{0.3886}, & if\, 1538.5 < x \le 1541.5 \\ \frac{\Phi(x)-0.7124}{0.3886}, & if\, 1541.5 < x \le 1550 \end{cases}$$

where Φ is the uncertainty distributions $N(1540, 3)$. Then, using the observation y_1, y_2, the unknown parameter can be estimated.

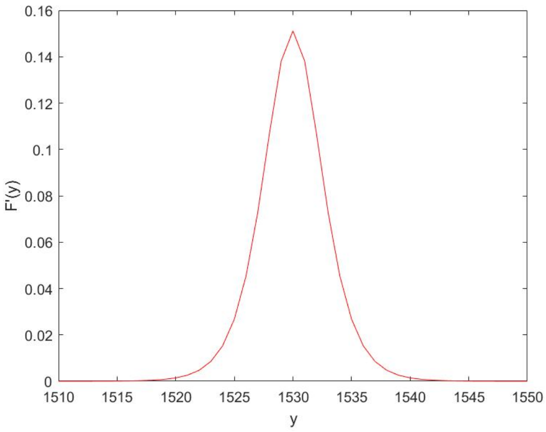

Figure 3: Function F' in Example 2

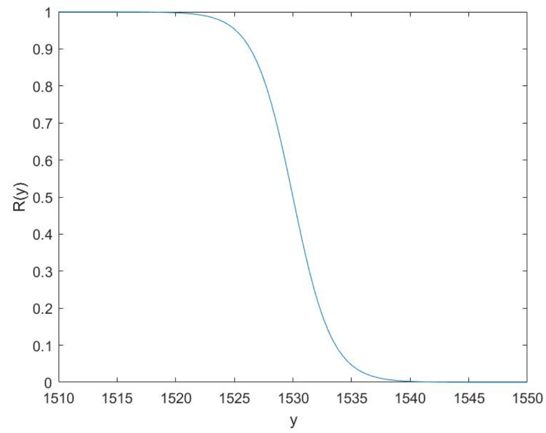

Figure 4: Uncertainty reliability R in Example 2

$$\hat{\xi}_E = E(\xi) = 1543.38$$

Moreover, the uncertainty distribution of the unit and its derivative F' can be obtained(see Figure 6).

$$F(y|\hat{\xi}_E) = \begin{cases} 0 & , if\, y \le 1523.38 \\ \frac{y-1523.38}{30} & , if\, 1523.38 < y \le 1553.38 \\ 1 & , if\, y > 1553.38 \end{cases}$$

$$F'(y|\hat{\xi}_E) = \begin{cases} \frac{y}{30} & , if\, 1523.38 < y \le 1553.38 \\ 0 & , otherwise \end{cases}$$

By substituting the estimation results into Eq.(10) and Eq.(11) , the $MTBF$ and $R(1520)$ can be obtained. As shown in Figure 5, it can be seen that the uncertainty reliability of the unit decreases with the increase in the expected time of failure.

$$\begin{aligned} MTBF &= \int_0^{+\infty} (1 - F(y|\hat{\xi}_E))dy \\ &= \int_0^1 F^{-1}(\alpha)d\alpha \\ &= \int_0^1 (1-\alpha)(\hat{\xi}_E - c) + \alpha \\ &= \int_0^1 1523.38 + 30\alpha d\alpha \\ &= 1538.38 \end{aligned}$$

where F^{-1} is the inverse uncertainty distributions of F.

$$R(1520) = 1 - F(1520) \approx 88.74\%$$

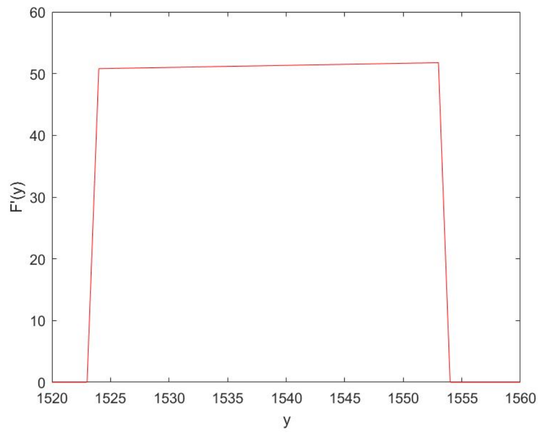

Figure 5: Function F' in Example 3

6 Conclusion

This paper studied the reliability evaluation of the system based on uncertain Bayesian rule, and the following conclusions can be drawn:

(i)Bayesian estimation method of uncertainty parameter was first proposed. There was not enough data to obtain the probability distribution of the lifetime, so the stochastic method did not apply to our research. Thus, the lifetime of the unit was regarded as an uncertain variable, where two types of the uncertainty distribution (linear and normal) were considered. It can be extended to Bayesian parameter testing and decision-making, providing a basic method for the research of uncertain statistical inference.

(ii)Based on the method of uncertain Bayesian parameter estimation, reliability evaluation was derived by calculating two reliability indexes ($MTBF$

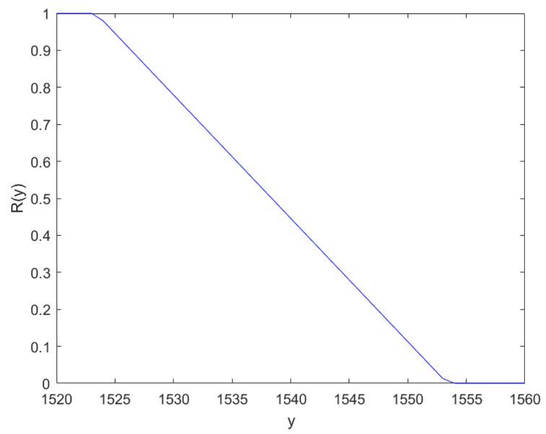

Figure 6: Uncertainty reliability R in Example 3

and $R(T)$). Finally, Some numerical examples were given to illustrate the application of the new method. This method can be applied to reliability evaluation in the engineering field, mainly aiming at the shortage of failure data and considering the reliability of experts. It can also be applied to project evaluation and decision-making in the economic field.

For future works, the uncertain hypothesis testing and decision-making for the unknown uncertainty parameters in uncertain Bayesian statistics will be studied. When the operational data is fully obtained, some random lifetime can be considered, such as Weibull distribution.

Contribution of individual authors to the creation of a scientific article

Chunxiao Zhang proposed the idea of the method and checked the correctness of the manuscript.

Yuanyuan Wang gave the method and wrote the article.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.