Convergence of Iterative Scheme for Asymptotically Nonexpansive Mapping in Hadamard Spaces

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Abstract: -In this paper, we introduce and prove the convergence of a novel iterative scheme for asymptotically nonexpansive mapping under some suitable conditions in the context of Hadamard spaces. We also present a numerical experiment in which the rate of convergence of the new iterative scheme is compared to that of an existing iterative scheme.

Key-Words: Hadamard space, CAT(0) space, Asymptotically nonexpansive mappings, Weak and strong convergence

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1 Introduction

A geodesic triangle $\Delta(u_1, u_2, u_3)$ in a geodesic metric space (X, d) consists of three points in X (called vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle $\Delta(u_1, u_2, u_3)$ in (X, d) is a triangle $\bar{\Delta}(u_1, u_2, u_3) := \Delta(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{u}_i, \bar{u}_j) = d(u_i, u_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists, [1].

Let Δ be a geodesic triangle in X and Δ its comparison triangle in $\mathbb{R}^2$. Then Δ is said to satisfy CAT(0) inequality if for all $u, v \in \Delta$ and all comparison points $u, v \in \Delta$, $d(u, v) \leq d_{\mathbb{R}^2}(u, v)$. A geodesic metric space X is called a CAT(0) space if all geodesic triangles satisfy the above comparison axiom (i.e. CAT(0) inequality). Some well known examples of CAT(0) spaces are complete. The complete CAT(0) spaces are often called Hadamard spaces.

Fixed point theory in a CAT(0) space has been first studied by Kirk (see [2]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point.

Let $\mathcal{K}$ be a nonempty closed subset of a CAT(0) space X, and $\mathcal{T}$ be a self map defined on $\mathcal{K}$. Then $\mathcal{T}$ is said to be:

- nonexpansive if
$$d(\mathcal{T}u, \mathcal{T}v) \leq d(u, v), \quad \forall\, u, v \in \mathcal{K},$$
- asymptotically nonexpansive if there exists a sequence$\{\zeta_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} \zeta_n = 1$ such that
$$d(\mathcal{T}^n u, \mathcal{T}^n v) \leq \zeta_n d(u, v),\ \forall\, u, v \in \mathcal{K},\ \forall\, n \geq 1,$$
- uniformly $\mathcal{L}$-Lipschitzian if there exists a constant $\mathcal{L} > 0$ such that
$$d(\mathcal{T}^n u, \mathcal{T}^n v) \leq \mathcal{L}(u, v), \quad \forall\, u, v \in \mathcal{K} \ \ \forall\, n \geq 1.$$

Moreover, every asymptotically nonexpansive mapping is a uniformly $\mathcal{L}$-Lipschitzian mapping with $\mathcal{L} = \sup_{n\in\mathbb{N}}\{\zeta_n\}$.

A mapping $\mathcal{T}$ is said to have a fixed point u^* if $\mathcal{T}u^* = u^*$ and a sequence $\{u_n\}$ is said to be asymptotic fixed point sequence if
$$\lim_{n\to\infty} d(u_n, \mathcal{T}u_n) = 0.$$

Authors create many new iterative processes to achieve a relatively effective rate of convergence and overcome such difficulties (see, e.g., Mann [4], Ishikawa [5], Agarwal et al. [6], Noor [7], Abbas and Nazir [8] and Thakur et al. [9]).

Şahin and Basarir, [10], suggested an effective two step iterative scheme for approximating fixed points of asymptotically quasi-nonexpansive mapping and sequence $\{u_n\}$ as follows:
$$\begin{cases} u_1 \in \mathcal{K}, \\ v_n = (1-\rho_n)u_n \oplus \rho_n \mathcal{T}^n u_n, \\ u_{n+1} = (1-\lambda_n)\mathcal{T}^n u_n \oplus \lambda_n \mathcal{T}^n v_n, \quad \forall\, n \geq 1, \end{cases} \tag{1}$$
where and throughout the paper $\{\lambda_n\}, \{\rho_n\}$ are the sequence such that $0 \leq \lambda_n, \rho_n \leq 1$ for all $n \geq 1$. They established some strong convergence results under some suitable conditions such that generalizing some results of Khan and Abbas, [11].

Niwongsa and Panyanak, [12], suggested an effective two step iterative scheme for approximating fixed points of asymptotically nonexpansive mapping and sequence $\{u_n\}$ as follows:

$$\begin{cases} u_1 \in \mathcal{K}, \\ y_n = \tau_n \mathcal{T}^n u_n \oplus (1-\tau_n) u_n, \\ v_n = \rho_n \mathcal{T}^n y_n \oplus (1-\rho_n) u_n, \\ u_{n+1} = \lambda_n \mathcal{T}^n v_n \oplus (1-\lambda_n) u_n, \quad \forall\, n \geq 1, \end{cases} \tag{2}$$

where $\{\lambda_n\}$, $\{\rho_n\}$ and $\{\tau_n\}$ are real sequence in $[0, 1]$. They proved Δ and strong convergence theorems of the following Noor iteration for an asymptotically nonexpansive mapping in CAT(0) spaces.

Recently, Yambangwai et al., [13], suggested an effective three step iterative scheme for approximating fixed points of asymptotically nonexpansive mapping and sequence $\{u_n\}$ as follows:

$$\begin{cases} u_1 \in \mathcal{K}, \\ y_n = \tau_n \mathcal{T}^n u_n \oplus (1-\tau_n) u_n, \\ v_n = \rho_n \mathcal{T}^n y_n \oplus (1-\rho_n) y_n, \\ u_{n+1} = \lambda_n \mathcal{T}^n v_n \oplus (1-\lambda_n) \mathcal{T}^n y_n, \quad \forall\, n \geq 1, \end{cases} \tag{3}$$

where $\{\lambda_n\}$, $\{\rho_n\}$ and $\{\tau_n\}$ are real sequence in $[0, 1]$. They established some convergence theorems to approximate the fixed points of asymptotically nonexpansive mapping in the setting CAT(0) spaces.

Motivated by the preceding work, we present a new iterative scheme, which is defined as follows:

$$\begin{cases} u_1 \in \mathcal{K}, \\ y_n = \mathcal{T}^n(\tau_n \mathcal{T}^n u_n \oplus (1-\tau_n) u_n), \\ v_n = \mathcal{T}^n(\rho_n \mathcal{T}^n y_n \oplus (1-\rho_n) y_n), \\ u_{n+1} = \mathcal{T}^n(\lambda_n \mathcal{T}^n v_n \oplus (1-\lambda_n) v_n), \quad \forall\, n \geq 1, \end{cases} \tag{4}$$

where $\{\lambda_n\}$, $\{\rho_n\}$ and $\{\tau_n\}$ are real sequence in $[0, 1]$ and $\mathcal{T}$ is an asymptotically nonexpansive mapping on a nonempty closed bounded and convex subset of a Hadamard space X.

2 Preliminaries

Definition 2.1. [14] *A sequence $\{u_n\}$ in X is said to Δ-converge to $u^* \in X$ if u^* is the unique asymptotic center of $\{w_n\}$ for every subsequence $\{w_n\}$ of $\{u_n\}$. In this case we*

write Δ-$\lim_{n\to\infty} u_n = u^*$ and we call u^* the Δ-$\lim_{n\to\infty} u_n = u^*$

Lemma 2.2. [14]

(i) *Every bounded sequence in X has Δ-convergence subsequence.*

(ii) *If $\mathcal{K}$ is a closed convex subset of X and if $\{u_n\}$ is a bounded sequence in $\mathcal{K}$, then the asymptotic center of $\{u_n\}$ is in $\mathcal{K}$.*

The asymptotic radius $r(\{u_n\})$ of $\{u_n\}$ is given by

$$r(\{u_n\}) = \inf\{r(u, \{u_n\}) : u \in X\},$$

and the asymptotic center $\mathcal{A}(\{u_n\})$ of $\{u_n\}$ is the set

$$\mathcal{A}(\{u_n\}) = \{u \in X : r(u, \{u_n\}) = r(\{u_n\})\}.$$

Lemma 2.3. [15] *Let X be a complete CAT(0) space and $\{u_n\}$ be a bounded sequence in X. If $\mathcal{A}(\{u_n\}) = \{u\}$, $\{w_n\}$ is a subsequence of $\{u_n\}$ such that $\mathcal{A}(\{w_n\}) = \{w\}$ and $d(u_n, w)$ converges, then $u = w$.*

Lemma 2.4. [16] *Let $\mathcal{K}$ be a closed and convex subset of a complete CAT(0) space X and $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ be an asymptotically nonexpansive mapping. Let $\{u_n\}$ be a bounded sequence in $\mathcal{K}$ such that $\lim_{n\to\infty} d(u_n, \mathcal{T}u_n) = 0$ and Δ-$\lim_{n\to\infty} u_n = u^*$. Then $u^* = \mathcal{T}u^*$.*

Lemma 2.5. [17] *Let X be a CAT(0) space, $u, v, w \in X$ and $t \in [0, 1]$. Then*

(i) $d((1-t)u \oplus tv, w) \leq (1-t)d(u, w) + td(v, w)$.

(ii) $d^2((1-t)u \oplus tv, w) \leq (1-t)d^2(u, w) + td^2(v, w) - t(1-t)d^2(u, v)$.

Lemma 2.6. [18] *Let $\{\delta_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$\delta_{n+1} \leq (1+\sigma_n)\delta_n,\ n \geq 1.$$

If $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} \delta_n$ exists.

Lemma 2.7. [19] *Let X be a complete CAT(0) space and let $u^* \in X$. Suppose $\{\alpha_n\}$ is a sequence in $[e, f]$ for some $e, f \in (0, 1)$ and $\{u_n\}$, $\{w_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(u_n, u^*) \leq l$, $\limsup_{n\to\infty} d(w_n, u^*) \leq l$ and $\lim_{n\to\infty} d((1-\alpha_n)u_n \oplus \alpha_n w_n, u^*) = l$ for some $l \geq 0$. Then $d(u_n, w_n) = 0$.*

Theorem 2.8. [20] *Let $\mathcal{K}$ be a nonempty bounded closed and convex subset of a complete CAT(0) space X and $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ be asymptotically nonexpansive. Then $\mathcal{T}$ has a fixed point.*

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3 Main results

Theorem 3.1. *Let $\mathcal{K}$ be a closed bounded and convex subset of a Hadamard space X and a self map $\mathcal{T}$ defined on $\mathcal{K}$ be an asymptotically nonexpansive mapping with $\{\zeta_n\}$. Assume that the following conditions hold:*

(i) $\{\zeta_n\} \geq 1$ *and* $\sum_{n=1}^{\infty}(\zeta_n - 1) < \infty$,

(ii) *there exist constants c_1, c_2 with $0 < c_1 \leq \tau_n \leq c_2 < 1$ for each $n \in \mathbb{N}$,*

(iii) *there exist constants b_1, b_2 with $0 < b_1 \leq \rho_n \leq b_2 < 1$ for each $n \in \mathbb{N}$,*

(iv) *there exist constants* a_1, a_2 *with* $0 < a_1 \leq \lambda_n \leq a_2 < 1$ *for each* $n \in \mathbb{N}$.

For the sequence $\{u_n\}$ *given by* (4). *Then* $\lim_{n\to\infty} d(u_n, u^*)$ *exists for all* $u^* \in \mathcal{F}(\mathcal{T})$.

Proof. Using Theorem 2.8, we note that $\mathcal{F}(\mathcal{T}) \neq \emptyset$. Putting $\zeta_n = 1+\kappa_n$ for all $n \geq 1$. Using $\sum_{n=1}^{\infty}(\zeta_n - 1) < \infty$, we have $\sum_{n=1}^{\infty} \kappa_n < \infty$. For each $u^* \in \mathcal{F}(\mathcal{T})$, we obtain that

$$\begin{aligned}
&d(y_n, u^*) &(5)\\
&= d(\mathcal{T}^n(\tau_n\mathcal{T}^n u_n \oplus (1-\tau_n)u_n), u^*)\\
&\leq (1+\kappa_n)d(\tau_n\mathcal{T}^n u_n \oplus (1-\tau_n)u_n, u^*)\\
&\leq (1+\kappa_n)[\tau_n d(\mathcal{T}^n u_n, u^*)\\
&\quad + (1-\tau_n)d(u_n, u^*)]\\
&\leq (1+\kappa_n)[\tau_n(1+\kappa_n)d(u_n, u^*)\\
&\quad + (1-\tau_n)d(u_n, u^*)]\\
&= (1+\kappa_n)(1+\tau_n\kappa_n)d(u_n, u^*)\\
&\leq (1+\kappa_n)^2 d(u_n, u^*) &(6)
\end{aligned}$$

and

$$\begin{aligned}
&d(v_n, u^*) &(7)\\
&= d(\mathcal{T}^n(\rho_n\mathcal{T}^n y_n \oplus (1-\rho_n)y_n), u^*)\\
&\leq (1+\kappa_n)d(\rho_n\mathcal{T}^n y_n \oplus (1-\rho_n)y_n, u^*)\\
&\leq (1+\kappa_n)[\rho_n d(\mathcal{T}^n y_n, u^*)\\
&\quad + (1-\rho_n)d(y_n, u^*)]\\
&\leq (1+\kappa_n)[\rho_n(1+\kappa_n)d(y_n, u^*)\\
&\quad + (1-\rho_n)d(y_n, u^*)]\\
&= (1+\kappa_n)(1+\rho_n\kappa_n)d(y_n, u^*)\\
&\leq (1+\kappa_n)^2 d(y_n, u^*). &(8)
\end{aligned}$$

Using (5) and (7), we have

$$\begin{aligned}
&d(u_{n+1}, u^*) &(9)\\
&= d(\mathcal{T}^n(\lambda_n\mathcal{T}^n v_n \oplus (1-\lambda_n)v_n), u^*)\\
&\leq (1+\kappa_n)d(\lambda_n\mathcal{T}^n v_n \oplus (1-\lambda_n)v_n, u^*)\\
&\leq (1+\kappa_n)[\lambda_n d(\mathcal{T}^n v_n, u^*)\\
&\quad + (1-\lambda_n)d(\mathcal{T}^n v_n, u^*)]\\
&\leq (1+\kappa_n)[\lambda_n(1+\kappa_n)d(v_n, u^*)\\
&\quad + (1-\lambda_n)d(v_n, u^*)]\\
&= (1+\kappa_n)(1+\lambda_n\kappa_n)d(v_n, u^*)\\
&\leq (1+\kappa_n)^2 d(v_n, u^*)\\
&\leq (1+\kappa_n)^4 d(y_n, u^*)\\
&\leq (1+\kappa_n)^6 d(u_n, u^*). &(10)
\end{aligned}$$

Because $\sum_{n=1}^{\infty} \kappa_n < \infty$ and using Lemma 2.6, we have $\lim_{n\to\infty} d(u_n, u^*)$ exists. □

Theorem 3.2. *Let* $\mathcal{K}$ *be a closed bounded and convex subset of a Hadamard space* X *and a self map* $\mathcal{T}$ *defined on* $\mathcal{K}$ *be an asymptotically nonexpansive mapping with* $\{\zeta_n\}$. *Assume that the following conditions hold:*

(i) $\{\zeta_n\} \geq 1$ *and* $\sum_{n=1}^{\infty}(\zeta_n - 1) < \infty$,

(ii) *there exist constants* c_1, c_2 *with* $0 < c_1 \leq \tau_n \leq c_2 < 1$ *for each* $n \in \mathbb{N}$,

(iii) *there exist constants* b_1, b_2 *with* $0 < b_1 \leq \rho_n \leq b_2 < 1$ *for each* $n \in \mathbb{N}$,

(iv) *there exist constants* a_1, a_2 *with* $0 < a_1 \leq \lambda_n \leq a_2 < 1$ *for each* $n \in \mathbb{N}$.

For the sequence $\{u_n\}$ *given by* (4). *Then* $\lim_{n\to\infty} d(u_n, \mathcal{T}u_n) = 0$.

Proof. For each $u^* \in \mathcal{F}(\mathcal{T})$. Putting $\zeta_n = 1 + \kappa_n$ for all $n \geq 1$. Using $\sum_{n=1}^{\infty}(\zeta_n - 1) < \infty$, we have $\sum_{n=1}^{\infty} \kappa_n < \infty$. From Theorem 3.1, we support that

$$\lim_{n\to\infty} d(u_n, u^*) = l \geq 0. \tag{11}$$

From (5), we have

$$\limsup_{n\to\infty} d(y_n, u^*) \leq l. \tag{12}$$

Because $\mathcal{T}$ be an asymptotically nonexpansive

$$d(\mathcal{T}y_n, u^*) \leq (1+\kappa_n)d(y_n, u^*) \tag{13}$$

Using (12) and (13), we have

$$\limsup_{n\to\infty} d(\mathcal{T}^n y_n, u^*) \leq l. \tag{14}$$

Similar to that,

$$\limsup_{n\to\infty} d(\mathcal{T}^n u_n, u^*) \leq l \tag{15}$$

and

$$\limsup_{n\to\infty} d(\mathcal{T}^n v_n, u^*) \leq l. \tag{16}$$

Because

$$d(u_{n+1}, u^*) \leq (1+\kappa_n)^4 d(y_n, u^*). \tag{17}$$

Taking limit infimum both sides, we obtain,

$$l \leq \liminf_{n\to\infty} d(y_n, u^*). \tag{18}$$

Using (12) and (18), we obtain that

$$\begin{aligned}
l &= \lim_{n\to\infty} d(y_n, u^*)\\
&= \lim_{n\to\infty} d(\mathcal{T}^n(\tau_n\mathcal{T}^n u_n \oplus (1-\tau_n)u_n), u^*).
\end{aligned} \tag{19}$$

Also,

$$d(\mathcal{T}^n(\tau_n\mathcal{T}^n u_n \oplus (1-\tau_n)u_n), u^*) \\ \leq (1+\kappa_n)d(\tau_n\mathcal{T}^n u_n \oplus (1-\tau_n)u_n, u^*)$$

and

$$l \leq \liminf_{n\to\infty} d(\mathcal{T}^n(\tau_n\mathcal{T}^n u_n \oplus (1-\tau_n)u_n), u^*) \\ \leq \liminf_{n\to\infty} d(\tau_n\mathcal{T}^n u_n \oplus (1-\tau_n)u_n, u^*). \quad (20)$$

Using (11) and (15), we have

$$d(\tau_n\mathcal{T}^n u_n \oplus (1-\tau_n)u_n, u^*) \\ \leq (1+\kappa_n)[\tau_n d(\mathcal{T}^n u_n, u^*) + (1-\tau_n)d(u_n, u^*)]$$

and

$$\limsup_{n\to\infty} d(\tau_n\mathcal{T}^n u_n \oplus (1-\tau_n)u_n, u^*) \leq l. \quad (21)$$

Using (20) and (21), we have

$$\lim_{n\to\infty} d(\tau_n\mathcal{T}^n u_n \oplus (1-\tau_n)u_n, u^*) = l. \quad (22)$$

Using (11), (15), (22) and Lemma 2.7, we have

$$\lim_{n\to\infty} d(u_n, \mathcal{T}^n u_n) = 0. \quad (23)$$

From (9), we have

$$d(u_{n+1}, u^*) \leq (1+\kappa_n)^2 d(v_n, u^*)$$

and

$$l \leq \liminf_{n\to\infty} d(u_{n+1}, u^*) \leq \liminf_{n\to\infty} d(v_n, u^*) \quad (24)$$

From (7), we have

$$d(v_n, u^*) \leq (1+\kappa_n)^2 d(y_n, u^*)$$

and

$$\limsup_{n\to\infty} d(v_n, u^*) \leq \limsup_{n\to\infty} d(y_n, u^*) \leq l. \quad (25)$$

Using (24) and (25), we have

$$\lim_{n\to\infty} d(v_n, u^*) = l \quad (26)$$

and

$$l = \lim_{n\to\infty} d(v_n, u^*) \\ = \lim_{n\to\infty} d(\mathcal{T}^n(\rho_n\mathcal{T}^n y_n \oplus (1-\rho_n)y_n), u^*)$$

which

$$d(\mathcal{T}^n(\rho_n\mathcal{T}^n y_n \oplus (1-\rho_n)y_n), u^*) \\ \leq (1+\kappa_n)d(\rho_n\mathcal{T}^n y_n \oplus (1-\rho_n)y_n, u^*).$$

Taking limit infimum both sides, we obtain,

$$l \leq \liminf_{n\to\infty} d(\rho_n\mathcal{T}^n y_n \oplus (1-\rho_n)y_n, u^*). \quad (27)$$

Also,

$$d(\rho_n\mathcal{T}^n y_n \oplus (1-\rho_n)y_n, u^*) \\ \leq \rho_n d(\mathcal{T}^n y_n, u^*) + (1-\rho_n)d(y_n, u^*)$$

Using (12) and (14), we have

$$\limsup_{n\to\infty} d(\rho_n\mathcal{T}^n y_n \oplus (1-\rho_n)y_n, u^*) \leq l. \quad (28)$$

Using (27) and (28), we have

$$\limsup_{n\to\infty} d(\rho_n\mathcal{T}^n y_n \oplus (1-\rho_n)y_n, u^*) = l. \quad (29)$$

Using (12), (14), (29) and Lemma 2.7, we have

$$\lim_{n\to\infty} d(y_n, \mathcal{T}^n y_n) = 0. \quad (30)$$

From (9), we have

$$l = \lim_{n\to\infty} d(u_{n+1}, u^*) \\ = \lim_{n\to\infty} d(\mathcal{T}^n(\lambda_n\mathcal{T}^n v_n \oplus (1-\lambda_n)v_n), u^*)$$

and

$$d(\mathcal{T}^n(\lambda_n\mathcal{T}^n v_n \oplus (1-\lambda_n)v_n), u^*) \\ \leq (1+\kappa_n)d(\lambda_n\mathcal{T}^n v_n \oplus (1-\lambda_n)v_n, u^*).$$

Taking limit infimum both sides, we obtain,

$$l \leq \liminf_{n\to\infty} d(\lambda_n\mathcal{T}^n v_n \oplus (1-\lambda_n)v_n, u^*). \quad (31)$$

Also,

$$d(\lambda_n\mathcal{T}^n v_n \oplus (1-\lambda_n)v_n, u^*) \\ \leq \lambda_n d(\mathcal{T}^n v_n, u^*) + (1-\lambda_n)d(v_n, u^*).$$

Using (16) and (25), we have

$$\limsup_{n\to\infty} d(\lambda_n\mathcal{T}^n v_n \oplus (1-\lambda_n)v_n, u^*) \leq l. \quad (32)$$

Using (31) and (32), we have

$$\lim_{n\to\infty} d(\lambda_n\mathcal{T}^n v_n \oplus (1-\lambda_n)v_n, u^*) = l. \quad (33)$$

Using (16), (25), (33) and Lemma 2.7, we have

$$\lim_{n\to\infty} d(v_n, \mathcal{T}^n v_n) = 0. \quad (34)$$

In addition, using (23), we have

$$\begin{aligned} &d(y_n, u_n) \\ &= d(\mathcal{T}^n(\tau_n \mathcal{T}^n u_n \oplus (1-\tau_n)u_n), u_n) \\ &\leq d(\mathcal{T}^n(\tau_n \mathcal{T}^n u_n \oplus (1-\tau_n)u_n), \mathcal{T}^n u_n) \\ &\quad + (1+\kappa_n) d(\mathcal{T}^n u_n, u_n) \\ &\leq (1+\kappa_n)[d(\tau_n \mathcal{T}^n u_n \oplus (1-\tau_n)u_n, u_n)] \\ &\quad + (1+\kappa_n) d(\mathcal{T}^n u_n, u_n) \\ &\leq (1+\kappa_n)[\tau_n d(\mathcal{T}^n u_n, u_n) + (1-\tau_n) d(u_n, u_n)] \\ &\quad + (1+\kappa_n) d(\mathcal{T}^n u_n, u_n) \\ &= (1-\kappa_n)(1-\tau_n) d(\mathcal{T}^n u_n, u_n) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{35}$$

Also,

$$\begin{aligned} &d(y_n, \mathcal{T}^n u_n) \qquad (36) \\ &= d(\mathcal{T}^n(\tau_n \mathcal{T}^n u_n \oplus (1-\tau_n)u_n), \mathcal{T}^n u_n) \\ &\leq (1+\kappa_n) d(\tau_n \mathcal{T}^n u_n \oplus (1-\tau_n)u_n, u_n) \\ &\leq (1+\kappa_n)[\tau_n d(\mathcal{T}^n u_n, u_n) + (1-\tau_n) d(u_n, u_n)] \\ &\leq (1+\kappa_n)\tau_n d(\mathcal{T}^n u_n, u_n) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{37}$$

Using (23), (36) and (30), we have

$$\begin{aligned} &d(u_n, \mathcal{T}^n y_n) \\ &\leq d(u_n, \mathcal{T}^n u_n) + d(\mathcal{T}^n u_n, y_n) + d(y_n, \mathcal{T}^n y_n) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{38}$$

Using (38) and (35), we have

$$\begin{aligned} &d(v_n, \mathcal{T}^n u_n) \\ &= d(\mathcal{T}^n(\rho_n \mathcal{T}^n y_n \oplus (1-\rho_n)y_n), \mathcal{T}^n u_n) \\ &\leq (1+\kappa_n) d(\rho_n \mathcal{T}^n y_n \oplus (1-\rho_n)y_n, u_n) \\ &\leq (1+\kappa_n)[\rho_n d(\mathcal{T}^n y_n, u_n) + (1-\rho_n) d(y_n, u_n)] \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{39}$$

Using (23), (39) and (34), we have

$$\begin{aligned} &d(u_n, \mathcal{T}^n v_n) \\ &\leq d(u_n, \mathcal{T}^n u_n) + d(\mathcal{T}^n u_n, v_n) + d(v_n, \mathcal{T}^n v_n) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{40}$$

Using (38), (35) and (36), we have

$$\begin{aligned} &d(v_n, u_n) \\ &= d(\mathcal{T}^n(\rho_n \mathcal{T}^n y_n \oplus (1-\rho_n)y_n), u_n) \\ &\leq d(\mathcal{T}^n(\rho_n \mathcal{T}^n y_n \oplus (1-\rho_n)y_n), \mathcal{T}^n u_n) \\ &\quad + (1+\kappa_n) d(\mathcal{T}^n u_n, y_n) \\ &\leq (1+\kappa_n) d(\rho_n \mathcal{T}^n y_n \oplus (1-\rho_n)y_n), u_n) \\ &\quad + (1+\kappa_n) d(\mathcal{T}^n u_n, y_n) \\ &\leq (1+\kappa_n)[\rho_n d(\mathcal{T}^n y_n, u_n) + (1-\rho_n) d(y_n, u_n)] \\ &\quad + (1+\kappa_n) d(\mathcal{T}^n u_n, y_n) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{41}$$

Using (40), (41) and (39), we have

$$\begin{aligned} &d(u_{n+1}, u_n) \\ &= d(\mathcal{T}^n(\lambda_n \mathcal{T}^n v_n \oplus (1-\lambda_n)v_n), u_n) \\ &\leq d(\mathcal{T}^n(\lambda_n \mathcal{T}^n v_n \oplus (1-\lambda_n)v_n), \mathcal{T}^n u_n) \\ &\quad + (1+\kappa_n) d(\mathcal{T}^n u_n, v_n) \\ &\leq (1+\kappa_n) d(\lambda_n \mathcal{T}^n v_n \oplus (1-\lambda_n)v_n, u_n) \\ &\quad + (1+\kappa_n) d(\mathcal{T}^n u_n, v_n) \\ &\leq 1+\kappa_n)[\lambda_n d(\mathcal{T}^n v_n, u_n) + (1-\lambda_n) d(v_n, u_n)] \\ &\quad + (1+\kappa_n) d(\mathcal{T}^n u_n, v_n) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{42}$$

Using (42) and (23), we have

$$\begin{aligned} &d(u_{n+1}, \mathcal{T}^n u_{n+1}) \\ &\leq d(u_{n+1}, u_n) + d(u_n, \mathcal{T}^n u_n) + d(\mathcal{T}^n u_n, \mathcal{T}^n u_{n+1}) \\ &\leq d(u_{n+1}, u_n) + (1+\kappa_n) d(u_{n+1}, u_n) + d(u_n, \mathcal{T}^n u_n) \\ &\leq (2+\kappa_n) d(u_{n+1}, u_n) + d(u_n, \mathcal{T}^n u_n). \end{aligned} \tag{43}$$

Using (23) and (43), we have

$$\begin{aligned} &d(u_{n+1}, \mathcal{T} u_{n+1}) \\ &\leq d(u_{n+1}, \mathcal{T}^{n+1} u_{n+1}) + d(\mathcal{T}^{n+1} u_{n+1}, \mathcal{T} u_{n+1}) \\ &\leq d(u_{n+1}, \mathcal{T}^{n+1} u_{n+1}) + (1+\kappa_1) d(\mathcal{T}^n u_{n+1}, u_{n+1}) \\ &\to 0 \quad \text{as} \quad n \to \infty, \end{aligned} \tag{44}$$

which implies $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

Theorem 3.3. *Let $\mathcal{K}$ be a closed bounded and convex subset of a Hadamard space X and a self map $\mathcal{T}$ defined on $\mathcal{K}$ be an asymptotically nonexpansive mapping with $\{\zeta_n\}$. Assume that the following conditions hold:*

(i) $\{\zeta_n\} \geq 1$ *and* $\sum_{n=1}^{\infty}(\zeta_n - 1) < \infty$,

(ii) *there exist constants* c_1, c_2 *with* $0 < c_1 \leq \tau_n \leq c_2 < 1$ *for each* $n \in \mathbb{N}$,

(iii) *there exist constants* b_1, b_2 *with* $0 < b_1 \leq \rho_n \leq b_2 < 1$ *for each* $n \in \mathbb{N}$,

(iv) *there exist constants* a_1, a_2 *with* $0 < a_1 \leq \lambda_n \leq a_2 < 1$ *for each* $n \in \mathbb{N}$.

For the sequence $\{u_n\}$ *given by* (4). *Then* $\{u_n\}$ Δ-*converges to a fixed point of* $\mathcal{T}$.

Proof. Let $u \in \omega_\Delta(u_n) = \bigcup \mathcal{A}(\{w_n\})$. So, there exists subsequence $\{w_n\}$ of $\{u_n\}$ such that $\mathcal{A}(\{w_n\}) = \{u\}$. By Lemmas 2.2 (i), (ii) and Theorem 2.8 there exists a subsequence $\{s_n\}$ of $\{w_n\}$ such that Δ-$\lim_{n\to\infty} s_n = s \in \mathcal{K}$. From Lemma 2.4, we have $s \in \mathcal{F}(\mathcal{T})$. Because $\{d(w_n, s)\}$ converges and using Lemma 2.3, we have $u = s$. This implies that $\omega_\Delta(u_n) \subseteq \mathcal{F}(\mathcal{T})$.

Next, we show that $\omega_\Delta(u_n)$ consists of exactly one point. Let $\{w_n\}$ be a subsequence of $\{u_n\}$ with $\mathcal{A}(\{w_n\}) = \{u\}$ and $\mathcal{A}(\{u_n\}) = \{p\}$. We have that $u = s$ and $s \in \mathcal{F}(\mathcal{T})$. Finally, because $\{d(u_n, s)\}$ converges and using Lemma 2.3, we have $p = s \in \mathcal{F}(\mathcal{T})$. This shows that $\omega_\Delta(u_n) = p$. □

4 Numerical example

Let $X = \mathbb{R}$ with usual metric and $\mathcal{K} = [1, 11]$. Let a self map $\mathcal{T}$ on $\mathcal{K}$ as follows:

$$\mathcal{T}u = \sqrt[4]{u^3 + 8}, \quad \forall u \in \mathcal{K}.$$

It is undeniable that $\mathcal{F}(\mathcal{T}) = \{2\}$. Next, We demonstrate that $\mathcal{T}$ is asymptotically nonexpansive mapping on $[1, 11]$.

We can see that the function $f(u) = \sqrt[4]{u^3 + 8} - u, \quad \forall u \in [1, 11]$ has the derivative

$$f'(u) = \frac{3u^2}{4(u^3 + 8)^{3/4}} - 1, \quad \forall u \in [1, 11].$$

Because $1 \leq u$, we have $f'(u) = f'(u) = \frac{3u^2}{4(u^3+8)^{3/4}} \leq 1$ and so

$$f'(u) \leq 0, \quad \forall u \in [1, 11],$$

which shows that the above function is decreasing on $[1, 5]$. Let $u, v \in [1, 5]$ with $u \leq v$ shows that

$$f(v) \leq f(u),$$

we obtain that

$$\sqrt[4]{v^3 + 8} - v \leq \sqrt[4]{u^3 + 8} - u$$

and change it as

$$\sqrt[4]{v^3 + 8} - \sqrt[4]{u^3 + 8} \leq v - u$$
$$\left|\sqrt[4]{v^3 + 8} - \sqrt[4]{u^3 + 8}\right| \leq |v - u|$$
$$\left|\sqrt[4]{u^3 + 8} - \sqrt[4]{v^3 + 8}\right| \leq |u - v|.$$

Therefore, we obtain that

$$\|\mathcal{T}u - \mathcal{T}v\| \leq \|u - v\|.$$

Thus, $\mathcal{T}$ satisfies asymptotically nonexpansive mapping because it is a nonexpansive mapping.
Using the initial value $u_1 = 9$ and the specified stopping criteria $\|u_n - 2\| \leq 10^{-15}$. For two choices, calculate the values of iterative scheme (3) and iterative scheme (4).

Choice 1: $\tau_n = \frac{9n}{\sqrt{100n^2+4}}$, $\rho_n = \frac{4n}{5n+4}$ and $\lambda_n = \frac{n}{2n+4}$.

Choice 2: $\tau_n = 1 - \frac{n}{5n+4}$, $\rho_n = \frac{3n}{4n+4}$ and $\lambda_n = \frac{7n}{10n+4}$.

The results of choice 1 are shown in Table 1 and Figure 1, as are the results of choice 2 in Table 2 and Figure 2,.

Table 1: Sequences of comparison for Choice 1.

Number of Iterations	Iterative scheme (3) CPU Time (0.08 Sec.)	Iterative scheme (4) CPU Time (0.03 Sec.)
1	9.000000000000000	9.000000000000000
2	3.932991495161055	2.211043635365002
3	2.436259801588707	2.002827615845894
4	2.077070700749241	2.000032201213855
5	2.012069935390678	2.000000343639970
6	2.001794224194666	2.000000003507589
7	2.000258150269182	2.000000000034648
8	2.000036209681632	2.000000000000334
9	2.000004972343876	2.000000000000003
10	2.000000670525931	2.000000000000000
11	2.000000089009141	2.000000000000000
12	2.000000011653699	2.000000000000000
13	2.000000001507295	2.000000000000000
14	2.000000000192849	2.000000000000000
15	2.000000000024434	2.000000000000000
16	2.000000000003069	2.000000000000000
17	2.000000000000382	2.000000000000000
18	2.000000000000047	2.000000000000000
19	2.000000000000006	2.000000000000000
20	2.000000000000000	2.000000000000000

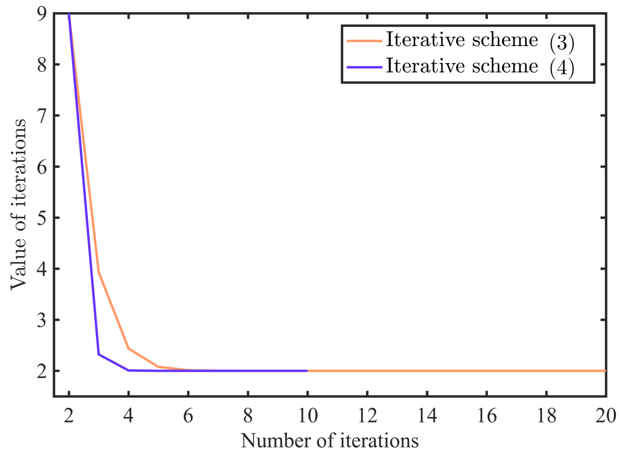

Figure 1: Sequences of comparison for Choice 1.

5 Conclusions

We introduced a novel iterative scheme (4) for asymptotically nonexpansive mapping in Hadamard spaces under certain conditions. The demonstrated that our new type of iteration is more efficient than iterative scheme (3). In addition, We have presented a numer-

Table 2: Sequences of comparison for Choice 2.

Number of Iterations	Iterative scheme (3) CPU Time (0.10 Sec.)	Iterative scheme (4) CPU Time (0.04 Sec.)
1	9.000000000000000	9.000000000000000
2	3.389076649070681	2.170966511771209
3	2.225543383619792	2.001921267657179
4	2.029729550343821	2.000019117614857
5	2.003699439414234	2.000000181840148
6	2.000453631686722	2.000000001678215
7	2.000055244686556	.000000000015154
8	2.000006696261599	2.000000000000135
9	2.000000808714998	2.000000000000000
10	2.000000097384327	2.000000000000000
11	2.000000011698706	2.000000000000000
12	2.000000001402528	2.000000000000000
13	2.000000000167858	2.000000000000000
14	2.000000000020060	2.000000000000000
15	2.000000000002394	2.000000000000000
16	2.000000000000285	2.000000000000000
17	2.000000000000034	2.000000000000000
18	2.000000000000004	2.000000000000000
19	2.000000000000000	2.000000000000000

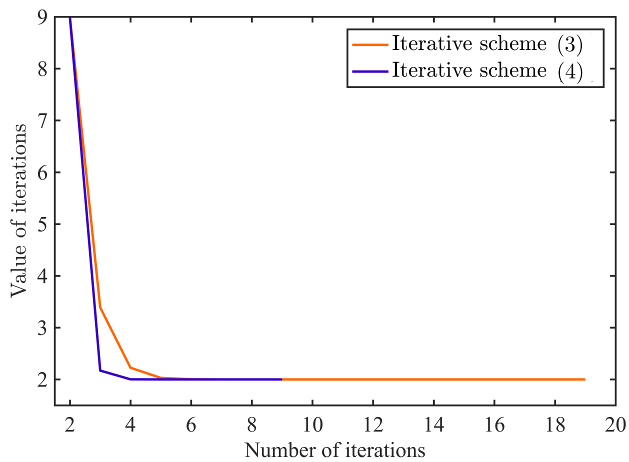

Figure 2: Sequences of comparison for Choice 2.

ical experiment to the reader to support our claim.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.