Exhaustive Nets on Function Spaces

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Abstract: The observation of α-convergence is done in many publications in different contexts and tools. Recently, we defined the d_a-convergence in metric space and we emphasized that in every case *d* -converge or so-called locally uniform convergence opens new ways for studying relationships in different spaces. In this paper, we extend the convergence of sequences to the net's convergence and arrived to relativize some known propositions.

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1- Exhaustive Functions

We now introduce a new notion of exhaustiveness close to the idea of equicontinuity, [1].

Definition 1.1 Let (X, d), (Y, p) be metric spaces, x ∈ X, and Φ be a family of functions from X to Y. Let $f_n: X \to Y$, n ∈ ℕ be the functional sequence.

(1) If Φ is infinite, we call this family exhaustive at x∈X if for each ε > 0 there exists δ> 0 and any finite set A as a subset of Φ such that: for each y ∈S (x, δ) and for each f ∈Φ\A we have that $p\big(f(x), f(y)\big) < \varepsilon$

(2) In the case where Φ is finite we define that Φ is exhaustive at x if each member of Φ, is a continuous function at X.

(3) Φ is called exhaustive if Φ is exhaustive at every x∈X.

(4) The sequence $(f_n)_{n\in N}$ is called exhaustive at x if for all ε> 0 there exist δ >0 and n_0 ∈N, such that for every y∈S(x,δ) and every $n>n_0$ we have that $p\Big(f_n(y),\ f_n(x)\Big) < \varepsilon$.

(5) The sequence $(f_n)_{n\in N}$i s called exhaustive if it is exhaustive at every x ∈ X, [1].

Notice that in the most interesting case where$(f_n)_{n\in N}$ is a sequence of functions such that for m≠n we have $f_n \neq f_m$ for which, then the family Φ is exhaustive at some point x, if and only if the sequence is $(f_n)_{n\in N}$ is exhaustive at point x.

Part of this paper will be a special convergence commonly called α- convergence.

It was originally called *continuous convergence* and has been known since the beginning of the 20th century. This concept, from the literature, written in the 1950 year we knew that [2] and [5], bring some facts about it, and by them, it is possible to characterize this type of convergence, [2], [5].

Expressed non-rigorously, α-convergence is characterized by two convergences. If $x_n \to x$, then $f(x_n) \to f(x)$. It is understood that when function f maps two spaces, f: (X, d)→(Y, p), this means that when $x_n \xrightarrow{d} x$ then

$f_n(x_n) \xrightarrow{p} f(x)$.

Definition 1.2 Let us have the functions $f_n, f: (X, d) \to (Y, \rho)$.

(1)A sequence of functions $(f_n)_{n\in N}$, all having the same domain X and codomain Y is said to converge pointwise at point x∈X to a given function f (often written as $f_n \xrightarrow{pw} f$) if and only if, for every ε>0, there exists natural number p(ε, x) ∈N, such that for every n≥p(ε,x) we have, that $\rho\left(f_n(x), f(x)\right) < \varepsilon$.

(2) Let (X, d), (Y, p) be two metric spaces, x∈X and f_n, f: X→Y.

It is said that the sequence $\left(f_n(x)\right)_{n\in N}$ is locally uniformly strongly convergent (or shortly δ_a-convergence) if for every $n_0(\varepsilon, x) \in N$ and δ>0, such that, for every n≥n_0(ε,x) and y ∈S(x,δ) we have $p(f_n(y), f(x)) < \varepsilon$, [3].

Proposition 1.3 Let f and f_n be the functions that map metric spaces (X, d) to (Y, p). If the sequence $\left(f_n(x)\right)_{n\in N}$ is α-convergent to the function f then this sequence is also δ_a- convergent to this function.

Proof. Let the sequence $\left(f_n(x)\right)_{n\in N}$ be α-convergent to f. That means that if x_n →x on X, then $f_n(x_n)$→f(x) on Y. Since the sequence x_n →x then for every δ there exists an n_0(x,δ) that x_n ∈S(x,δ). By the second convergence $f_n(x_n)$→f(x) we have that for n≥n_0(x,δ) that $p(f_n(x_n), f(x)) < \varepsilon.$ That means that the sequence$\left(f_n(x)\right)_{n\in N}$ is δ_a- convergent to the point x∈X if we substitute the x_n with y in the definition of δ_a-convergence.

It shows that δ_a –convergence is wider than α-convergence.

Example 1.4 We denote by K, the Cantor continuum in [0,1], and from the construction of the Cantor function, we know that its values are of the form $f_n^k = \dfrac{2k-1}{2^n}$ for k, n∈N, and k≤n. We denote $f(x) = \inf\{f_n^k(t) : t \in [0,1] \setminus K\}$. Let y∈K and y∈S(x,δ) we have that $|f_n^k(y) - f(x)|$→0, from which it follows that the f_n^k is δ_a sequent to f.

Proposition 1.5 Let be (X, d), (Y, p) two metric spaces and the function f: X→Y. If the sequence of functions f_n: X→Y for every x∈X, is δ_a – convergent than

(a) The sequence of the functions $(f_n(x))_{n\in N}$ is exhaustive
(b) f(x) is continuous in every x ∈X.

Proof. (a) By the definition of δ_a-convergence we have that for every ε>0 and ε/2>0 for every x∈X, there exists the natural number n_0(ε,x)∈N and δ>0 such that for n≥ n_0(ε,x) and y∈S(x,δ), holds that

(1) $$p(f_n(y), f(x)) < \frac{\varepsilon}{2}$$

We can write that

$$p(f_n(y), f_n(x)) \le p(f_n(y), f_n(x)) + $$
$$+ p(f(x), f_n(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

F_n are exhaustive for n≥ n_0(ε,x) and y∈S(x,δ).

(b) In the inequation (1), if we take instead of ε>o, ε/3>0 for every x∈X there exist n_1(ε,x)∈N and δ_1>0 such that for every n> n_1(ε,x) and y∈S(x,δ) we have that $p(f_n(y), f(x)) < \dfrac{\varepsilon}{3}$

Also, there exist $n_2(\varepsilon, x) \in N$ and δ_2>0, such that for every $n \ge n_2(\varepsilon, x)$ we have

$p(f_n(y), f(x)) < \dfrac{\varepsilon}{3}$ and $p(f_n(x), f(x)) < \dfrac{\varepsilon}{3}$.

The second inequation shows that from δ_a-convergence derives the pointwise convergence. This allows us to write that from every ε/3>0, there exists n_0(ε,y), such that for every n≥n_0(ε,y) we have that $p(f_n(y), f(y)) < \dfrac{\varepsilon}{3}$.

Thus, for every x∈X and y∈S(x,δ), where δ=min{δ_1,δ_2} if n*=max{n_0,n_1,n_3}, for every n≥n* we take

$$p(f(x), f(y)) \le p(f(x), f_{n_3}(x)) +$$
$$+ p(f_{n_3}(x), f_{n_3}(y)) + p(f_{n_3}(y), f(y)) < \varepsilon.$$

Definition 1.6 The net on metric(topologic) space X is called a function that maps an ordered set of D to X. The net is denoted (x_σ:σ∈D) where x_σ is a point of the space X defined by an element σ of D. In the case when D is a denumerable set, we use the symbol{x_i:i∈I} or $(x_\sigma)_{\sigma\in D}$. The ordered relation of D is denoted ≤ for two elements σ_1,σ_2 from D that

$\sigma_1 \leq \sigma_2$ when is σ_1 before σ_2. The point x from X is called the limit of the net $(x_\sigma)_{\sigma\in D}$ if and only if for every neighborhood U of the point x there exists an $\sigma_0 \in D$ such that $x_\sigma \in U$ for every $\sigma \geq \sigma_0$. Then we can say that the net $(x_\sigma)_{\sigma\in D}$ converges to the point x. Unlike the sequences of elements in metric space, the net can converge to many points at the same time.

2 Main Results

Definition 2.1, [1], Let X be a topological space, (Y,p)be a metric space, a function $f: X \to Y$ and a net $(f_i)_{i\in I}$ of functions from X to Y. We say that the net $(f_i)_{i\in I}$ α −converges to f, if for all $x \in X$, and all nets $(x_k)_{k\in K}$ in x such that $x_k \to x$ the net $\left(y(i,k)\right)_{(i,k)\in(I,K)}$defined by $y(i,k) = f_i(x_k)$converges to $f(x)$(in Y),i.e., for all $i \in I$, and $k \in K$ with $i_0 \preccurlyeq i$ and $k_0 \preccurlyeq k$, we have that $p\left(f_i(x_k), f(x)\right) < \varepsilon$. As before we shall write $f_i \overset{\alpha}{\to} f$ in the case where $(f_i)_{i\in I}$ α −converges to f.

Definition 2.2 Let X be a topological space, be a metric space, a function $f: X \to Y$ and a net $(f_i)_{i\in I}$ of functions from X to Y. We say that the net $(f_i)_{i\in I}$ δ_a −converges to f if for all $x \in X$ and every $\delta > 0$ and $n > n_0(\varepsilon, x)$, we have that for $y \in S(x,\delta)$, the net $f_i(y)$ converges to $f(x)$(in Y), i.e., for all $\varepsilon > 0$ there exists $i_0 \in I$, that for all $i \in I$ with $i_0 \preccurlyeq i$, we have that $p\left(f_i(y), f(x)\right) < \varepsilon$.

Proposition 2.3, [1], Let X and Y be metric spaces and also let functions $f_n, f: X \to Y$. The net $(f_n)_{n\in\mathbb{N}}$ α −converges to f (in the sense of the previous definition) if and only if $(f_n)_{n\in\mathbb{N}}$ α −converges to f as a sequence (i.e., in the sense of definition 2.1).

Proposition 2.5 Let X and Y be two metric spaces and also $f_n, f: X \to Y$. The net (f_n) is δ_a-convergent to f if and only if $(f_n)_{n\in N}$ is δ_a-convergent to f as a sequence according to definition 1.1.

Proof: It is evident that the δ_a-convergence of the nets $(f_i)_{i\in I}$ derives as a special case of the δ_a-convergence of the sequence $(f_n)_{n\in N}$.

Let us suppose that the sequence$(f_n)_{n\in N}$ δ_a-converges to f and y be an element such that $y \in S(x,\delta)$ we want to prove that the nets $(f_i)_{i\in I}$ δ_a-converges to $f(x)$.

Let's consider the opposite assertion. The sequence $(f_i)_{i\in I}$ doesn't converge to $f(x)$. This means that there exists an $\varepsilon_0 > 0$ and $x \in X$, such that for every $0<\delta<1$ and $n_1 > n_0$, there exists $y \in S(x, \delta)$ for which $p(f_{n_1}(y), f(x)) \geq \varepsilon_0$. By repeating the same procedure $\delta < \frac{1}{2}$ and $n_2 \geq n_1$ will find an element y such that $y \in S\left(x, \frac{1}{2}\right)$ to derive $\left(f_{n_2}(y), f(x)\right) \geq \varepsilon_0$, and same for $\delta_k \leq \frac{1}{k}$, we will find y such that $y \in S\left(x, \frac{1}{k}\right)$ to derive $\left(f_{n_k}(y), f(x)\right) \geq \varepsilon_0$, and proceed in this way indefinitely.

By an appropriate recount, we mark $m=n_k$ when $k \to \infty$ and $m_0 = \sup\{n_k\}$ then for, $m > m_0$ there exists a δ such that for $y \in S(x,\delta)$ it follows $\left(f_m(y), f(x)\right) \geq \varepsilon_0$ that contradicts our assumption of δ_a-convergence of the sequence $(f_n)_{n\in N}$.

Theorem 2.6 Let $f: X \to Y$ and the net of functions $(f_i)_{i\in I}$ such that map X to Y. The following propositions are equivalent:

1)$f_i \overset{\delta_a}{\to} f$;

2) $f_i \overset{pw}{\longrightarrow} f$ and the net $(f_i)_{i\in I}$ is exhaustive.

Corollary 2.7 If the net $(f_i)_{i\in I}$ is exhaustive and f is δ_a-limit then the function f is a continuous function.

Proof
The only issue that is not the same in this proof as compared to that in Theorem 1, 3.3, [3], is the implication (1) ⇒ (2).
We suppose that $f_i \overset{\delta_a}{\to} f$ meanwhile, the net of functions f_i is not exhaustive to any point $x_0 \in X$. Then, for every $\varepsilon > 0$ we will have that for all the open neighborhood V of the point x_0 and all elements $i \in I$, will find an element $x \in V$, such that for every $k \in I$, such that for $i \leq k$, we have:

$$p\big(f_k(x), f_k(x_0)\big) \geq \varepsilon \qquad (*)$$

We mark with $\varsigma_{x_0}$the family of all the open neighborhoods of point x_0. For two open neighborhoods $U_1, U_2 \in \varsigma_{x_0}$ we define $U_1 \leq U_2$ if and only if $U_2 \subset U_1$.Thus the set ς_{x_0} is ordered. We denote by $M = \left\{(k, V) \in I \times \varsigma_{x_0}\right\}$, where there exists $x \in V$ such that $p\big(f_k(x), f_k(x_0)\big) \geq \varepsilon_0$. The set $I \times \varsigma_{x_0}$is partially ordered. Let's prove that the set M is ordered with respect to the relation$(*)$. Actually, for $\in I$ and for all $V \in \varsigma_{x_0}$, there exists $k \in I$ such that $i \leq k$ and

$$(k, V) \in M$$

By the axiom of choice, we can find a net $\big(x_{(k,V)}\big)_{(k,V)\in M}$ in X such that $p\left(f_k\big(x_{(k,V)}\big), f_k(x_0)\right) \geq \varepsilon_0$ and $x_{(k,V)} \in V$ for all $(k, V) \in M$.

It is obvious that for $x_{(k,V)} \xrightarrow{(k,V)\in M} x_0$ and from the hypothesis of δ_a-convergence of the net

$$f_i(x_{(k,V)}) \xrightarrow{(k,V)\in I\times M} f(\, x_0)$$

This means that for x_0, there exists V and k_0 such that $k \leq k_0$ and $x_{(k,V)} \in \varsigma_{x_0}$ then

$$p\left(f_k\big(x_{(k,v)}\big), f_k(x_0)\right) < \varepsilon.$$

This contradicts the assumption and we proved that the net $(f_i)_{i\in I}$ is an exhaustive family.

It seems that the two concepts of α-convergence and δa-convergence are equivalent. Actually, the concepts of α-convergence used in [4], in arrays in topology, it has been shown that two convergences $x_n \xrightarrow{\sigma_1} x$ and $f_m(x_n) \xrightarrow{\sigma_2} f(x)$ are very different.

Theorem 2.8 Let X be locally compact and (Y, p) a metric space and the functions $f_i, f\colon X \to Y, i \in I$, where I is an ordered set. The following propositions are equivalent:

(1) The net $(f_i)_{i\in I}$ is δ_a-convergent to the function f.

(2) Function f is continuous and for every compact set $K \subseteq X$, the net of functions $(f_i)_{i\in I}$ converges uniformly to f in K.

From this Theorem derives that when X is a locally compact space then δ_a-convergence in $C(X, Y)$, which is provided from the topology of the uniform convergence in the compact sets. In the proof of the theorem, we have to use regular spaces. If set I is ordered and J⊆I is cofinal in I then J is also ordered. This means that when $(x_i)_{i\in I}$ is a net then for every J we are in conditions of the subnet $\big(x_j\big)_{j\in J}$.

Proof.
(1) ⇒ (2) From the corollary (2.3), the function f is continuous. Let K be a compact subset of X. If the net $(f_i)_{i\in I}$ will not converge uniformly to the function f in K, then for every $\varepsilon > 0$ and for every $i \in I$, there exist an element $j \in J$, such that

$i \leq j$ and $x \in K$, to have that $p\left(f_j(x), f(x)\right) \geq \varepsilon$.

We denote that $J = \left\{j \in I\text{: that exists any element of } x \in K \text{ such that } p\left(f_j(x), f(x)\right) \geq \varepsilon\right\}$

The set J is cofinal in I. According to the Axiom of Choice, there exist the net $\big(x_j\big)_{j\in J}$ in K such that $p\left(f_j\big(x_j\big), f\big(x_j\big)\right) \geq \varepsilon$ for every $j \in J$. $(*)$

Since K is a compact space, we find a subnet $\left(x_{j_\mu}\right)_{\mu\in M}$and $x_0 \in K$ such that

$x_{j_\mu} \xrightarrow{\mu\in M} x_0$. As we mentioned above, $\delta_a -$ convergence, is also applied even for subnets, so we have $f_{j_\mu} \xrightarrow{\delta_a} f$ which implies that $f_{j_\mu}(x_{j_\mu}) \xrightarrow{\mu\in M} f(x_0)$. Regarding the continuity of the function f, it follows the result $f(x_{j_\mu}) \xrightarrow{\mu\in M} f(\, x_0)$. This means that $p(f_{j_\mu}\left(x_{j_\mu}\right), f\left(x_{j_\mu}\right)) \xrightarrow{\mu\in M} 0$ which contradicts $(*)$.

(2) ⇒ (1) We consider the net $\left(x_\mu\right)_{\mu\in M}$in X which converges to the point x_0. Since X is a locally

compact space, there exists a neighborhood of a point x_0, which we mark with U_0, for which its closure $\overline{U_0}$ is a compact space. We choose a $\mu_0 \in M$, such that for all $\mu \in M$ that $\mu_0 \leq \mu$, we have $x_\mu \in M$. We define $L = \{x_\mu : \mu_0 \leq \mu\}$ and K=$\overline{L}$. Then the space K is a closed subset of $\overline{U_0}$, which is compact. It follows from the hypothesis that the net $(f_i)_{i\in I}$converges uniformly in K. As we know f is a continuous function, so we have $p\left(f_i(x_\mu), f(x_0)\right) \leq \varepsilon$ for $x_\mu \in U_0$ and $\mu_0 \leq \mu$ that proves that

$$f_i \overset{\delta_a}{\rightarrow} f .$$

Example 2.8.1
Let's show a case that α-convergence is different from δ_a-convergence.
We get a sequence (x $_n$) that tends to zero in a special way
$n = \frac{(i+k-1)(i+k-2)}{2} + i$ which is formed if we go along the diagonals in the infinite table below.
For example, the third diagonal is
a_{31}, a_{22}, a_{13} the sum of the indices is 4. Thus, we have constructed an order σ_1 with the pair of indices in this way: $(i,k) \prec (i',k')$ if $i+k \leq i'+k'$.
If as a sequence (x $_n$), in X we take $x_n = \frac{1}{i+k}$ for the elements in a diagonal of the table below, for example, i + k=4, we will have that $x_{31} \prec x_{22} \prec x_{13}$. These terms x $_n$ can have the value 1/4 for n=4,5,6.
Thus, the sequence x_{11}, x_{12}, x_{21}, x_{31}, x_{22} is non-decreasing monotone sequence
for $n = i + k \leq i' + k'$ ... It is easy to prove that $n \to \infty \Leftrightarrow i \to \infty \vee k \to \infty$.
In the case when n→∞, then x $_n$→0. Let's mark $f_n(x_n) = x_n$ that goes to zero in order σ_1.
Let's go back to the infinite table:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \dots \end{matrix}$$

Now we can choose another order σ_2. The sequence sorting (x $_n$), we do by taking the indices n according to the "determinant rule", which allows δ_a-convergence.

For example, we have a part of a net
$x_{11} \prec x_{21} \prec x_{22} \prec x_{12} \prec \dots$ corresponding to ranking σ_2, the values of the net are $x_{11} = \frac{1}{2} \prec x_{21} = \frac{1}{3} \prec x_{22} = \frac{1}{4} \prec x_{12} = \frac{1}{3}$.
It is clear that in this case too $n \to \infty \Leftrightarrow i \to \infty \vee k \to \infty$ but
if we fix $\varepsilon = \frac{1}{99}$, we find one number, e.g., n_0=100=k+i=50+50, which corresponds to e.g., $x_{50,50}$. We see that for such n>100 $|f_n(x_n) - 0| = x_n = \frac{1}{50+50} < \frac{1}{99}$ but the smallest term $x_n = \frac{1}{50+49} \prec x_{n+1} = \frac{1}{50+50}$ and $x_n > x_{n+1}$ because

(i, k)$\prec$(i, k-1) = 50+49 < n_0 which defines the definition of α-convergence.

3 Conclusion

We can substitute α-convergence with δ_a-convergence even in the following statement, by specifying some relations attributed to α-convergence and δ_a-convergence and how it is treated to the proof of [1].

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