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First-exit problems for integrated diffusion processes with state-dependent jumps

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Abstract: Let $\mathrm{d}X(t) = -Y(t)\,\mathrm{d}t$, where $Y(t)$ is a one-dimensional diffusion process. First-exit problems from $C \in \mathbb{R}^2$ are studied for the degenerate two-dimensional diffusion process $(X(t), Y(t))$ when the process leaves C not later than after a random time having an exponential distribution. When $Y(t)$ is a standard Brownian motion, the Laplace transform of the moment-generating function M of the first-exit time is computed explicitly, as well as the Laplace transforms of the mean exit time m and the probability p of leaving C through a given part of its boundary. When $Y(t)$ is a geometric Brownian motion, the functions M, m and p are obtained by making use of the method of similarity solutions to solve the various partial differential equations, subject to the appropriate boundary conditions.

Key-Words: Kolmogorov backward equation, Brownian motion, geometric Brownian motion, method of similarity solutions.

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1 Introduction

We consider degenerate two-dimensional diffusion processes $(X(t), Y(t))$ defined by

$$\mathrm{d}X(t) = -Y(t)\,\mathrm{d}t, \tag{1}$$

$$\mathrm{d}Y(t) = f[Y(t)]\,\mathrm{d}t + \{v[Y(t)]\}^{1/2}\,\mathrm{d}B(t), \tag{2}$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion and the functions f and v are such that $\{Y(t), t \geq 0\}$ is a diffusion process. The process $\{X(t), t \geq 0\}$ (multiplied by -1) is known as an *integrated diffusion process*, because we can write that

$$X(t) = X(0) - \int_0^t Y(s)\,\mathrm{d}s. \tag{3}$$

Numerous papers have been written on first-exit problems for integrated diffusion processes. For the case when $\{Y(t), t \geq 0\}$ is a Wiener process, see McKean [12], Goldman [2], Gor'kov [3], Lachal [5] and Lefebvre [8]. Moreover, Lefebvre [7], Makasu [11], Metzler [13], Caravelli *et al.* [1] and Levy [10] considered the case of integrated geometric Brownian motions. Finally, Lefebvre [6] and Hesse [4] studied problems for integrated Ornstein-Uhlenbeck processes.

Next, let $(X(0), Y(0)) = (x, y) \in C \subset \mathbb{R}^2$ and define the first-exit time from C

$$T(x, y) = \inf\{t > 0 : (X(t), Y(t)) \notin C\}. \tag{4}$$

Notice that if $Y(t)$ is always positive in C, then $X(t)$ will be strictly decreasing with time. Therefore, this type of process can serve as model in applications where the variable of interest cannot increase, for instance when $X(t)$ represents the remaining lifetime of a certain device.

Assume that $\alpha > 0$. The function

$$M(x, y) := E\left[e^{-\alpha T(x,y)}\right] \tag{5}$$

is the moment-generating function of the random variable $T(x, y)$. It satisfies the Kolmogorov backward equation (see Lefebvre [9], for a generalization of this result)

$$\frac{1}{2}v(y)\,M_{yy} + f(y)\,M_y - y\,M_x = \alpha\,M \tag{6}$$

for $(x, y) \in C$, where $M_{yy} := \frac{\partial^2}{\partial y^2}M$, etc. Furthermore, since $T(x, y) = 0$ if $(x, y) \notin C$, we have the boundary condition

$$M(x, y; \alpha) = 1 \quad \text{for } (x, y) \notin C. \tag{7}$$

Now, suppose that at a random time τ having an exponential distribution with parameter λ, if $(X(t), Y(t))$ is still inside C a random quantity Z is added to the value of $Y(t)$ at that time instant, so that the process $(X(t), Y(t))$ will exit the continuation region C immediately. We say that the process is *killed* not later than at time τ.

We can prove the following proposition.

Proposition 1.1. *The function $M(x,y)$ satisfies the partial differential equation (PDE)*

$$\begin{aligned}\alpha M(x,y) \;=\; & \frac{1}{2}v(y)\,M_{yy}(x,y)+f(y)\,M_y(x,y)\\ & -\,y M_x(x,y)+\lambda[1-M(x,y)]. \quad (8)\end{aligned}$$

The equation is valid for $(x,y)\in C$ and is subject to the boundary condition (7).

We also have the following corollaries.

Corollary 1.1. *If it exists, the function $m(x,y) := E[T(x,y)]$ satisfies the PDE*

$$\begin{aligned}-1 \;=\; & \frac{1}{2}v(y)\,m_{yy}(x,y)+f(y)\,m_y(x,y)\\ & -\,y m_x(x,y)-\lambda m(x,y) \qquad (9)\end{aligned}$$

for $(x,y)\in C$. The boundary condition is

$$m(x,y)=0 \quad \text{for } (x,y)\notin C. \qquad (10)$$

Corollary 1.2. *Let*

$$p(x,y) := P[(X(T),Y(T))\in \partial C_0], \qquad (11)$$

where ∂C_0 is a subset of the boundary ∂C of C. For $(x,y)\in C$, the probability $p(x,y)$ is a solution of the PDE

$$\begin{aligned}0 \;=\; & \frac{1}{2}v(y)\,p_{yy}(x,y)+f(y)\,p_y(x,y)\\ & -\,y p_x(x,y)+\lambda[p_0-p(x,y)], \qquad (12)\end{aligned}$$

where $p_0 := P[(X(\tau),Y(\tau))\in\partial C_0]$. Furthermore, the boundary condition is

$$p(x,y)=\begin{cases}1 & \text{if } (x,y)\in\partial C_0,\\ 0 & \text{if } (x,y)\in\partial D,\end{cases} \qquad (13)$$

where $\partial D := \partial C\setminus\partial C_0$.

In the next section, we will compute the Laplace transform of the functions $M(x,y)$, $m(x,y)$ and $p(x,y)$ when $\{Y(t), t\geq 0\}$ is a standard Brownian motion and $T(x,y)$ is the first time that the two-dimensional process $((X(t),Y(t))$ leaves a rectangle located in the first quadrant. In Section 3, the set C will be the region between two straight lines and $\{Y(t), t\geq 0\}$ will be a particular geometric Brownian motion. With the help of the method of similarity solutions, we will be able to get the exact solutions to the PDE's (8), (9) and (12). We will end this paper with a few remarks in Section 4.

2 Integrated Brownian motion

In this section, we assume that $v(y)\equiv 1$ and $f(y)\equiv 0$, so that $\{Y(t), t\geq 0\}$ is a standard Brownian motion, and we define

$$T_1(x,y)=\inf\{t>0 : X(t)=0 \text{ or } Y(t)=a \text{ or } b\}, \qquad (14)$$

where $x>0$ and $0\leq a<y<b$. Moreover, the quantity Z added to $Y(t)$ at time τ is

$$Z=\begin{cases} b-Y(\tau) & \text{with probability } p_1\in(0,1),\\ a-Y(\tau) & \text{with probability } 1-p_1.\end{cases} \qquad (15)$$

Because $Y(t)$ is always positive in C and $\mathrm{d}X(t) = -Y(t)\,\mathrm{d}t$, $X(t)$ will be strictly decreasing with time. Moreover, C is actually a finite rectangle located in the first quadrant.

Solving the PDE (8) is not an easy task. We can however obtain the Laplace transform of the function $M(x,y)$ explicitly. Indeed, let

$$L(y) := \int_0^\infty e^{-\beta x}\,M(x,y)\,\mathrm{d}x, \qquad (16)$$

where $\beta>0$. Since $M(x,y)\in(0,1)$ for $(x,y)\in C$ and $M(0,y)=1$, we have

$$\begin{aligned}\int_0^\infty e^{-\beta x}\,M_x(x,y)\,\mathrm{d}x \;&=\; e^{-\beta x}M(x,y)\Big|_0^\infty+\beta L\\ &=\; -1+\beta L. \qquad (17)\end{aligned}$$

It follows, taking the Laplace transform of each side of Eq. (8), that

$$\begin{aligned}\alpha L(y) \;=\; & \frac{1}{2}L''(y)-y[-1+\beta L(y)]\\ & +\lambda[(1/\beta)-L(y)]. \qquad (18)\end{aligned}$$

The above ordinary differential equation (ODE) is subject to the boundary conditions

$$L(a)=L(b)=1/\beta. \qquad (19)$$

Making use of the mathematical software program *Maple*, we find that the general solution of Eq. (18) can be written as follows:

$$\begin{aligned}L(y) \;=\; & c_1\,\mathrm{Ai}(\eta)+c_2\,\mathrm{Bi}(\eta)\\ & +\frac{2^{2/3}\pi}{\beta^{4/3}}\left\{\mathrm{Ai}(\eta)\int \mathrm{Bi}(\eta)\,(\beta y+\lambda)\,\mathrm{d}y\right.\\ & \left.-\,\mathrm{Bi}(\eta)\int \mathrm{Ai}(\eta)\,(\beta y+\lambda)\,\mathrm{d}y\right\}, \qquad (20)\end{aligned}$$

where Ai and Bi are *Airy functions* and

$$\eta := \frac{2^{1/3}}{\beta^{2/3}}\,(\beta y+\lambda+\alpha). \qquad (21)$$

The arbitrary constants c_1 and c_2 can be determined from the boundary conditions (19) for any choice of a and b. The expression obtained is rather involved. Therefore, inverting the Laplace transform to obtain the function $M(x,y)$ would be really difficult.

Next, let

$$L_1(y) := \int_0^{\infty} e^{-\beta x}\, m(x,y)\,\mathrm{d}x. \tag{22}$$

Proceeding as above, we find that Eq. (9) becomes the ODE

$$-\frac{1}{\beta} = \frac{1}{2}\,L_1''(y) - (y\beta+\lambda)\,L_1(y), \tag{23}$$

whose general solution is

$$\begin{aligned} L_1(y) &= c_1\,\mathrm{Ai}(\xi) + c_2\,\mathrm{Bi}(\xi) \qquad (24)\\ &+ \frac{2^{2/3}\pi}{\beta^{4/3}}\left\{\mathrm{Ai}(\xi)\int \mathrm{Bi}(\xi)\,\mathrm{d}y - \mathrm{Bi}(\xi)\int \mathrm{Ai}(\xi)\,\mathrm{d}y\right\},\end{aligned}$$

where

$$\xi := \frac{2^{1/3}}{\beta^{2/3}}\,(\beta y+\alpha). \tag{25}$$

The constants c_1 and c_2 must be chosen so that $L_1(a) = L_1(b) = 0$.

Finally, let

$$p_1(x,y) := P[X(T)=0]. \tag{26}$$

Since $p_0 = P[X(\tau)=0] = 0$, we may write that the function

$$L_2(y) := \int_0^{\infty} e^{-\beta x}\, p_1(x,y)\,\mathrm{d}x \tag{27}$$

satisfies the ODE

$$0 = \frac{1}{2}\,L_2''(y) - (y\beta+\lambda)\,L_2(y) + y. \tag{28}$$

Indeed, we calculate

$$\begin{aligned}\int_0^{\infty} e^{-\beta x}\,\frac{\partial}{\partial x} p_1(x,y)\,\mathrm{d}x &= e^{-\beta x} p_1\Big|_0^{\infty} + \beta L_2\\ &= [0 - p_1(0,y)] + \beta L_2\\ &= -1 + \beta L_2. \qquad (29)\end{aligned}$$

Maple gives us the following expression for the general solution of Eq. (28):

$$\begin{aligned} L_2(y) &= c_1\,\mathrm{Ai}(\xi) + c_2\,\mathrm{Bi}(\xi) \qquad (30)\\ &+ \frac{2^{2/3}\pi}{\beta^{1/3}}\left\{\mathrm{Ai}(\xi)\int \mathrm{Bi}(\xi)\,\mathrm{d}y - \mathrm{Bi}(\xi)\int \mathrm{Ai}(\xi)\,\mathrm{d}y\right\}.\end{aligned}$$

Notice that the functions $L_1(y)$ and $L_2(y)$ are almost the same. Moreover, the constants c_1 and c_2 are also such that $L_2(a) = L_2(b) = 0$.

In the next section, we will be able to obtain the exact expressions for the functions of interest.

3 Integrated geometric Brownian motion

Suppose that the diffusion process $\{Y(t), t \geq 0\}$ is defined by

$$\mathrm{d}Y(t) = kY(t)\,\mathrm{d}t + Y(t)\,\mathrm{d}B(t). \tag{31}$$

That is, $\{Y(t), t \geq 0\}$ is a geometric Brownian motion with infinitesimal mean ky and infinitesimal variance y^2. This process can be expressed as the exponential of a Wiener process with infinitesimal parameters $k - \frac{1}{2}$ and 1. Therefore, if $Y(0) > 0$, the process will remain positive for any $t > 0$, and $X(t)$ will be strictly decreasing with t.

Let

$$T_2(x,y) := \inf\{t > 0 : X(t)/Y(t) = k_1 \text{ or } k_2\}, \tag{32}$$

where $0 < k_1 < x/y < k_2$. Thus, the set C is the region between two straight lines going through the origin.

Next, assume that

$$Z = \begin{cases} \frac{1}{k_1}X(\tau) - Y(\tau) & \text{with probability } p_1,\\ \frac{1}{k_2}X(\tau) - Y(\tau) & \text{with probability } 1-p_1, \end{cases} \tag{33}$$

so that the process $(X(t), Y(t))$ will indeed leave the continuation region C not later than at time τ.

The PDE given in (8) becomes

$$\alpha M = \frac{1}{2}y^2 M_{yy} + ky M_y - y M_x + \lambda[1 - M(x,y)]. \tag{34}$$

It is subject to the boundary conditions $M(x,y) = 1$ if $x/y = k_1$ or k_2.

Now, based on Eq. (34) and the boundary conditions, we will look for a solution of the form $M(x,y) = N(r)$, where $r := x/y$ is called a *similarity variable*. We find that Eq. (8) is transformed into the ODE

$$\frac{1}{2}r^2 N'' + (r - kr - 1)N' - (\alpha+\lambda)N + \lambda = 0. \tag{35}$$

The boundary conditions become $N(k_1) = N(k_2) = 1$.

The general solution of Eq. (35) is

$$\begin{aligned} N(r) &= r^{k-\frac{\zeta+1}{2}} e^{-2/r}\left\{c_1 M\left(\frac{1+\zeta}{2}+k, 1+\zeta, \frac{2}{r}\right)\right.\\ &\left. + c_2 U\left(\frac{1+\zeta}{2}+k, 1+\zeta, \frac{2}{r}\right)\right\} + \frac{\lambda}{\alpha+\lambda}, \qquad (36)\end{aligned}$$

where

$$\zeta := \sqrt{4(k^2-k) + 8(\alpha+\lambda) + 1}, \tag{37}$$

$M(\cdot,\cdot,\cdot)$ and $U(\cdot,\cdot,\cdot)$ are *Kummer functions*, and the constants c_1 and c_2 are determined from the boundary conditions $N(k_1) = N(k_2) = 1$. In the special case when $k = 1$, the solution can be expressed in terms of the *Bessel functions* $I_\nu(\cdot)$ and $K_\nu(\cdot)$ as follows:

$$N(r) = \Big\{ c_1 \left[(\gamma+1)r+2\right] I_{\frac{\gamma}{2}}(1/r) + 2 I_{\frac{\gamma}{2}+1}(1/r) + c_2 \left[(\gamma+1)r+2\right] K_{\frac{\gamma}{2}}(1/r) - 2 K_{\frac{\gamma}{2}+1}(1/r) \Big\} \times \frac{e^{-1/r}}{\sqrt{r}} + \frac{\lambda}{\alpha+\lambda}, \tag{38}$$

where

$$\gamma := \sqrt{8(\alpha+\lambda)+1}. \tag{39}$$

Next, we assume that the function $m_2(x,y) := E[T_2(x,y)]$ can be written as $N_1(r)$. Proceeding as above, we find that the function $N_1(r)$ satisfies the ODE

$$-1 = \frac{1}{2} r^2 N_1''(r) + (r - kr - 1) N_1'(r) - (\alpha+\lambda) N_1(r), \tag{40}$$

subject to $N_1(k_1) = N_1(k_2) = 0$. The general solution of Eq. (40) is

$$N_1(r) = \left\{ c_1 M\left(\frac{1+\zeta_0}{2}+k, 1+\zeta_0, \frac{2}{r}\right) + c_2 U\left(\frac{1+\zeta_0}{2}+k, 1+\zeta_0, \frac{2}{r}\right) \right\} \times r^{k-\frac{\zeta_0+1}{2}} e^{-2/r} + \frac{1}{\lambda}, \tag{41}$$

where

$$\zeta_0 := \sqrt{4(k^2-k)+8\lambda+1}. \tag{42}$$

This solution can also be expressed in terms of Bessel functions when $k = 1$.

Finally, we define

$$p_2(x,y) = P[X(T_2)/Y(T_2) = k_1]. \tag{43}$$

We assume that $p_2(x,y) = N_2(r)$. To obtain the function $N_2(r)$, we must find the solution of the ODE

$$0 = \frac{1}{2} r^2 N_2''(r) + (r - kr - 1) N_2'(r) - \lambda N_2(r) + \lambda p_1 \tag{44}$$

that is such that $N_2(k_1) = 1$ and $N_2(k_2) = 0$. We find that the general solution of the above equation is

$$N_2(r) = \left\{ c_1 M\left(\frac{1+\zeta_0}{2}+k, 1+\zeta_0, \frac{2}{r}\right) + c_2 U\left(\frac{1+\zeta_0}{2}+k, 1+\zeta_0, \frac{2}{r}\right) \right\} \times r^{k-\frac{\zeta_0+1}{2}} e^{-2/r} + p_1. \tag{45}$$

As in the previous cases, the function $N_2(r)$ can be expressed in terms of Bessel functions when $k = 1$.

4 Conclusion

First-exit problems for diffusion processes have applications in many areas. To solve such problems, one usually needs to find the solution of a differential equation that satisfies certain boundary conditions. In two or more dimensions, the equation to be solved is a PDE. Therefore, the problem is difficult.

In this paper, we have added the constraint that the two-dimensional process leaves the continuation region at the latest at a random time that follows an exponential distribution, which further increases the difficulty of the problems considered.

In Section 2, we treated the case where the diffusion process $\{Y(t), t \geq 0\}$ is a standard Brownian motion and the continuation region C is a rectangle located in the first quadrant. We have succeeded in obtaining the Laplace transform of the three functions of interest.

In Section 3, $\{Y(t), t \geq 0\}$ was a geometric Brownian motion and the set C was the region between two straight lines that intersect at the origin. In this case, we obtained, using the method of similarity solutions, the exact expressions for the functions of interest.

As a continuation of this work, we could try to solve this type of problem when the jumps are continuous rather than discrete random variables. We could also try to solve stochastic optimal control problems defined in terms of the processes studied.

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References:

[1] Caravelli, F., Mansour, T., Sindoni, L. and Severini, S., On moments of the integrated exponential Brownian motion, *The European Physical Journal Plus*, Vol. 131, 2016, Article 245. https://doi.org/10.1140/epjp/i2016-16245-9

[2] Goldman, M., On the first passage of the integrated Wiener process, *Annals of Mathematical Statistics*, Vol. 42, No. 6, 1971, pp. 2150–2155. https://doi.org/10.1214/aoms/1177693084

[3] Gor'kov, Yu. P., A formula for the solution of a certain boundary value problem for the stationary equation of Brownian motion, *Soviet Mathematics. Doklady*, Vol. 16, 1975, pp. 904–908.

[4] Hesse, C. H., The one-sided barrier problem for an integrated Ornstein-Uhlenbeck process, *Communications in Statistics. Stochastic Models*, Vol. 7, No. 3, 1991, pp. 447–480. https://doi.org/10.1080/15326349108807200

[5] Lachal, A., L'intégrale du mouvement brownien, *Journal of Applied Probability*, Vol. 30, No. 1, 1993, pp. 17–27. https://doi.org/10.2307/3214618

[6] Lefebvre, M., Moment generating function of a first hitting place for the integrated Ornstein-Uhlenbeck process, *Stochastic Processes and their Applications*, Vol. 32, No. 2, 1989, pp. 281–287. https://doi.org/10.1016/0304-4149(89)90080-X

[7] Lefebvre, M., First hitting time and place for the integrated geometric Brownian motion, *International Journal of Differential Equations and Applications*, Vol. 9, No. 4, 2004, pp. 365–374.

[8] Lefebvre, M., Moments of first-passage places and related results for the integrated Brownian motion, *ROMAI Journal*, Vol. 2, No. 2, 2006, pp. 101–108.

[9] Lefebvre, M., A first-passage problem for exponential integrated diffusion processes, *Journal of Stochastic Analysis*, Vol. 3, No. 3, 2022, Article 2. https://doi.org/10.31390/josa.3.3.02

[10] Levy, E., On the moments of the integrated geometric Brownian motion, *Journal of Computational and Applied Mathematics*, Vol. 342, 2018, pp. 263–273. https://doi.org/10.1016/j.cam.2018.04.005

[11] Makasu, C., Exit probability for an integrated geometric Brownian motion, *Statistics & Probability Letters*, Vol. 79, No. 11, 2009, pp. 1363–1365. https://doi.org/10.1016/j.spl.2009.02.009

[12] McKean, H. P., A winding problem for a resonator driven by a white noise, *Journal of Mathematics of Kyoto University*, Vol. 2, 1963, pp. 227–235. https://doi.org/10.1215/kjm/1250524936

[13] Metzler, A., The Laplace transform of hitting times of integrated geometric Brownian motion, *Journal of Applied Probability*, Vol. 50, No. 1, 2013, pp. 295–299. https://doi.org/10.1239/jap/1363784440

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